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NONLINEAR BOUNDARY VALUE PROBLEMS FOR SECOND ORDER DIFFERENTIAL INCLUSIONS

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Abstract. In this paper we study two boundary value problems for second order strongly nonlinear differential inclusions involving a maximal monotone term. The first is a vector problem with Dirichlet boundary conditions and a nonlinear differential operator of the form $x \mapsto a(x, x')'$. In this problem the maximal monotone term is required to be defined everywhere in the state space $\mathbb{R}^N$. The second problem is a scalar problem with periodic boundary conditions and a differential operator of the form $x \mapsto (a(x)x')'$. In this case the maximal monotone term need not be defined everywhere, incorporating into our framework differential variational inequalities. Using techniques from multivalued analysis and from nonlinear analysis, we prove the existence of solutions for both problems under convexity and nonconvexity conditions on the multivalued right-hand side.

Keywords: measurable multifunction, usc and lsc multifunction, maximal monotone operator, pseudomonotone operator, generalized pseudomonotone operator, coercive operator, surjective operator, eigenvalue, eigenfunction, Rayleigh quotient, p-Laplacian, Yosida approximation, periodic problem.

MSC 2000: 34B15

Introduction

The purpose of this paper is to study boundary value problems for a large class of nonlinear second order differential inclusions. In the last decade there have been several papers dealing with boundary value problems for differential inclusions. We mention the papers of Erbe-Krawcewicz [10], [11], [12], Erbe-Krawcewicz-Peschke [13], Frigon [15], [16], Frigon-Granas [17], Halidias-Papageorgiou [20], [21], Kandilakis-Papageorgiou [25] and Pruszko [34]. All these papers (with the exception of Halidias-Papageorgiou [21]) deal with the semilinear problem, i.e. the differential operator is $x \mapsto x''$ and there is no multivalued maximal monotone operator present in the equation (as it is here, see problems (1), (2) below). Of the above mentioned papers, Frigon [15], [16] and Frigon-Granas [17] study scalar problems with certain Sturm-Liouville type boundary conditions and employ the method of upper and lower solutions appropriately modified to fit the set-valued character of the problem. The other papers examine the vector problem and in Erbe-Krawcewicz [10], [11], Halidias-Papageorgiou [20], [21] and Kandilakis-Papageorgiou [25], the authors use general nonlinear boundary conditions which include as special cases the classical Dirichlet, Neumann and periodic boundary conditions. In addition our work is related to the recent papers which examine single-valued differential equations involving the one-dimensional p-Laplacian. We mention the work of Boccardo-Drábek-Giachetti-Kučera [2], Dang-Oppenheimer [7], Del Pino-Elgueta-Manasevich [8], Drábek [9], Fabry-Fayyad [14], Guo [19], Manasevich-Mawhin [28] and the references therein. All these papers (with the exception of Manasevich-Mawhin [28]) study scalar problems. In Boccardo-Drábek-Giachetti-Kučera [2], Del Pino-Elgueta-Manasevich [8], the boundary conditions are Dirichlet. In Fabry-Fayyad [14] and Manasevich-Mawhin [28] the authors investigate the periodic problem, while Guo [19] deals with both the periodic and the Neumann problems and finally Dang-Oppenheimer [7] examine all three problems (Dirichlet, Neumann and periodic). It should be mentioned here that Dang-Oppenheimer [7] and Manasevich-Mawhin [28] use a general p-Laplacian-like differential operator which is not necessarily homogeneous and has no growth restrictions. However, their operator is independent of x and depends only on x' and in their problem there is no multivalued maximal monotone operator (so they can not treat variational inequalities).

In this paper we study the following multivalued boundary value problem:

$$(1)\qquad \left\{\begin{array}{l} (a(x(t),x'(t)))' - A(x(t)) - F(t,x(t),x'(t)) \ni 0 \ \text{ a.e. on } \ T=[0,b], \\ x(0)=x(b)=0. \end{array}\right\}$$

Here $A\colon \mathbb{R}^N \to 2^{\mathbb{R}^N} \setminus \{\emptyset\}$ is a maximal monotone map and $a\colon \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}^N} \setminus \{\emptyset\}$, $F\colon T \times \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}^N} \setminus \{\emptyset\}$ are set-valued functions. The presence of the operator A in (1) incorporates into our framework second order systems with an autonomous, convex, in general nonsmooth potential. Semilinear such systems with smooth (but possibly time-varying and nonconvex) potential can be found in the book of Mawhin-Willem [31].

By a solution of (1) we mean a function $x \in W_0^{1,p}(T,\mathbb{R}^N)$ such that there exist $f \in L^q(T,\mathbb{R}^N)$ with $f(t) \in F(t,x(t),x'(t))$ a.e. on T and $g \in W^{1,q}(T,\mathbb{R}^N)$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$ with $g(t) \in a(x(t),x'(t))$ a.e. on T such that $g'(t)-f(t) \in A(x(t))$ a.e. on T.

In Section 4 we deal with the following scalar periodic problem:

$$\text{(2)} \qquad \left\{ \begin{array}{l} (a(x(t))x'(t))' \in A(x(t)) + F(t, x(t), x'(t)) \text{ a.e. on } T = [0, b], \\ x(0) = x(b), \ \ x'(0) = x'(b). \end{array} \right\}$$

Now a is single-valued, $A\colon \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone map and in contrast to problem (1), $\operatorname{dom} A = \{x \in \mathbb{R}\colon \ A(x) \neq \emptyset\} \neq \mathbb{R}$. So our formulation incorporates in particular second order differential inequalities. Note that because the problem is scalar, $A = \partial\varphi$ with $\varphi\colon \mathbb{R} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ proper, convex and lower semicontinuous (see for example Brezis [3, p. 43] or Hu-Papageorgiou [23, p. 348]). Vectorial versions of problem (2), with the differential operator being $x \mapsto a(x')'$ not necessarily homogeneous and with no growth restrictions on $a(\cdot)$, can be found in the very recent work of Kyritsi-Matzakos-Papageorgiou [27].

By a solution of (2) we mean a function $x \in C^1(T)$ such that $x(0) = x(b)$, $x'(0) = x'(b)$ and there exists $f \in L^q(T, \mathbb{R}^N)$ with $f(t) \in F(t, x(t), x'(t))$ for which we have $(a(x(t))x'(t))' - f(t) \in A(x(t))$ a.e. on T.

In Section 2 we provide the mathematical background needed to follow the arguments of this paper. In Section 3 we study problem (1) and finally in Section 4 we deal with problem (2).

2. Mathematical preliminaries

Our approach will be based on notions and results from multivalued analysis and the theory of nonlinear operators of monotone type. So for the convenience of the reader, in this section we recall some basic definitions and facts from these areas. Our main sources are the books of Klein-Thompson [26], Hu-Papageorgiou [23] and Zeidler [36] and the paper of Browder-Hess [4] for pseudomonotone operators.

Let (Ω, Σ) be a measurable space and X a separable Banach space. We introduce the following notations:

$$P_{f(c)}(X) = \{A \subseteq X\colon \text{ nonempty closed (and convex)}\} \text{ and}$$
$$P_{(w)k(c)}(X) = \{A \subseteq X \text{ nonempty, (weakly-) compact (and convex)}\}.$$

A multifunction $F\colon \Omega \to P_f(X)$ is said to be *measurable*, if for all $x \in X$, $\omega \mapsto d(x, F(\omega))$ is measurable. A multifunction $F\colon \Omega \to 2^X \setminus \{\emptyset\}$ is said to be *graph measurable*, if $\operatorname{Gr} F = \{(\omega, x) \in \Omega \times X\colon \ x \in F(\omega)\} \in \Sigma \times B(X)$, with $B(X)$ being the Borel σ-field of X. For $P_f(X)$-valued multifunctions, measurability implies graph measurability and the converse is true if Σ is complete (i.e. $\Sigma = \hat{\Sigma} =$ the universal σ-field; this is the case if for example there exists a σ-finite measure μ on Ω with respect to which Σ is complete, see Cohn [6]). Let μ be a finite measure on (Ω,Σ). For a multifunction $F\colon \Omega \to 2^X \setminus \{\emptyset\}$ and $1 \leqslant p \leqslant \infty$, we define $S_F^p = \{f \in L^p(\Omega,X)\colon f(\omega) \in F(\omega)$ for μ-a.a. $\omega \in \Omega\}$. This set may be empty. For a graph measurable multifunction it is nonempty if and only if $\inf\{\|x\|\colon x \in F(\omega) \leqslant \varphi(\omega)\}$ for μ-*a.a.* $\omega \in \Omega$, with $\varphi \in L^p(\Omega)$. If $\{A_n\}_{n\geqslant 1} \subseteq 2^X \setminus \{\emptyset\}$, then we define

$$w\text{-}\limsup_{n\to\infty} A_n = \{x \in X\colon\ x = w\text{-}\lim x_{n_k},\ \ x_{n_k} \in A_{n_k},\ \ n_1 < \ldots < n_k < \ldots\}.$$

Here w- stands for the weak topology on X. So $w\text{-}\limsup\limits_{n\to\infty} A_n$ is the set of all weak subsequential limits of all sequences $\{x_n\}_{n\geqslant 1}$ with $x_n \in A_n$, $n \geqslant 1$. Moreover, if $C \subseteq X$, then $\overline{\mathrm{conv}}\, C$ denotes the closed convex hull of C.

In our analysis we shall need the Yankov-von Neumann-Aumann selection theorem. For the convenience of the reader, we recall a version of this result that we shall use in the sequel. For the general form of the theorem and a proof of it we refer to Hu-Papageorgiou [23, p. 158], Klein-Thompson [26, p. 166] or Wagner [35].

