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A NOTE ON THE INDEPENDENT DOMINATION
NUMBER OF SUBSET GRAPHXUE-GANG CHEN, Shantou, DE-XIANG MA, Taian,
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Abstract. The independent domination number $i(G)$ (independent number $\beta(G)$) is the minimum (maximum) cardinality among all maximal independent sets of G . Haviland (1995) conjectured that any connected regular graph G of order n and degree $\delta \leq \frac{1}{2}n$ satisfies $i(G) \leq \lceil 2n/3\delta \rceil \frac{1}{2}\delta$. For $1 \leq k \leq l \leq m$, the subset graph $S_m(k, l)$ is the bipartite graph whose vertices are the k - and l -subsets of an m element ground set where two vertices are adjacent if and only if one subset is contained in the other. In this paper, we give a sharp upper bound for $i(S_m(k, l))$ and prove that if $k + l = m$ then Haviland's conjecture holds for the subset graph $S_m(k, l)$. Furthermore, we give the exact value of $\beta(S_m(k, l))$.

Keywords: independent domination number, independent number, subset graph

MSC 2000: 05C69, 05C35

1. INTRODUCTION

Let $G = (V, E)$ be a simple graph of order n . The degree, neighborhood and closed neighborhood of a vertex v in the graph G are denoted by $d(v)$, $N(v)$ and $N[v] = N(v) \cup \{v\}$ respectively. The minimum degree and maximum degree of the graph G are denoted by $\delta(G)$ and $\Delta(G)$ respectively. The graph induced by $S \subseteq V$ is denoted by $\langle S \rangle$. Let $\varepsilon(S, V - S)$ denote the number of edges between S and $V - S$.

For integers $1 \leq k \leq l \leq m$, we define the subset graph $S_m(k, l)$ to be the bipartite graph (χ, E, ψ) where the vertices of χ are the k -subsets of $[m] = \{1, 2, \dots, m\}$, the vertices of ψ are the l -subsets of $[m]$, and for $X \in \chi$ and $Y \in \psi$, X is adjacent to Y

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if and only if $X \subseteq Y$. Notice that the subset graph $S_m(k, k)$ is a matching with $\binom{m}{k}$ edges, and if $k + l = m$ then $S_m(k, l)$ is a regular graph.

An independent set is a set of pairwise nonadjacent vertices of G . The independent domination number $i(G)$ is the minimum cardinality of a maximal independent set of G , while the maximum cardinality of an independent set of vertices of G is the independent number of G and is denoted by $\beta(G)$.

A set of vertices is a dominating set if $N[S] = V$. The domination number of a graph G , denoted $\gamma(G)$, is the minimum cardinality of a dominating set in G , and the upper domination number $\Gamma(G)$ is the maximum cardinality of a dominating set in G .

For $x \in X \subseteq V$, if $N[x] - N[X - \{x\}] = \emptyset$, then x is said to be redundant in X . A set X containing no redundant vertex is called irredundant. The irredundant number of G , denoted by $ir(G)$, is the minimum cardinality taken over all maximal irredundant sets of G . The upper irredundant number of G , denoted by $IR(G)$, is the maximum cardinality of an irredundant set of G . Let $EPN(v, X) = \{u \in V - X : u \text{ is only adjacent to } v \text{ but to no other vertex of } X\}$.

The parameter $i(G)$ was introduced by Cockayne and Hedetniemi in [1] and some results on it can be found in [1]–[7]. Favaron [2] and Haviland [3] established upper bounds for $i(G)$ in terms of n and δ . For regular graphs of degree different from zero, we can prove that $i(G) \leq \frac{1}{2}n$. However, for most values of δ , this is far from the best possible. In [2], it was shown that for any graph with $\frac{1}{2}n \leq \delta \leq n$, we have $i(G) \leq n - \delta$, and this bound could be attained only by complete multipartite graphs with vertex classes all of the same order. By adapting the arguments from [3], the following results can readily be proved (see [4]).

Proposition 1.1. *Let G be a regular graph. If $\frac{1}{4}n \leq \delta \leq \frac{1}{2}(3 - \sqrt{5})n$, then $i(G) \leq n - \sqrt{n\delta}$, and if $\frac{1}{2}(3 - \sqrt{5})n \leq \delta \leq \frac{1}{2}n$, then $i(G) \leq \delta$.*

If $n = 2m\delta$, then $i(mK_{\delta,\delta}) = \frac{1}{2}n$ and $mK_{\delta,\delta}$ is disconnected for $m > 1$. Haviland [3] thought that if G was connected then the upper bound for $i(G)$ could be a function of n and δ . She also stated the following conjecture in [4].

Conjecture 1.2. *If G is a connected r -regular graph with $r = \delta \leq \frac{1}{2}n$, then $i(G) \leq \lceil 2n/3\delta \rceil \frac{1}{2}\delta$.*

However, Pear Che Bor Lam et al. [7] provided counterexamples to Conjecture 1.2.

In this paper, we give a sharp upper bound for $i(S_m(k, l))$ and prove that if $k + l = m$ then Haviland's conjecture holds for the subset graph $S_m(k, l)$. Furthermore, we give the exact value of $\beta(S_m(k, l))$.

2. MAIN RESULTS

By the definition of the subset graph, it is easy to prove the following two lemmas.

Lemma 1. *If $k = l$, then $i(S_m(k, l)) = \beta(S_m(k, l)) = \binom{m}{k}$.*

Lemma 2. *If $1 \leq k < l = m$, then $i(S_m(k, l)) = 1$ and $\beta(S_m(k, l)) = \binom{m}{k}$.*

Now, we give the main results of this paper.

Theorem 1. *If $1 \leq k \leq l \leq m$, then $i(S_m(k, l)) \leq \binom{m-l+k}{k}$ and the bound is sharp.*

Proof. Let $d = l - k$. By Lemma 1 and Lemma 2, if $k = l$ or $l = m$, Theorem 1 holds. So, we only consider $1 \leq k < l < m$. Let

$$A = \{X \in \chi : i \notin X \text{ for } m - d \leq i \leq m\}$$

and

$$B = \{Y \in \psi : i \in Y \text{ for } m - d \leq i \leq m\}.$$

We have the following claims.

Claim 1. *$A \cup B$ is an independent set of $S_m(k, l)$.*

Let t_1 and t_2 be arbitrary two vertices of $A \cup B$. If $t_1, t_2 \in A$ or $t_1, t_2 \in B$, then it is obvious that t_1 is not adjacent to t_2 . Without loss of generality, we assume that $t_1 \in A$ and $t_2 \in B$. Let $t_1 = \{x_1, x_2, \dots, x_k\}$ and $t_2 = \{y_1, y_2, \dots, y_l\}$ where $1 \leq x_1 < x_2 < \dots < x_k < m$ and $1 \leq y_1 < y_2 < \dots < y_l < m$. Since $i \notin t_1$ and $i \in t_2$ for $m - d \leq i \leq m$, $\{y_1, y_2, \dots, y_l\}$ has at most $l - (d + 1)$ elements which are identical to elements of $\{x_1, x_2, \dots, x_k\}$. Since $l - (d + 1) = k - 1 < k$, it follows that $\{x_1, x_2, \dots, x_k\} \not\subseteq \{y_1, y_2, \dots, y_l\}$. Hence, t_1 is not adjacent to t_2 . Since t_1 and t_2 are arbitrary two vertices of $A \cup B$, $A \cup B$ is an independent set of $S_m(k, l)$.

Claim 2. *$A \cup B$ is a dominating set of $S_m(k, l)$.*

For an arbitrary vertex $t \in (V(S_m(k, l)) - (A \cup B))$, we prove that t is dominated by at least one vertex of $A \cup B$.