Theorem A. *If (Ω,Σ,μ) is a complete, σ-finite measure space, X is a separable complete metric space and $F\colon \Omega \to 2^X \setminus \{\emptyset\}$ is a multifunction such that $\mathrm{Gr}\, F = \{(\omega,x) \in \Omega \times X\colon x \in F(\omega)\} \in \Sigma \times B(X)$ (i.e. F is graph measurable), then there exists a Σ-measurable function $f\colon \Omega \to X$ such that $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$.*

Another auxiliary result that will be useful in our analysis is the next Proposition which gives information about the pointwise behavior of a weakly convergent sequence in the Lebesgue-Bochner space $L^p(\Omega,X)$, $1 \leqslant p < \infty$. For a proof of this result, we refer to the paper of Papageorgiou [33] or the book of Hu-Papageorgiou [23, p. 694].

Proposition B. *If (Ω,Σ,μ) is a finite measure space, X is a Banach space, $\{f_n,f\}_{n\geqslant 1} \subseteq L^p(\Omega,X)$, $1 \leqslant p < \infty$, $f_n \xrightarrow{w} f$ in $L^p(\Omega,X)$ and for μ-almost all $\omega \in \Omega$ there exists a nonempty, weakly compact set $G(\omega)$ such that $f_n(\omega) \in G(\omega)$ for all $n \geqslant 1$, then $f(\omega) \in \overline{\mathrm{conv}}[w\text{-}\limsup\limits_{n\to\infty}\{f_n(\omega)\}]$ μ-a.e. on Ω.*

Let Y, Z be Hausdorff topological spaces. A multifunction $G\colon Y \to 2^Z \setminus \{\emptyset\}$ is said to be *lower semicontinuous* (lsc) (resp. *upper semicontinuous* (usc)), if for all $C \subseteq Z$ closed, the set $G^+(C) = \{y \in Y\colon G(y) \subseteq C\}$ (resp. $G^-(C) = \{y \in Y\colon G(y) \cap C \neq \emptyset\}$) is closed. An usc multifunction has closed graph in $Y \times Z$, while the converse is true if G is locally compact (i.e. for every $y \in Y$ there exists a neighborhood U of y such that $\overline{G(U)}$ is compact in Z). Also a $P_k(Z)$-valued multifunction G which is usc, maps compact sets of Y into compact sets of Z. A multifunction which is both usc and lsc, is said to be *continuous* (or sometimes Vietoris continuous). If Z is a metric space and $A, C \subseteq Z$, we set $h^*(A, C) = \sup[d(a, C)\colon\ a \in A]$ (the excess of A over C) and $h(A, C) = \max\{h^*(A, C), h^*(C, A)\}$ (the Hausdorff distance between A and C). We know that $h(\cdot, \cdot)$ is a metric on $P_f(Z)$ and if Z is a complete metric space, then so is $(P_f(Z), h)$. A multifunction $G\colon Y \to P_f(Z)$ which is continuous into the metric space $(P_f(Z), h)$ is said to be *h-continuous*. For $P_k(Z)$-valued multifunctions continuity and h-continuity are equivalent notions. Moreover, if $G\colon Y \to P_k(Z)$, then G is lsc if and only if for every $y_0 \in Y$ the function $y \to h^*(G(y_0), G(y))$ is continuous.

Next let X be a reflexive Banach space and X^* its topological dual. A map $A\colon D \subseteq X \to 2^{X^*}$ is said to be *monotone*, if for all $x^* \in A(x)$, $y^* \in A(y)$, we have $(x^* - y^*, x - y) \geqslant 0$ (here by $(\cdot, \cdot)$ we denote the duality brackets for the pair (X, X^*)). If $(x^* - y^*, x - y) = 0$ implies $x = y$, we say that A is *strictly monotone*. The map A is said to be *maximal monotone*, if $(x^* - y^*, x - y) \geqslant 0$ for all $x \in D$, $x^* \in A(x)$, imply $y \in D$ and $y^* \in A(y)$, i.e. the graph of A is maximal with respect to inclusion among the graphs of all monotone maps. It is easy to see that the graph of a maximal monotone map A is sequentially closed in $X \times X^*_w$ and in $X_w \times X^*$ (here by X_w and X^*_w we denote the spaces X and X^* furnished with their respective weak topologies). A monotone map is locally bounded at every point in the interior of its domain D (recall $D = \{x \in X\colon\ A(x) \neq \emptyset\}$) and if A is maximal monotone, then $A\big|_{\operatorname{int} D}$ is usc into X^* furnished with the weak topology. A map $A\colon D \subseteq X \to 2^{X^*}$ is said to be *coercive*, if D is bounded or D is unbounded and $\inf\{\|x^*\|\colon\ x^* \in A(x)\} \to \infty$ as $\|x\| \to \infty$. A maximal monotone, coercive map is surjective. If A is monotone, $D = X$ and for all $x, y \in X$, $\lambda \mapsto A(x + \lambda y)$ is usc from $[0, 1]$ into $X^*_{w^*}$ ($= X^*$ with the w^*-topology), then A is maximal monotone.

Let $X = H$ be a Hilbert space and $A\colon D \subseteq H \to 2^H$ a maximal monotone operator. We know that for every $x \in D$, $A(x)$ is nonempty, closed and convex. So the set $A(x)$ contains an element of minimum norm (the projection of the origin on the set $A(x)$). This unique element is denoted by $A^0(x)$. We have $A^0(x) \in A(x)$ and $\|A^0(x)\| = \inf[\|x^*\|\colon\ x^* \in A(x)]$. Also the set $\overline{D}$ is convex. Now for $\lambda > 0$ we define the following well-known operators:

$$J_\lambda = (I + \lambda A)^{-1} \quad \text{(the resolvent of } A)$$

and

$$A_\lambda = \frac{1}{\lambda}(I - J_\lambda) \quad \text{(the Yosida approximation of } A).$$

Recall that according to Minty's theorem (see for example Brezis [3, p. 23]), A is maximal monotone if and only if for every $\lambda > 0$ (equivalently for some $\lambda > 0$), we have $R(I + \lambda A) = H$, i.e. the operator $I + \lambda A$ is surjective. So we see that the operators J_λ and A_λ are defined on all of H and it is easy to see that they are single-valued. Several properties of J_λ and A_λ are collected in the proposition that follows (see Brezis [3, pp. 23 and 28] and Hu-Papageorgiou [23, p. 325]).

Proposition C. *If $A\colon D \subseteq H \to 2^H$ is a maximal monotone operator, then for every $\lambda > 0$ we have*
(a) *J_λ is nonexpansive (i.e. Lipschitz continuous with constant 1);*
(b) *$A_\lambda(x) \in A(J_\lambda(x))$ for all $x \in H$;*
(c) *A_λ is monotone and Lipschitz continuous with constant $1/\lambda$ (therefore A_λ is maximal monotone);*
(d) *$\|A_\lambda(x)\| \leqslant \|A^0(x)\|$ and $A_\lambda(x) \to A^0(x)$ as $\lambda \to 0$ for all $x \in D$;*
(e) *$\overline{D}$ is convex and $J_\lambda(x) \to \operatorname{proj}(x; \overline{D})$ as $\lambda \to 0$ for all $x \in H$.*

Here $\operatorname{proj}(x; \overline{D})$ denotes the metric projection of x on the closed, convex set $\overline{D}$. So according to (e) $J_\lambda(\cdot)$ can be viewed as an approximation of the identity operator. Indeed note that if $D = H$, then $J_\lambda(x) \to x$ for all $x \in H$.

By $\Gamma_0(\mathbb{R}^N)$ we denote the functions $\varphi\colon \mathbb{R}^N \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ which are convex, lower semicontinuous and proper (i.e. not identically $+\infty$). By $\partial\varphi$ we denote the subdifferential of φ, i.e.

$$\partial\varphi(x) = \{x^* \in \mathbb{R}^N\colon (x^*, y - x)_{\mathbb{R}^N} \leqslant \varphi(y) - \varphi(x) \quad \text{for all } y \in \mathbb{R}^N\}.$$

It is well-known that $\partial\varphi\colon \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is maximal monotone. Moreover, if φ is continuous, then $\partial\varphi(x) \neq \emptyset$ for all $x \in \mathbb{R}^N$. In addition φ is locally Lipschitz. For locally Lipschitz (not necessarily convex functions), we have an extension of the notion of subdifferential, which is due to Clarke [5]. Namely let $\varphi\colon X \to \mathbb{R}$ be locally Lipschitz. We introduce the generalized directional derivative

$$\varphi^0(x; h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

The function $h \to \varphi^0(x; h)$ is continuous, sublinear. The Clarke subdifferential of φ at x is defined by

$$\partial\varphi(x) = \{x^* \in X^*\colon (x^*, h) \leqslant \varphi^0(x; h) \text{ for all } h \in \mathbb{R}^N\}.$$

If φ is also convex, the two subdifferentials coincide and for this reason we use the same notation. If $\varphi \in C^1(X)$, then $\partial\varphi(x) = \{\varphi'(x)\}$. The Clarke subdifferential is monotone if and only if the function is convex.

Given $\varphi \in \Gamma_0(\mathbb{R}^N)$, the Moreau-Yosida approximation (or regularization) of φ is defined by

$$\varphi_\lambda(x) = \inf\left[\varphi(y) + \frac{1}{2\lambda}\|x-y\|^2 \colon\ y \in \mathbb{R}^N\right] \text{ for all } x \in \mathbb{R}^N,\ \ \lambda > 0.$$

The function φ_λ is convex, differentiable and

$$(\partial\varphi)_\lambda = \partial\varphi_\lambda \text{ for all } \lambda > 0.$$

Moreover, $\varphi_\lambda \uparrow \varphi$ as $\lambda \downarrow 0$.