Case 1: $t \in (\chi - A)$. Let $t = \{x_1, x_2, \dots, x_k\}$ where $1 \leq x_1 < x_2 < \dots < x_k < m$. Then there exists a x_i such that $x_i \in \{m - d, m - d + 1, \dots, m\}$. Without loss of generality, we assume that x_s is the first number such that $x_s \in \{m - d, m - d + 1, \dots, m\}$. Let $C = \{x_1, x_2, \dots, x_{s-1}, x_{m-d}, \dots, x_m\}$. Since $k \geq s$, $|C| = s - 1 + d +$

$1 = s + d = s + l - k = l - (k - s) \leq l$. So there exists a vertex $Y \in \psi \cap B$ such that $\{x_1, x_2, \dots, x_k\} \subseteq \{x_1, x_2, \dots, x_{s-1}, x_{m-d}, \dots, x_m\} \subseteq Y$. Hence t is adjacent to Y .

Case 2: $t \in (\psi - B)$. Let $t = \{y_1, y_2, \dots, y_l\}$ where $1 \leq y_1 < y_2 < \dots < y_l < m$. Then there exists an $i \in \{m-d, \dots, m\}$ such that $y_j \neq i$ for $1 \leq j \leq l$. Let y_s be the first number that belongs to $\{m-d, \dots, m\}$ and let $C = \{y_1, y_2, \dots, y_{s-1}\}$. Since $l - (s-1) < d+1 = l-k+1$, it follows that $k < s$ and $|C| \geq k$. Hence if X is a k -subset of C , then $X \subseteq A$ and X is adjacent to t . Since t is an arbitrary vertex, by Case 1 and Case 2, it follows that $A \cup B$ is a dominating set of $S_m(k, l)$. By Claim 1 and Claim 2, $A \cup B$ is an independent dominating set of $S_m(k, l)$. Hence,

$$\begin{aligned} i(S_m(k, l)) &\leq |A \cup B| = |A| + |B| = \binom{m - (d+1)}{k} + \binom{m - (d+1)}{l - (d+1)} \\ &= \binom{m - (d+1)}{k} + \binom{m - (d+1)}{k-1} = \binom{m-l+k}{k}. \end{aligned}$$

Corollary 1. *If $1 \leq k \leq l < m$ and $k+l = m$, then $i(S_m(k, l)) \leq \binom{2k}{k}$ and the bound is sharp.*

The sharpness of Theorem 1 and Corollary 1 can be seen from the following result.

Theorem 2. *If $1 < l < m$ and $l+1 = m$, then $i(S_m(1, l)) = 2 = \binom{2}{1} = \binom{2k}{k}$.*

Proof. Since $1 < l < m$, it follows that $m \geq 3$ and $S_m(1, l)$ is not a star. Hence, $\gamma(S_m(1, l)) \geq 2$. Since $2 \leq \gamma(S_m(1, l)) \leq i(S_m(1, l)) \leq 2$, it follows that $i(S_m(1, l)) = 2$. \square

Theorem 3. $i(S_5(2, 3)) = 6 = \binom{4}{2} = \binom{2k}{k}$.

Proof. Since $S_5(2, 3)$ is a 3-regular graph,

$$i(S_5(2, 3)) \geq \gamma(S_5(2, 3)) \geq \frac{|V(S_5(2, 3))|}{\Delta(S_5(2, 3)) + 1} = \frac{2\binom{5}{2}}{4} = 5.$$

If $\gamma(S_5(2, 3)) = 5$, then let I be a dominating set of $S_5(2, 3)$ with cardinality 5. We have the following claims.

Claim 1. I is an independent set of $S_5(2, 3)$.