An operator $A\colon\ X \to 2^{X^*}$ is said to be *pseudomonotone*, if

(a) for every $x \in X$, $A(x) \in P_{wkc}(X^*)$,

(b) A is usc from every finite dimensional subspace Z of X into X^*_w and

(c) if $x_n \xrightarrow{w} x$, $x_n^* \in A(x_n)$ and $\overline{\lim}(x_n^*, x_n - x) \leqslant 0$, then for every $y \in X$, there exists $x^*(y) \in A(x)$ such that $(x^*(y), x - y) \leqslant \underline{\lim}(x_n^*, x_n - y)$.

If A is bounded (i.e. maps bounded sets into bounded sets) and satisfies condition (c), then it satisfies condition (b) too. An operator $A\colon\ X \to 2^{X^*}$ is said to be *generalized pseudomonotone*, if for all $x_n^* \in A(x_n)$, $n \geqslant 1$, which satisfy $x_n \xrightarrow{w} x$ in X, $x_n^* \xrightarrow{w} x^*$ in X^* and $\overline{\lim}(x_n^*, x_n - x) \leqslant 0$, we have $x^* \in A(x)$ and $(x_n^*, x_n) \to (x^*, x)$.

The following theorem relates the notions of maximal monotonicity, pseudomonotonicity and generalized pseudomonotonicity. For details we refer to the paper of Browder-Hess [4] and the book of Hu-Papageorgiou [23, Section III.6].

Theorem D. *If X is a reflexive Banach space and $A\colon\ X \to 2^{X^*}$, then*

(a) *if A is maximal monotone, it is also generalized pseudomonotone;*

(b) *if A is pseudomonotone, it is also generalized pseudomonotone;*

(c) *if A is generalized pseudomonotone, bounded and for every $x \in X$ we have $A(x) \in P_{wkc}(X^*)$, then A is pseudomonotone;*

(d) *if A is pseudomonotone and coercive, then it is surjective;*

(e) *the sum of pseudomonotone maps is pseudomonotone too.*

Let Y, Z be Banach spaces and $K\colon\ Y \to Z$. We say that

(a) K is *completely continuous*, if $y_n \xrightarrow{w} y$ in Y implies $K(y_n) \to K(y)$ in Z, and

(b) K is *compact*, if it is continuous and maps bounded sets into relatively compact sets.

In general these two notions are distinct. However, if Y is reflexive, then complete continuity implies compactness. Moreover, if Y is reflexive and K is linear, then the two notions are equivalent. Also a multivalued map $F\colon\ Y \to 2^Z \setminus \{\emptyset\}$ is said to be compact, if it is usc and maps bounded sets in Y into relatively compact sets in Z.

In Section 4 we will need the following multivalued generalization of the classical Leray-Schauder alternative principle, which is due to Bader [1]. Let X, Y be Banach spaces, $G\colon X \to P_{wkc}(Y)$ be usc from X into Y_w, $K\colon Y \to X$ be completely continuous and $\Phi = K \circ G$.

Proposition 1. *If X, Y, Φ are as above and Φ is compact, then either*
(a) $S = \{x \in X\colon\ x \in \beta\Phi(x)$ *for some* $0 < \beta < 1\}$ *is unbounded, or*
(b) Φ *has a fixed point.*

In what follows we employ on $\mathbb{R}^N$ the Euclidean norm denoted by $\|\cdot\|$ and the usual inner product denoted by $(\cdot,\cdot)_{\mathbb{R}^N}$. Also by $\|\cdot\|_p$ $(1 \leqslant p \leqslant \infty)$ we denote the L^p norm. Moreover, for the Sobolev spaces $W^{1,p}(T,\mathbb{R}^N)$ the norm will be denoted by $\|\cdot\|$. There will be no confusion with the $\mathbb{R}^N$-norm since it will be clear from the context which one is used. Finally by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $(W^{1,p}(T,\mathbb{R}^N), W^{1,p}(T,\mathbb{R}^N)^*)$ or for $(W_0^{1,p}(T,\mathbb{R}^N), W^{-1,q}(T,\mathbb{R}^N))$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$ and by $(\cdot,\cdot)_{pq}$ the duality brackets for the pair $(L^p(T,\mathbb{R}^N), L^q(T,\mathbb{R}^N))$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$.

3. The vector Dirichlet problem

In this section we study the Dirichlet problem (1). The hypotheses on the data of the problem are the following:

$\mathrm{H(a)}_1$: $a\colon \mathbb{R}^N \times \mathbb{R}^N \to P_{kc}(\mathbb{R}^N)$ is a multifunction such that
(i) for all $x \in \mathbb{R}^N$, $y \to a(x,y)$ is maximal monotone and strictly monotone, for every $y \in \mathbb{R}^N$ $x \to a(x,y)$ is lsc and $(x,y) \to a(x,y)$ has closed graph;
(ii) for all $x,y \in \mathbb{R}^N$ and all $v \in a(x,y)$, $\|v\| \leqslant c_1(1+\|x\|^{p-1}+\|y\|^{p-1})$, $2 \leqslant p < \infty$, $c_1 > 0$;
(iii) for all $x,y \in \mathbb{R}^N$ and all $v \in a(x,y)$, $(v,y)_{\mathbb{R}^N} \geqslant c_2\|y\|^p - c_3$, with $c_2, c_3 > 0$.

Consider the differential operator $x \to -(\|x'(\cdot)\|^{p-2}x'(\cdot))'$ (the vectorial p-Laplacian, see Manasevich-Mawhin [28]) and consider the nonlinear eigenvalue problem:

$$\left\{\begin{aligned} &-(\|x'(t)\|^{p-2}x'(t))' = \mu\|x(t)\|^{p-2}x(t) \ \text{ a.e. on } T,\\ &x(0)=x(b)=0,\ \ \mu\in\mathbb{R}. \end{aligned}\right\} \tag{3}$$

If for some $\mu \in \mathbb{R}$, problem (3) has a nontrivial solution $x \in C^1(T,\mathbb{R}^N)$, then μ is said to be an eigenvalue of the p-Laplacian differential operator. It is well-known (see for example Fučík-Nečas-Souček-Souček [18] (scalar problems) and Manasevich-Mawhin [29] (vector problems)), that all the eigenvalues of (3) form a countable set $\{\mu_k\}_{k\geqslant 1}$ such that $0 < \mu_1 < \mu_2 < \ldots < \mu_k < \ldots$ and $\lim\limits_{n\to\infty} \mu_n = \infty$. Moreover, $\mu_1 =$

$\inf\left[\|x'\|_p^p/\|x\|_p^p\colon\ x\in W_0^{1,p}(T,\mathbb{R}^N),\ x\neq 0\right]$ (Rayleigh quotient) and μ_1 is simple. The infimum in the Rayleigh quotient is attained at w_1 the normalized first eigenfunction. Also $w_1(t)\neq 0$ for all $t\in(0,b)$.

$\mathrm{H(F)}_1$: $F\colon T\times\mathbb{R}^N\times\mathbb{R}^N\to P_{kc}(\mathbb{R}^N)$ is a multifunction such that

(i) for all $x,y\in\mathbb{R}^N$, $t\to F(t,x,y)$ is measurable;

(ii) for almost all $t\in T$, $(x,y)\to F(t,x,y)$ has a closed graph;

(iii) for almost all $t\in T$, all $x,y\in\mathbb{R}^N$ and all $g\in F(t,x,y)$, we have

$$\|g\|\leqslant\gamma_1(t,\|x\|)+\gamma_2(t,\|x\|)\|y\|^{p-1}$$

with

$$\sup_{0\leqslant r\leqslant k}\gamma_1(t,r)\leqslant\eta_{1,k}(t),\quad \eta_{1,k}\in L^q(T)\quad\Big(\frac{1}{p}+\frac{1}{q}=1\Big)$$

and

$$\sup_{0\leqslant r\leqslant k}\gamma_2(t,r)\leqslant\eta_{2,k}(t)\quad\text{a.e. on } T,\quad \eta_{2,k}\in L^\infty(T)$$

for all $k>0$;

(iv) if $m(t,x,y)=\inf[(g,x)_{\mathbb{R}^N}\colon\ g\in F(t,x,y)]$, then

$$\varliminf_{\|x\|\to\infty}\Big[\inf_{y\in\mathbb{R}^N}\frac{m(t,x,y)}{\|x\|^p}\Big]\geqslant -k_1(t)$$

uniformly for almost all $t\in T$, with $k_1\in L^\infty(T)$ and $0\leqslant k_1(t)\leqslant c_2\mu_1$ a.e. on T; the inequality is strict on a set of positive Lebesgue measure and $c_2>0$ as in hypothesis $\mathrm{H(a)}_1$ (iii).

Remark. Hypothesis $\mathrm{H(F)}_1$ (iv) is a kind of a nonresonance condition.

$\mathrm{H(A)}_1$: $A\colon\mathbb{R}^N\to 2^{\mathbb{R}^N}$ is maximal monotone, with $\operatorname{dom}A=\{x\in\mathbb{R}^N\colon\ A(x)\neq\emptyset\}=\mathbb{R}^N$ and $0\in A(0)$.

Fix $x\in W_0^{1,p}(T,\mathbb{R}^N)$ and let $\varepsilon_x\colon W_0^{1,p}(T,\mathbb{R}^N)\to 2^{W^{-1,q}(T,\mathbb{R}^N)}$ be defined by

$$\varepsilon_x(v)=\big\{-w'\colon\ w\in S^q_{a(x(\cdot),v'(\cdot))}\big\}.$$

Lemma 2. *If hypotheses* $\mathrm{H(a)}_1$ *hold then* $\varepsilon_x\colon W_0^{1,p}(T,\mathbb{R}^N)\to 2^{W^{-1,q}(T,\mathbb{R}^N)}$ *is maximal monotone.*

P r o o f. Because of hypotheses $\mathrm{H(a)}_1$, it is clear that for all $v\in W_0^{1,p}(T,\mathbb{R}^N)$, $S^q_{a(x(\cdot),v'(\cdot))}\in P_{fc}(L^q(T,\mathbb{R}^N))$ and so ε_x has nonempty, closed and convex values. Also $\varepsilon_x(\cdot)$ is monotone. So according to what was said in Section 2 we know that in order to prove the desired maximality of ε_x, it suffices to show that for every $z,y\in W_0^{1,p}(T,\mathbb{R}^N)$, the multifunction $r\to\varepsilon_x(z+ry)$ is usc from $[0,1]$ into $W^{-1,q}(T,\mathbb{R}^N)_w$. Since ε_x is bounded (hypothesis $\mathrm{H(a)}_1$ (ii)), to show upper semicontinuity of ε_x, it is enough to show that $\operatorname{Gr}\varepsilon_x$ is sequentially closed in $W_0^{1,p}(T,\mathbb{R}^N)\times W^{-1,q}(T,\mathbb{R}^N)_w$ (recall that since $W_0^{1,p}(T,\mathbb{R}^N)$ is separable, there is a metric on $W^{-1,q}(T,\mathbb{R}^N)$ which generates a topology weaker than the weak topology and coincides with it on bounded sets; for this reason and because of hypothesis $\mathrm{H(a)}_1$ (ii) it suffices to work with sequences and establish the sequential closedness of $\operatorname{Gr}\varepsilon_x$). So let $r_n\to r$ in $[0,1]$, $h_n\xrightarrow{w}h$ in $W^{-1,q}(T,\mathbb{R}^N)$ and $h_n\in\varepsilon_x(z+r_ny)$, $n\geqslant 1$. By definition we have $h_n=-g_n'$, $g_n\in S^q_{a(x(\cdot),(z+t_ny)'(\cdot))}$. Hypothesis H(a) (ii), allows us to assume (by passing to a subsequence if necessary) that $g_n\xrightarrow{w}g$ in $L^q(T,\mathbb{R}^N)$, hence $h=-g'$. Recall that since $a(x(t),\cdot)$ is maximal monotone and $\operatorname{dom}a(x(t),\cdot)=\mathbb{R}^N$, the multifunction $u\to a(x(t),u)$ is usc from $\mathbb{R}^N$ into $\mathbb{R}^N$. Then using Proposition B we have that

$$g(t)\in\overline{\operatorname{conv}}\,\overline{\lim}\,a(t,x(t),(z+r_ny)'(t))\subseteq a(t,x(t),(z+ry)'(t))\quad\text{a.e. on } T,$$

where the last inclusion is a consequence of the closed graph of $(x,y)\to a(x,y)$. Thus $g\in S^q_{a(x(\cdot),(z+ry)'(\cdot))}$ and so $h\in\varepsilon_x(z+ry)$ which completes the proof of the Lemma. $\square$

Now let $K\colon W_0^{1,p}(T,\mathbb{R}^N)\to P_{fc}(L^q(T,\mathbb{R}^N))$ be defined by

$$K(x)=S^q_{a(x(\cdot),x'(\cdot))}.$$

Then let $\alpha\colon W_0^{1,p}(T,\mathbb{R}^N)\to P_{fc}(W^{-1,q}(T,\mathbb{R}^N))$ be defined by

$$\alpha(x)=\{-g'\colon\ g\in K(x)\}.$$

Using Lemma 2, we can prove the following useful property for the operator $\alpha(\cdot)$.

Proposition 3. *If hypotheses* $\mathrm{H(a)}_1$ *hold, then*

$$\alpha\colon W_0^{1,p}(T,\mathbb{R}^N)\to P_{fc}(W^{-1,q}(T,\mathbb{R}^N))$$

is bounded, pseudomonotone.

P r o o f. The boundedness of α follows from hypothesis $\mathrm{H(a)}_1$ (ii). Since α is bounded, in order to prove that it is pseudomonotone it suffices to show that it is generalized pseudomonotone (see Section 2). To this end let $x_n \xrightarrow{w} x$ in $W_0^{1,p}(T,\mathbb{R}^N)$, $h_n \xrightarrow{w} h$ in $W^{-1,q}(T,\mathbb{R}^N)$, $h_n \in \alpha(x_n)$, $n \geqslant 1$, and assume $\overline{\lim}\, \langle h_n, x_n - x\rangle \leqslant 0$ (here by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $(W_0^{1,p}(T,\mathbb{R}^N), W^{-1,q}(T,\mathbb{R}^N))$. We need to show that $h \in \alpha(x)$ and $\langle h_n, x_n\rangle \to \langle h, x\rangle$. By definition $h_n = -g_n'$ with $g_n \in K(x_n)$, $n \geqslant 1$. From hypothesis $\mathrm{H}(\alpha)_1$ (ii), we see that $\{g_n\}_{n\geqslant 1} \subseteq L^q(T,\mathbb{R}^N)$ is bounded and so by passing to a subsequence if necessary, we may assume that $g_n \xrightarrow{w} g$ in $L^q(T,\mathbb{R}^N)$. Hence $h = -g'$. We have to show that $-g' \in \alpha(x)$. To this end let $z \in W_0^{1,p}(T,\mathbb{R}^N)$ and consider the multifunction $\xi\colon T \to P_{kc}(\mathbb{R}^N)$ defined by $\xi(t) = a(x(t), z'(t))$. By virtue of hypothesis H(a) (i) α has a closed graph, hence ξ is graph measurable and so Lebesgue measurable (by virtue of the completeness of the Lebesgue σ-field, see Section 2). Using the Yankov-von Neumann-Aumann Selection Theorem (see Theorem A), we can find $w\colon T \to \mathbb{R}^N$ a measurable map such that $w(t) \in a(x(t), z'(t))$ a.e. on T.

Also let $\Gamma_n(t) = \{y \in \alpha(x_n(t), z'(t))\colon \|w(t) - y\| = d(w(t), \alpha(x_n(t), z'(t)))\}$. Evidently for almost all $t \in T$, $\Gamma_n(t) \neq \emptyset$. By redefining Γ_n on the exceptional Lebesgue-null set, we may assume that for all $t \in T$, $\Gamma_n(t) \neq \emptyset$. Note that since $a(\cdot,\cdot)$ has closed graph (Hypothesis H(a) (i)) it is graph measurable and so $t \to a(x_n(t), z'(t))$ is measurable. Therefore $t \to d(w(t), a(x_n(t), z'(t)))$ is a measurable $\mathbb{R}_+$-valued function and so $(t,y) \to u_n(t,y) = \|w(t) - y\| - d(w(t), a(x_n(t), z'(t)))$ is a Caratheodory function (i.e. measurable in $t \in T$ and continuous in $y \in \mathbb{R}^N$), thus it is jointly measurable. Therefore it follows that $\operatorname{Gr}\Gamma_n = \{(t,y) \in \operatorname{Gr} a(x_n(\cdot), z'(\cdot))\colon u_n(t,y) = 0\} \in \mathcal{L}(T) \times \mathcal{B}(\mathbb{R}^N)$ with $\mathcal{L}(T)$ being the Lebesgue σ-field of T and $\mathcal{B}(\mathbb{R}^N)$ the Borel σ-field of $\mathbb{R}^N$. So we can apply Theorem A and obtain $w_n \in S^q_{a(x_n(\cdot), z'(\cdot))}$, $n \geqslant 1$, such that

$$\begin{aligned}\|w(t) - w_n(t)\| &= d(w(t), a(x_n(t), z'(t))) \\ &\leqslant h^*(a(x(t), z'(t)), a(x_n(t)), z'(t))) \quad \text{a.e. on } T.\end{aligned} \tag{4}$$

Because $x_n \xrightarrow{w} x$ in $W_0^{1,p}(T,\mathbb{R}^N)$, we have $x_n \to x$ in $C(T,\mathbb{R}^N)$ ($W_0^{1,p}(T,\mathbb{R}^N)$ is embedded compactly in $C(T,\mathbb{R}^N)$). Also from the lower semicontinuity of $a(\cdot, z'(t))$ (hypothesis $\mathrm{H(a)}_1$ (i)), we have that

$$h^*(a(x(t), z'(t)), a(x_n(t), z'(t))) \to 0 \quad \text{a.e. on } T \text{ as } n \to \infty$$

(see Section 2 where for compact valued multifunctions we gave an equivalent definition of lower semicontinuity in terms of h^*).

So from (4) it follows that $w_n(t) \to w(t)$ a.e. on T and by the dominated convergence theorem, we have $w_n \to w$ in $L^q(T, \mathbb{R}^N)$, hence $w_n' \to w'$ in $W^{-1,q}(T, \mathbb{R}^N)$ as $n \to \infty$. Exploiting the monotonicity of $a(x_n(\cdot), \cdot)$ (hypothesis $\mathrm{H(a)}_1$ (i)), we have

$$
\begin{aligned}
0 &\leqslant \int_0^b (g_n(t) - w_n(t), x_n'(t) - z'(t))_{\mathbb{R}^N} \, \mathrm{d}t \\
&= \int_0^b (g_n(t), x_n'(t) - x'(t))_{\mathbb{R}^N} \, \mathrm{d}t + \int_0^b (g_n(t), x'(t) - z'(t))_{\mathbb{R}^N} \, \mathrm{d}t \\
&\quad + \int_0^b (w_n(t), z'(t) - x_n'(t))_{\mathbb{R}^N} \, \mathrm{d}t \\
&= \langle -g_n', x_n - x \rangle + \int_0^b (g_n(t), x'(t) - z'(t))_{\mathbb{R}^N} \, \mathrm{d}t + \langle -w_n', z - x_n \rangle \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by integration by parts).}
\end{aligned}
$$

Passing to the limit as $n \to \infty$ we obtain

$$
0 \leqslant \int_0^b (g(t), x'(t) - z'(t))_{\mathbb{R}^N} \, \mathrm{d}t + \langle -w', z - x \rangle = \langle -g' - w', x - z \rangle .
$$