Otherwise, if I is not an independent set, then there exists at least one edge in $\langle I \rangle$. Hence, $\varepsilon(I, V - I) \leq \sum_{v \in I} d(v) - 2 = 3 \times 5 - 2 = 13 < 15 = |V(S_5(2, 3)) - I|$. So there exists a vertex $v \in V - I$ such that v is not dominated by I , which is a contradiction.

Claim 2. For each vertex $v \in I$, $|\text{EPN}(v, I)| = 3$.

By Claim 1, I is an independent dominating set of $S_5(2, 3)$. If there exists a vertex $v \in I$ such that $|\text{EPN}(v, I)| < 3$, then I dominates at most $\sum_{v \in I} d(v) - 1 = 3 \times 5 - 1 = 14 < 15 = |V(S_5(2, 3)) - I|$ vertices, which is a contradiction.

Let $A = \chi \cap I$ and $B = \psi \cap I$. Since $|I| = 5$, without loss of generality, we assume that $|A| \geq 3$. It is obvious that $|A| \leq 4$. So, $3 \leq |A| \leq 4$.

Case 1: If $|A| = 4$, then by Claim 1 and Claim 2 the set A dominates 12 vertices of ψ , which is a contradiction since ψ has 10 vertices.

Case 2: If $|A| = 3$, then by Claim 1 and Claim 2 the set A dominates 9 vertices of ψ . So, there is only one vertex of ψ that belongs to I , which is a contradiction with $|B| = 2$.

Hence, $\gamma(S_5(2, 3)) \geq 6$. Since $6 \leq \gamma(S_5(2, 3)) \leq i(S_5(2, 3)) \leq 6$, it follows that $i(S_5(2, 3)) = 6$.

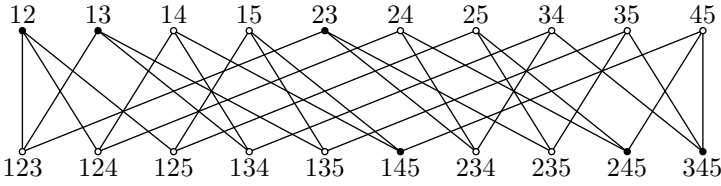


Figure 1.

By Figure 1, it is easy to see that the black vertices form an independent dominating set of $S_5(2, 3)$ with cardinality 6.

The following theorem proves that conjecture 1.2 holds for the subset graph $S_m(k, l)$ if $1 \leq k < l < m$ and $k + l = m$.

Theorem 4. If $1 \leq k < l < m$ and $k + l = m$, then Conjecture 1.2 holds for the subset graph $S_m(k, l)$.

Proof. Let $d = l - k$. If $1 \leq k < l < m$ and $k + l = m$ then $S_m(k, l)$ is a connected regular graph with $n = |V(S_m(k, l))| = 2^{\binom{m}{k}}$ and $\delta \leq \frac{1}{2}n$. By Corollary 1, $i(S_m(k, l)) \leq \binom{2k}{k}$. It follows that

$$\frac{i(S_m(k, l))}{|V(S_m(k, l))|} \leq \frac{\binom{2k}{k}}{2^{\binom{m}{k}}} = \frac{\binom{2k}{k}}{2^{\binom{2k+d}{k}}} \leq \frac{\binom{2k}{k}}{2^{\binom{2k+1}{k}}} = \frac{(2k)! k!(k+1)!}{2k!k! (2k+1)!} = \frac{k+1}{2(2k+1)} \leq \frac{1}{3}.$$

Hence,

$$i(S_m(k, l)) \leq \frac{|V(S_m(k, l))|}{3} = \frac{n}{3} \leq \frac{2n}{3} \frac{\delta}{2} \leq \left\lceil \frac{2n}{3\delta} \right\rceil \frac{\delta}{2}.$$

The exact value of $\beta(S_m(k, l))$ is given by the following result.

Lemma 3 [5]. For every r -regular graph $G = (V, E)$ of order n , $\text{IR}(G) \leq \frac{1}{2}n$.