Since $(z, w') \in \operatorname{Gr} \varepsilon_x$ was arbitrary and ε_x is maximal monotone (Lemma 2), we infer that $-g' = h \in \varepsilon_x(x) = \alpha(x)$. As above we can find $u_n \in S^q_{a(x_n(\cdot), x'(\cdot))}$ such that $u_n \to g$ in $L^q(T, \mathbb{R}^N)$. Then $u_n' \to g'$ in $W^{-1,q}(T, \mathbb{R}^N)$ and using the monotonicity of $a(x_n(t), \cdot)$ we have

$$
\begin{aligned}
&\langle h_n, x_n - x \rangle = \langle h_n + u_n', x_n - x \rangle - \langle u_n', x_n - x \rangle \\
&= \int_0^b (g_n(t) - u_n(t), x_n'(t) - x'(t))_{\mathbb{R}^N} \, \mathrm{d}t - \int_0^b (u_n'(t), x_n(t) - x(t))_{\mathbb{R}^N} \, \mathrm{d}t \\
&\geqslant \int_0^b (u_n(t), x_n'(t) - x'(t))_{\mathbb{R}^N} \, \mathrm{d}t = \langle -u_n', x_n - x \rangle \to 0 \quad \text{as } n \to \infty, \\
&\implies \underline{\lim} \, \langle h_n, x_n \rangle \geqslant \langle h, x \rangle .
\end{aligned}
$$

On the other hand from the choice of the sequences $\{x_n\}_{n \geqslant 1}$, $\{h_n\}_{n \geqslant 1}$ we have $\overline{\lim} \, \langle h_n, x_n \rangle \leqslant \langle h, x \rangle$, hence $\langle h_n, x_n \rangle \to \langle h, x \rangle$. This proves that α is generalized pseudomonotone, thus pseudomonotone (see Theorem D (c)). $\square$

For $\lambda > 0$, let $A_\lambda \colon \mathbb{R}^N \to \mathbb{R}^N$ be the Yosida approximation of A and let $\hat{A}_\lambda \colon L^p(T, \mathbb{R}^N) \to L^q(T, \mathbb{R}^N)$ be the corresponding Nemyckii operator, i.e. $\hat{A}_\lambda(x)(\cdot) = A_\lambda(x(\cdot))$. Note that since by hypothesis $0 \in A(0)$, we have $A_\lambda(0) = 0$ and because A_λ is Lipschitz continuous for every $x \in L^p(T, \mathbb{R}^N)$, we have $\|A_\lambda(x(t))\| \leqslant \lambda^{-1} \|x(t)\|$ a.e. on T. Therefore $\hat{A}_\lambda$ takes values into $L^p(T, \mathbb{R}^N) \subseteq L^q(T, \mathbb{R}^N)$

(since $2 \leqslant p < \infty$, see hypothesis $\mathrm{H(a)}_1$ (ii)). Clearly $\hat{A}_\lambda$ is continuous and of course so is $\hat{A}_\lambda$ as a map from $W_0^{1,p}(T,\mathbb{R}^N)$ into $W^{-1,q}(T,\mathbb{R}^N)$. Let $N\colon W_0^{1,p}(T,\mathbb{R}^N) \to P_{fc}(L^q(T,\mathbb{R}^N))$ be the multivalued Nemyckii operator corresponding to F, i.e. $N(x) = S^q_{F(\cdot,x(\cdot),x'(\cdot))}$. We know that N is usc from $W_0^{1,p}(T,\mathbb{R}^N)$ into $L^q(T,\mathbb{R}^N)_w$ (see Halidias-Papageorgiou [20] or Frigon [16]). Then let $V_\lambda\colon W_0^{1,p}(T,\mathbb{R}^N) \to P_{fc}(W^{-1,q}(T,\mathbb{R}^N))$ be defined by

$$V_\lambda(x) = \alpha(x) + \hat{A}_\lambda(x) + N(x).$$

Proposition 4. *If hypotheses* $\mathrm{H(a)}_1$, $\mathrm{H(F)}_1$, $\mathrm{H(A)}_1$ *hold, then* V_λ *is pseudomonotone, coercive for all* $\lambda > 0$.

P r o o f. Evidently V_λ is $P_{wkc}(W^{-1,q}(T,\mathbb{R}^N))$-valued and bounded. So as before in order to prove the pseudomonotonicity of V_λ, it suffices to show that it is generalized pseudomonotone. To this end let $x_n \xrightarrow{w} x$ in $W_0^{1,p}(T,\mathbb{R}^N)$, $v_n \xrightarrow{w} v$ in $W^{-1,q}(T,\mathbb{R}^N)$, $v_n \in V_\lambda(x_n)$, $n \geqslant 1$ and assume $\overline{\lim}\, \langle v_n, x_n - x\rangle \leqslant 0$. From the compact embedding of $W_0^{1,p}(T,\mathbb{R}^N)$ into $C(T,\mathbb{R}^N)$, we have $x_n \to x$ in $C(T,\mathbb{R}^N)$. Let $v_n = -g_n' + \hat{A}_\lambda(x_n) + u_n$, with $g_n \in K(x_n)$, $u_n \in N(x_n)$, $n \geqslant 1$. We have

$$\langle u_n, x_n - x\rangle = (u_n, x_n - x)_{pq} \to 0. \tag{5}$$

Here by $(\cdot,\cdot)_{pq}$ we denote the duality brackets for the dual pair $(L^p(T,\mathbb{R}^N), L^q(T,\mathbb{R}^N))$. Recall that $A(\cdot)$ is usc and $P_{kc}(\mathbb{R}^N)$-valued (being maximal monotone with $\operatorname{dom} A = \mathbb{R}^N$). So if $\varrho_1 = \sup_{n\geqslant 1} \|x_n\|_\infty$, then $A(\hat{B}_{\varrho_1})$ is compact (see Section 2). Therefore we have $\sup_{n\geqslant 1} \|\hat{A}_{\lambda_n}(x_n)\|_\infty \leqslant \sup[\|e\|\colon e \in A(\overline{B}_{\varrho_1})] = \beta_1 < \infty$. Thus we can say that

$$\left\langle \hat{A}_\lambda(x_n), x_n - x \right\rangle = (\hat{A}_\lambda(x_n), x_n - x)_{pq} \to 0 \quad \text{as } n \to \infty. \tag{6}$$

From (5) and (6) and recalling the choice of the sequences $\{x_n\}$ and $\{v_n\}_{n\geqslant 1}$, we obtain

$$\overline{\lim}\, \langle -g_n', x_n - x\rangle \leqslant 0. \tag{7}$$

But $\{g_n\}_{n\geqslant 1} \subseteq L^q(T,\mathbb{R}^N)$ is bounded (hypothesis $\mathrm{H}(\alpha)_1$ (ii)) and so by passing to a subsequence if necessary, we may assume that $g_n \xrightarrow{w} g$ in $L^q(T,\mathbb{R}^N)$ and so $-g_n' \xrightarrow{w} -g'$ in $W^{-1,q}(T,\mathbb{R}^N)$ as $n \to \infty$. From Proposition 3, we know that α is pseudomonotone. Since $-g_n' \in \alpha(x_n)$, $n \geqslant 1$, we infer that $-g' \in \alpha(x)$ and so $g \in K(x)$.

As before (see the proof of Proposition 3), with the use of the Yankov-von Neumann-Aumann Selection Theorem, we can produce $h_n \in L^q(T, \mathbb{R}^N)$ such that

$$h_n(t) \in a(x_n(t), x'(t)) \quad \text{and} \quad \|g(t) - h_n(t)\| = d(g(t), a(x_n(t), x'(t))) \quad \text{a.e. on } T.$$

Hence $\|g(t) - h_n(t)\| \leqslant h^*(a(x(t), x'(t)), a(x_n(t), x'(t))) \to 0$ a.e. on T (hypothesis $\mathrm{H(a)}_1$ (i)) and so $h_n \to g$ in $L^q(T, \mathbb{R}^N)$. Exploiting the monotonicity of $a(x, \cdot)$, we have

$$(g_n - h_n, x'_n - x')_{pq} = \int_0^b (g_n(t) - h_n(t), x'_n(t) - x'(t))_{\mathbb{R}^N} \, \mathrm{d}t \geqslant 0.$$

But from (7) and since $h_n \to g$ in $L^q(T, \mathbb{R}^N)$, we have

$$\overline{\lim}(g_n - h_n, x'_n - x')_{pq} \leqslant 0.$$

Therefore it follows that $(g_n - h_n, x'_n - x')_{pq} \to 0$ and from the monotonicity of $a(x, \cdot)$, we also have $\lim(g_n(t) - h_n(t), x'_n(t) - x'(t))_{\mathbb{R}^N} = 0$ a.e. on T. Let $\xi_n(t) = (g_n(t) - h_n(t), x'_n(t) - x'(t))_{\mathbb{R}^N}$. Using hypotheses H(a) (ii) and (iii), for all $t \in T \setminus N_1$, $|N_1| = 0$ ($|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$), we have

$$\begin{gathered} \xi_n(t) \geqslant c_2(\|x'_n(t)\|^p + \|x'(t)\|^p) - 2c_3 \\ -\|x'(t)\|(c_1 + c_1(\varrho_1^{p-1}\| + \|x'_n(t)\|^{p-1})) - \|x'_n(t)\|(c_1 + c_1(\varrho_1^{p-1}\| + \|x'(t)\|^{p-1})) \\ \Longrightarrow \{x'_n(t)\}_{n\geqslant 1} \subseteq \mathbb{R}^N \quad \text{is bounded for all } t \in T \setminus N_1, \ |N_1| = 0. \end{gathered}$$

So for every $t \in T \setminus N_1$, we can find a subsequence (in general depending on t) such that $x'_n(t) \to v_t$ in $\mathbb{R}^N$. Also for the same reason, we may assume that $g_n(t) \to w_t$ in $\mathbb{R}^N$. But recall that for all $t \in T \setminus N_1$, $g_n(t) \in a(x_n(t), x'_n(t))$ and by hypothesis $\mathrm{H(a)}_1$ (i) $a(\cdot, \cdot)$ has closed graph. So passing to the limit as $n \to \infty$ we obtain $w_t \in a(x(t), v_t)$. Also from the construction of the sequence $\{h_n\}_{n\geqslant 1} \subseteq L^q(T, \mathbb{R}^N)$, we have

$$\begin{gathered} \|g(t) - h_n(t)\| \leqslant h^*(a(x(t), x'(t)), a(x_n(t), x'_n(t))) \to 0 \\ \text{for } t \in T \setminus N_1 \quad \text{as } n \to \infty, \\ \Longrightarrow h_n(t) \to g(t) \quad \text{for all } t \in T \setminus N_1, \ |N_1| = 0. \end{gathered}$$

Since $\xi_n(t) \to 0$ for all $T \setminus N_1$ in the limit as $n \to \infty$, we have

$$(w_t - g(t), v_t - x'(t))_{\mathbb{R}^N} = 0$$

with $w_t \in a(x(t), v_t)$, $g(t) \in a(x(t), x'(t))$. Because $a(x, \cdot)$ is strictly monotone (hypothesis $\mathrm{H(a)}_1$ (ii)), from the last equality it follows that $v_t = x'(t)$ for all $t \in$

$T \setminus N_1$. Since every subsequence of $\{x_n'(t)\}_{n\geqslant 1}$, $t \in T\setminus N_1$, has a further subsequence converging to $x'(t)$, we infer that $x_n'(t) \to x'(t)$ in $\mathbb{R}^N$ for all $t \in T \setminus N_1$, $|N_1| = 0$. Then since $u_n \xrightarrow{w} u$ in $L^q(T, \mathbb{R}^N)$, invoking Proposition B, we have that

$$u(t) \in \overline{\mathrm{conv}}\, \overline{\lim}\, F(t, x_n(t), x_n'(t)) \subseteq F(t, x(t), x'(t)) \quad \text{a.e. on } T$$

the last inclusion following from hypothesis $\mathrm{H(F)}_1$ (ii). So $u \in N(x) = S^q_{F(\cdot, x(\cdot), x'(\cdot))}$. Also we have $\hat{A}_\lambda(x_n) \to \hat{A}_\lambda(x)$ in $L^q(T, \mathbb{R}^N)$ and $g_n' \xrightarrow{w} g'$ in $W^{-1,q}(T, \mathbb{R}^N)$. Thus

$$\begin{aligned} v_n = -g_n' + \hat{A}_{\lambda_n}(x_n) + u_n &\xrightarrow{w} -g' + \hat{A}_\lambda(x) + u \quad \text{in } W^{-1,q}(T, \mathbb{R}^N), \\ &\Longrightarrow v = -g' + \hat{A}_\lambda(x) + u \quad \text{with } g \in K(x),\ u \in N(x), \\ &\Longrightarrow v \in V_\lambda(x). \end{aligned}$$

Recall that from the choice of the sequence $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(T, \mathbb{R}^N)$, we have $\overline{\lim}\, \langle v_n, x_n\rangle \leqslant \langle v, x\rangle$. Also we have

$$\begin{aligned} \langle v_n, x_n - x\rangle &= \langle -g_n', x_n - x\rangle + (\hat{A}_\lambda(x_n), x_n - x)_{pq} + (u_n, x_n - x)_{pq} \\ \Longrightarrow \langle v_n, x_n - x\rangle &= \int_0^b (g_n(t), x_n'(t) - x'(t))_{\mathbb{R}^N}\, \mathrm{d}t + \eta_n, \end{aligned}$$

with

$$\eta_n = (\hat{A}_\lambda(x_n), x_n - x)_{pq} + (u_n, x_n - x)_{pq}.$$

We know that $\eta_n \to 0$, while from the previous considerations we have $\underline{\lim} \int_0^b (g_n(t), x_n'(t) - x'(t))_{\mathbb{R}^N}\, \mathrm{d}t \geqslant 0$, hence $\langle v, x\rangle \leqslant \underline{\lim}\, \langle v_n, x_n\rangle$. So finally $\langle v_n, x_n\rangle \to \langle v, x\rangle$. This proves the generalized pseudomonotonicity of V_λ, hence its pseudomonotonicity.

Next we show the coercivity of $V_\lambda(\cdot)$. To this end let $v \in V_\lambda(x)$. We have

$$\langle v, x\rangle = \langle -g', x\rangle + (\hat{A}_\lambda(x), x)_{pq} + (u, x)_{pq} \quad \text{with } g \in K(x),\ u \in N(x).$$

Because $\hat{A}_\lambda(0) = 0$, we have $(\hat{A}_\lambda(x), x)_{pq} \geqslant 0$ and so

$$\langle v, x\rangle \geqslant \langle -g', x\rangle + (u, x)_{pq} = \int_0^b (g(t), x'(t))_{\mathbb{R}^N}\, \mathrm{d}t + \int_0^b (u(t), x(t))_{\mathbb{R}^N}\, \mathrm{d}t.$$

From hypothesis $\mathrm{H(a)}_1$ (iii), we have

$$\int_0^b (g(t), x'(t))_{\mathbb{R}^N}\, \mathrm{d}t \geqslant c_2 \|x'\|_p^p - c_3 b. \tag{9}$$

Also by virtue of hypothesis $\mathrm{H(F)}_1$ (iv), given $\varepsilon > 0$, we can find $M = M(\varepsilon) > 0$ such that for almost all $t \in T$, all $\|x\| > M$, all $y \in \mathbb{R}^N$ and all $u \in F(t,x,y)$, we have $(u,x)_{\mathbb{R}^N} \geqslant -(k_1(t)+\varepsilon)\|x\|^p$. Moreover, from hypothesis $\mathrm{H(F)}_1$ (iii) we know that for almost all $t \in T$, all $\|x\| \leqslant M$, all $y \in \mathbb{R}^N$ and all $u \in F(t,x,y)$, we have

$$\begin{aligned}&\|u\| \leqslant \gamma_1(t,\|x\|) + \gamma_2(t,\|x\|)\|y\|^{p-1}\\ \Longrightarrow\ &(u,x)_{\mathbb{R}^N} \geqslant -M\eta_{1,M}(t) - M\eta_{2,M}(t)\|y\|^{p-1}.\end{aligned}$$

Therefore, for almost all $t \in T$, all $x, y \in \mathbb{R}^N$ and all $u \in F(t,x,y)$, we have

$$\begin{aligned}&(u,x)_{\mathbb{R}^N} \geqslant -(k_1(t)+\varepsilon)\|x\|^p - M\eta_{1,M}(t) - M\eta_{2,M}(t)\|y\|^{p-1}\\ \Longrightarrow\ &(u,x)_{\mathbb{R}^N} \geqslant -(k_1(t)+\varepsilon)\|x\|^p - \beta_1\|y\|^{p-1} - \beta_2(t)\end{aligned}$$

with $\beta_1 > 0$ and $\beta_2 \in L^q(T)_+$. So we obtain

$$\int_0^b (u(t),x(t))_{\mathbb{R}^N}\,\mathrm{d}t \geqslant -\int_0^b k_1(t)\|x(t)\|^p\,\mathrm{d}t - \varepsilon\|x\|_p^p - \beta_1\|x'\|_p^{p-1} - \|\beta_2\|_1. \tag{10}$$

Using (9) and (10) in (8), we have

$$\langle v,x\rangle \geqslant c_2\|x'\|_p^p - \int_0^b k_1(t)\|x(t)\|^p\,\mathrm{d}t - \varepsilon\|x\|_p^p - \beta_1\|x'\|_p^{p-1} - \beta_3 \tag{11}$$

for some $\beta_3 > 0$.

Set $\chi(x) = c_2\|x'\|_p^p - \int_0^b k_1(t)\|x(t)\|^p\,\mathrm{d}t$, $x \in W_0^{1,p}(T,\mathbb{R}^N)$.

Claim. *$\chi(x) \geqslant \beta_4\|x'\|_p^p$ for some $\beta_4 > 0$ and all $x \in W_0^{1,p}(T,\mathbb{R}^N)$.*

Suppose the claim is not true. Then by the positive p-homogeneity of $\chi(\cdot)$ and since $\|x'\|_p$, for $x \in W_0^{1,p}(T,\mathbb{R}^N)$, is an equivalent norm on $W_0^{1,p}(T,\mathbb{R}^N)$ (Poincaré inequality), we can find $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(T,\mathbb{R}^N)$, $\|x_n'\|_p = 1$ such that $\chi(x_n) \downarrow 0$ as $n \to \infty$. We may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(T,\mathbb{R}^N)$ and $x_n \to x$ in $C(T,\mathbb{R}^N)$. Exploiting the weak lower semicontinuity of the norm functional, we obtain

$$\begin{aligned}&c_2\|x'\|_p^p - \int_0^b k_1(t)\|x(t)\|^p\,\mathrm{d}t \leqslant 0\\ \Longrightarrow\ &c_2\|x'\|_p^p \leqslant \int_0^b k_1(t)\|x(t)\|^p\,\mathrm{d}t \leqslant c_2\mu_1\|x\|_p^p\\ \Longrightarrow\ &\|x'\|_p^p \leqslant \mu_1\|x\|_p^p.\end{aligned}$$

Since the opposite inequality is always true (Rayleigh quotient), we deduce that

$$\|x'\|_p^p = \mu_1\|x\|_p^p. \tag{12}$$

Because $\|x_n'\|_p = 1$, we have $\chi(x_n) = c_2 - \int_0^b k_1(t)\|x_n(t)\|^p \,\mathrm{d}t$, $n \geqslant 1$ and so in the limit as $n \to \infty$ we obtain $c_2 = \int_0^b k_1(t)\|x(t)\|^p \,\mathrm{d}t$, hence $x \neq 0$. Therefore from (12) we infer that $x = w_1 =$ the first eigenfunction of the vector p-Laplacian differential operator (recall that $\mu_1 > 0$ is simple). Then from hypothesis $\mathrm{H(F)}_1$ (iv) and since $w_1(t) \neq 0$ for all $t \in (0, b)$, we have

$$1 = \int_0^b \frac{k_1(t)}{c_2}\|w_1(t)\|^p \,\mathrm{d}t < \mu_1\|w_1\|_p^p = \|w_1'\|_p^p = \|x'\|_p^p,$$

a contradiction, since from $x_n' \xrightarrow{w} x'$ in $L^p(T, \mathbb{R}^N)$, we have $\|x'\|_p^p \leqslant \underline{\lim}\, \|x_n'\|_p^p = 1$. This proves the claim.