Lemma 4 [6]. If G is bipartite, then $\beta = \Gamma = \text{IR}$.

Theorem 5. If $1 \leq k < l \leq m$, then $\beta(S_m(k, l)) = \Gamma(S_m(k, l)) = \text{IR}(S_m(k, l)) = \max\{\binom{m}{k}, \binom{m}{l}\}$.

Proof. Since $S_m(k, l)$ is the bipartite graph, then by Lemma 4, $\beta(S_m(k, l)) = \Gamma(S_m(k, l)) = \text{IR}(S_m(k, l))$.

Case 1: If $k + l = m$, then $S_m(k, l)$ is a regular graph. By Lemma 3, $\text{IR}(G) \leq \frac{1}{2}|V(S_m(k, l))| = \binom{m}{k}$. Since $\beta(S_m(k, l)) \geq \binom{m}{k}$, it follows that $\beta(S_m(k, l)) = \Gamma(S_m(k, l)) = \text{IR}(S_m(k, l)) = \max\{\binom{m}{k}, \binom{m}{l}\}$.

Case 2: If $k + l \neq m$, then $\beta(S_m(k, l)) \geq \max\{\binom{m}{k}, \binom{m}{l}\}$. Without loss of generality, assume $\binom{m}{k} = \max\{\binom{m}{k}, \binom{m}{l}\}$. That is to say $|\chi| > |\psi|$ and $\beta(S_m(k, l)) \geq \binom{m}{k}$. For arbitrary vertices $X \in \chi$ and $Y \in \psi$, $d(X) < d(Y)$. If $\beta(S_m(k, l)) > \binom{m}{k} = |\chi|$, then let I be a maximal independent set with cardinality $\beta(S_m(k, l))$. Hence, I must contain some vertices of χ and some vertices of ψ . Let $X_1 = I \cap \chi$ and $Y_1 = I \cap \psi$. Let $X_2 = \chi - X_1$ and $Y_2 = \psi - Y_1$. So $X_i \neq \emptyset$ and $Y_i \neq \emptyset$ for $i = 1, 2$. Since

$$\varepsilon(Y_1, V - I) = \varepsilon(Y_1, X_2) = \sum_{Y \in Y_1} d(Y) = d(Y)|Y_1| \leq \sum_{X \in X_2} d(X) = d(X)|X_2|$$

and $d(X) < d(Y)$, it follows that $|Y_1| < |X_2|$. Hence $|I| = |X_1| + |Y_1| < |X_1| + |X_2| = |X|$, which is a contradiction. Hence, $\beta(S_m(k, l)) = \binom{m}{k}$. So, $\beta(S_m(k, l)) = \Gamma(S_m(k, l)) = \text{IR}(S_m(k, l)) = \max\{\binom{m}{k}, \binom{m}{l}\}$.

References

- [1] *E. J. Cockayne and S. T. Hedetniemi:* Independence graphs. Proc. 5th Southeast Conf. Comb. Graph Theor. Comput. Utilitas Math., Boca Raton, 1974, pp. 471–491.
- [2] *O. Favaron:* Two relations between the parameters of independence and irredundance. Discrete Math. 70 (1988), 17–20.
- [3] *J. Haviland:* On minimum maximal independent sets of a graph. Discrete Math. 94 (1991), 95–101.
- [4] *J. Haviland:* Independent domination in regular graphs. Discrete Math. 143 (1995), 275–280.
- [5] *M. A. Henning and P. J. Slater:* Inequality relating domination parameters in cubic graphs. Discrete Math. 158 (1996), 87–98.
- [6] *E. J. Cockayne, O. Favaron, C. Payan and A. G. Thomason:* Contributions to the theory of domination, independence and irredundance in graphs. Discrete Math. 33 (1981), 249–258.

- [7] *P. C. B. Lam, W. C. Shiu and L. Sun*: On independent domination number of regular graphs. *Discrete Math.* 202 (1999), 135–144.

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