Using the claim in (11), we can write

$$\{v, x\} \geqslant \Bigl(\beta_4 - \frac{\varepsilon}{\mu_1}\Bigr)\|x'\|_p^p - \beta_1\|x'\|_p^{p-1} - \beta_3. \tag{13}$$

Let $\varepsilon > 0$ such that $\mu_1\beta_4 > \varepsilon$. Then from the last inequality it follows that $V_\lambda(\cdot)$ is coercive. $\square$

Since $V_\lambda(\cdot)$ is pseudomonotone and coercive, it is surjective (see Theorem D (d)). So we can find $x \in W_0^{1,p}(T, \mathbb{R}^N)$ such that $0 \in V_\lambda(x)$. This means that $-g' + \hat{A}_\lambda(x) + u = 0$ for some $g \in K(x)$, and some $u \in N(x)$. Therefore

$$\left\{\begin{array}{l} a(x(t), x(t))' \ni A_\lambda(x(t)) + u(t) \ \text{ a.e. on } T, \\ x(0) = x(b) = 0, \ \ u(t) \in F(t, x(t), x'(t)) \ \text{ a.e. on } T. \end{array}\right\}$$

In what follows we will need the following auxiliary result on the maximal monotonicity of the lifting of A on $L^p(T, \mathbb{R}^N)$.

Lemma 5. *If hypotheses* $\mathrm{H(A)}_1$ *hold, then* $\hat{A}\colon D \subseteq L^p(T, \mathbb{R}^N) \to 2^{L^q(T, \mathbb{R}^N)}$, *defined by* $\hat{A}(x) = \{h \in L^q(T, \mathbb{R}^N)\colon h(t) \in A(x(t))$ *a.e. on* $T\}$ *for all* $x \in D = \{x \in L^p(T, \mathbb{R}^N)\colon$ *there exists* $h \in L^q(T, \mathbb{R}^N)$ *such that* $h(t) \in A(x(t))$ *a.e. on* $T\}$, *is maximal monotone.*

Proof. The monotonicity of $\hat{A}$ is clear. We will show that $R(\hat{A} + j) = L^q(T, \mathbb{R}^N)$, where $j\colon L^p(T, \mathbb{R}^N) \to L^q(T, \mathbb{R}^N)$ is defined by $j(x)(\cdot) = \|x(\cdot)\|^{p-2}x(\cdot)$. To this end let $h \in L^q(T, \mathbb{R}^N)$ and set

$$S(t) = \{(x, u) \in \mathrm{Gr}\, A\colon \ u + \psi(x) = h(t), \ \ \|x\| \leqslant \|h(t)\|^{1/(p-1)} + 1\}$$

where $\psi\colon \mathbb{R}^N \to \mathbb{R}^N$ is defined by $\psi(x) = \|x\|^{p-2}x$. The map $A + \psi$ is maximal monotone, coercive. So using Theorem A and Hu-Papageorgiou [23, p. 371], we see that $S(t) \neq \emptyset$ a.e. on T. Moreover, it is clear that $\operatorname{Gr} S \in \mathcal{L} \times B(\mathbb{R}^N)$, with $\mathcal{L}$ being the Lebesgue σ-field of T (recall that $\operatorname{Gr} A$ is closed). So we can apply the Yankov-von Neumann-Aumann Selection Theorem (see Theorem A) and obtain $x, u\colon T \to \mathbb{R}^N$ measurable maps such that $(x(t), u(t)) \in S(t)$ a.e. on T, hence $u(t) + \|x(t)\|^{p-2}x(t) = h(t)$ a.e. on T. Evidently $u \in L^q(T, \mathbb{R}^N)$. This proves that $R(\hat{A} + j) = L^q(T, \mathbb{R}^N)$. Now suppose that for some $y \in L^p(T, \mathbb{R}^N)$ and $v \in L^q(T, \mathbb{R}^N)$ we have

$$(u - v, x - y)_{pq} \geqslant 0 \quad \text{for all } (x, u) \in \operatorname{Gr} \hat{A}.$$

Let $x_1 \in D$ be such that $v + j(y) = u_1 + j(x_1)$, $u_1 \in \hat{A}(x_1)$ (it exists owing to the surjectivity of $\hat{A} + j$). So

$$0 \leqslant (u_1 - u_1 - j(x_1) + j(y), x_1 - y)_{pq} = (j(y) - j(x_1), x_1 - y)_{pq} \leqslant 0 \quad \text{i.e. } x_1 = y$$

(from the strict monotonicity of j). Hence $y \in D$ and $v = u_1 \in \hat{A}(x_1)$. This proves the maximality of $\hat{A}$. □

Remark. When $p = 2$, this result is well-known and in fact we do not need that $\operatorname{dom} A = \mathbb{R}^N$ (see for example Brezis [3, p. 25] or Hu-Papageorgiou [23, Example III.2.23, p. 328]). However, when $p > 2$, we were unable to find in the literature a corresponding result and the proof of the $p = 2$ case fails since the Yosida approximation of $\hat{A}$ is no longer equal to $\lambda^{-1}(I - J_\lambda)$.

Now we are ready to state and prove the first existence theorem for the problem (1).

Theorem 6. *If hypotheses* $\mathrm{H(a)}_1$, $\mathrm{H(F)}_1$, $\mathrm{H(A)}_1$ *hold, then the problem* (1) *has at least one solution* $x \in W_0^{1,p}(T, \mathbb{R}^N)$.

Proof. Let $\lambda_n \downarrow 0$ and consider the following sequence of auxiliary problems

$$\left\{\begin{aligned} & a(x_n(t), x_n'(t))' - A_{\lambda_n}(x_n(t)) - F(t, x_n(t), x_n'(t)) \ni 0 \quad \text{a.e. on } T, \\ & x_n(0) = x_n(b) = 0. \end{aligned}\right\} \tag{14}$$

We have already seen that for every $n \geqslant 1$, the problem (14) has a solution $x_n \in W_0^{1,p}(T, \mathbb{R}^N)$. We have

$$\begin{aligned} -g_n' + \hat{A}_{\lambda_n}(x_n) + u_n = 0, &\quad \text{with } g_n \in K(x_n),\ u_n \in N(x_n), \\ \Longrightarrow \Big(\beta_4 - \frac{\varepsilon}{\mu_1}\Big)\|x_n'\|_p^p \leqslant \beta_1 \|x_n'\|_p^{p-1} + \beta_3 &\quad \text{(see (13)).} \end{aligned}$$

Choosing $\varepsilon > 0$ so that $0 < \varepsilon < \beta_4 \mu_1$, we infer that $\{x_n\}_{n \geqslant 1} \subseteq W_0^{1,p}(T, \mathbb{R}^N)$ is bounded. Hence we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(T, \mathbb{R}^N)$ and $x_n \to x$ in

$C(T,\mathbb{R}^N)$. Then $\sup\limits_{n\geqslant 1}\|x_n\|_\infty = \beta_5 < \infty$ and $\sup\limits_{n\geqslant 1}\|\hat{A}_{\lambda_n}(x_n)\|_\infty \leqslant \|A^0(\overline{B}_{\beta_5})\| = \beta_6 < \infty$ and so we may assume that $\hat{A}_{\lambda_n}(x_n) \xrightarrow{w} w$ in $L^q(T,\mathbb{R}^N)$. Recall that $A_{\lambda_n}(x_n(t)) \in A(J_{\lambda_n}(x_n(t)))$ and

$$\begin{aligned}\|J_{\lambda_n}(x_n(t)) - x(t)\| &\leqslant \|J_{\lambda_n}(x_n(t)) - J_{\lambda_n}(x(t))\| + \|J_{\lambda_n}(x(t)) - x(t)\| \\ &\leqslant \|x_n(t) - x(t)\| + \|J_{\lambda_n}(x(t)) - x(t)\| \to 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{as } n\to\infty.\end{aligned}$$

Since $\|J_{\lambda_n}(x_n(t))\| \leqslant \|x_n(t)\| \leqslant \beta_5$ for all $n \geqslant 1$ and all $t\in T$, from the dominated convergence theorem, we see that $J_{\lambda_n}(x_n(\cdot)) = \hat{J}_{\lambda_n}(x_n) \to x$ in $L^q(T,\mathbb{R}^N)$. We know that $\hat{A}_{\lambda_n}(x_n) \in \hat{A}(\hat{J}_{\lambda_n}(x_n))$ and by Lemma 5, $\hat{A}$ is maximal monotone, hence $\operatorname{Gr}\hat{A}$ is sequentially closed in $L^p(T,\mathbb{R}^N)\times L^q(T,\mathbb{R}^N)_w$. Therefore $w\in\hat{A}(x)$, i.e. $w(t) \in A(x(t))$ a.e. on T.

Note that $\{u_n\}_{n\geqslant 1} \subseteq L^q(T,\mathbb{R}^N)$ is bounded (hypothesis $\mathrm{H(F)}_1$ (iii)) and so $(u_n, x_n - x)_{pq} \to 0$ as $n\to\infty$. Also $(\hat{A}_{\lambda_n}(x_n), x_n - x)_{pq} \to 0$ and so finally we have $\overline{\lim}\,\langle -g_n', x_n - x\rangle \leqslant 0$. Since $g_n \in K(x_n)$, $n\geqslant 1$, we may assume that $g_n \xrightarrow{w} g$ in $L^q(T,\mathbb{R}^N)$ (hypothesis $\mathrm{H(a)}_1$ (ii)) and so $g_n' \xrightarrow{w} g'$ in $W^{-1,q}(T,\mathbb{R}^N)$, with $-g'\in\alpha(x)$ (by Proposition 3). Arguing as in the proof of Proposition 4, we obtain $x_n'(t)\to x'(t)$ a.e. on T and so using Proposition B, we have $u\in N(x)$. Therefore in the limit as $n\to\infty$, we obtain

$$\begin{gathered}-g' + w + u = 0 \quad \text{with } g\in K(x),\ \ w\in\hat{A}(x),\ \ u\in N(x), \\ \Longrightarrow -a(x(t),x'(t))' - A(x(t)) - F(t,x(t),x'(t)) \ni 0 \quad \text{a.e. on } T, \\ x(0) = x(b) = 0,\end{gathered}$$

i.e. $x\in W_0^{1,p}(T,\mathbb{R}^N)$ solves (1). □

We can have a version of Theorem 6 in which F has nonconvex values (nonconvex problem). In this case the hypotheses on the multivalued $F(t,x,y)$ are the following:

$\mathrm{H(F)}_2$: $F\colon T\times\mathbb{R}^N\times\mathbb{R}^N \to P_{kc}(\mathbb{R}^N)$ is a multifunction such that
(i) $(t,x,y)\to F(t,x,y)$ is graph measurable;
(ii) for almost all $t\in T$, $(x,y)\to F(t,x,y)$ is lsc;
(iii) and (iv) are the same as $\mathrm{H(F)}_1$ (iii) and (iv).

Theorem 7. *If hypotheses* $\mathrm{H(a)}_1$, $\mathrm{H(F)}_2$, $\mathrm{H(A)}_1$ *hold, then the problem* (1) *has at least one solution* $x\in W_0^{1,p}(T,\mathbb{R}^N)$.

Proof. As before let $N\colon W_0^{1,p}(T,\mathbb{R}^N) \to P_f(L^q(T,\mathbb{R}^N))$ be the multivalued Nemyckii operator corresponding to F, i.e. $N(x) = S^q_{F(\cdot,x(\cdot),x'(\cdot))}$. We know that N is lsc (see Halidias-Papageorgiou [20] or Frigon [16] or Hu-Papageorgiou [24, p. 236]. So we can apply Theorem II.8.7, p. 245, of Hu-Papageorgiou [23] and obtain $u\colon W_0^{1,p}(T,\mathbb{R}^N) \to L^q(T,\mathbb{R}^N)$ a continuous map such that $u(x) \in N(x)$ for all $x \in W_0^{1,p}(T,\mathbb{R}^N)$. Let $V_\lambda\colon W_0^{1,p}(T,\mathbb{R}^N) \to 2^{W^{-1,q}(T,\mathbb{R}^N)} \setminus \{\emptyset\}$ be defined by $V_\lambda(x) = \alpha(x) + \hat{A}_\lambda(x) + u(x)$. By repeating the proof of Proposition 4, we can check that $V_\lambda(\cdot)$ is pseudomonotone and coercive, thus surjective. So we can find $x \in W_0^{1,p}(T,\mathbb{R}^N)$ such that $0 \in V_\lambda(x)$.

Now let $\lambda_n \downarrow 0$ and let $x_n \in W_0^{1,p}(T,\mathbb{R}^N)$ be such that $0 \in V_{\lambda_n}(x_n)$, $n \geqslant 1$. Then repeating the proof of Theorem 6 and exploiting the continuity of u, we can show that $x_n \to x$ in $W_0^{1,p}(T,\mathbb{R}^N)$ and x is a solution of (1). $\square$

Application 1. As an application of the existence result of this section, we consider the following nonlinear Dirichlet problem:

$$\left\{\begin{array}{l} (\hat{\alpha}(x(t))\|x'(t)\|^{p-2}x'(t))' \in \partial\varphi(x(t)) + \partial j(t,x(t)), \\ \qquad - c\,\dfrac{\mathrm{d}}{\mathrm{d}t}(\|x(t)\|^2)x'(t) \quad \text{a.e. on } T=[0,b], \\ x(0)=x(b), \quad x'(0)=x'(b),\ 2 \leqslant p < \infty,\ c \geqslant 0. \end{array}\right\}$$

This is a p-Lienard system. We assume the following:

(1) $\hat{\alpha} \in C(\mathbb{R}^N,\mathbb{R}^N)$, $\quad \hat{\alpha}(x) \geqslant \hat{c} > 0$ for all $x \in \mathbb{R}^N$.

(2) $\varphi\colon \mathbb{R}^N \to \mathbb{R}$ is continuous convex but not necessarily differentiable.

(3) $j\colon T \times \mathbb{R}^N \to \mathbb{R}$ is a function such that

  (i) for all $x \in \mathbb{R}^N$, $t \to j(t,x)$ is measurable;

  (ii) for almost all $t \in T$, $x \to j(t,x)$ is locally Lipschitz;

  (iii) for every $r > 0$, there exists $\alpha_r \in L^q(T)$ such that for almost all $t \in T$, all $\|x\| \leqslant r$ and all $u \in \partial j(t,x)$, we have
$$\|u\| \leqslant \alpha_r(t);$$

  (iv) $\limsup\limits_{\|x\|\to\infty} (u,x)_{\mathbb{R}^N}/\|x\|^p \leqslant k_1(t)$ uniformly for almost all $t \in T$ and all $u \in \partial j(t,x)$, with $k_1 \in L^\infty(T)$ as in hypothesis $\mathrm{H(F)}_1$ (iv).

The following nonsmooth locally Lipschitz integrands $j(t,x)$ satisfy the above hypotheses:

$$j(t,x) = \begin{cases} \tan^{-1}\|x\| & \text{if } \|x\| \leqslant 1, \\ \dfrac{k_1(t)}{p}\|x\|^p - \|x\|\ln\|x\| - \dfrac{k_1(t)}{p} + \dfrac{\pi}{4} & \text{if } \|x\| > 1 \end{cases}$$

and

$$j(t,x) = \begin{cases} \|x\| & \text{if } \|x\| \leqslant 1, \\ k_1(t)\|x\|^{p-1}\sin\|x\| - k_1(t)\sin 1 + 1 & \text{if } \|x\| > 1. \end{cases}$$

In this case

$$a(x,y) = \hat{\alpha}(x)\|y\|^{p-2}y, \quad A(x) = \partial\varphi(x)$$

and

$$F(t,x,y) = -\partial j(t,x) + 2c(y,x)_{\mathbb{R}^N}y.$$

Evidently all hypotheses of Theorem 6 are satisfied.

Problems like the above were studied recently in connection with problems in nonsmooth mechanics ("hemivariational inequalities", see Naniewicz-Panagiotopoulos [32]). When $\varphi \equiv 0$, $c = 0$ and $j(t,x)$ is C^1 and convex in the $x \in \mathbb{R}^N$ variable, the resulting problem was studied by Mawhin-Willem [31] using the Least Action Principle. The only other papers that we know which have results on p-Lienard systems are those by Manasevich-Mawhin [29, Section 7]. Our example partially extends their results.

4. The scalar periodic problem

In this section we turn our attention to the periodic problem. Serious technical difficulties force us to examine the scalar problem. It would be very interesting to know if our results can be extended to vector problems (i.e. to systems). A first step in that direction was taken by the work of Kyritsi-Matzakos-Papageorgiou [27], but there the differential operator is of the form $x \to (a(x'))'$ (i.e. independent of x), with no growth restrictions on $a(\cdot)$ (as in Dang-Oppenheimer [7] and Manasevich-Mawhin [28]). In this section we study problem (2). The presence of x in the differential operator raises nontrivial questions concerning the unicity of the solution of the auxiliary problem (15) below and distinguishes our work here from that of Kyritsi-Matzakos-Papageorgiou [27].

We make the following hypotheses on the data of (2).

$\mathrm{H(a)}_2$: $a\colon \mathbb{R} \to \mathbb{R}$ satisfies $|a(x)-a(y)| \leqslant k|x-y|$ for all $x,y \in \mathbb{R}$ and for some $k > 0$, and also $0 < c_1 \leqslant a(x)$ for all $x \in \mathbb{R}$.

$\mathrm{H(A)}_2$: $A\colon \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone map such that $0 \in A(0)$.

Remark. We know (see for example Brezis [3, p. 43] or Hu-Papageorgiou [23, p. 348]) that $A = \partial\varphi$ with $\varphi\colon \mathbb{R} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, proper convex and lower semicontinuous. So $A_\lambda = \partial\varphi_\lambda$, with φ_λ being the Moreau-Yosida approximation of φ.

We start by considering the following auxiliary problem:

$$\left\{\begin{array}{l} -(a(x(t))x'(t))' + x(t) + A_\lambda(x(t)) = h(t) \quad \text{a.e. on } T, \\ x(0) = x(b), \;\; x'(0) = x'(b), \;\; h \in L^2(T), \;\; \lambda > 0. \end{array}\right\} \tag{15}$$

Proposition 8. *If hypotheses* $\mathrm{H(a)}_2$, $\mathrm{H(A)}_2$ *hold, then for every* $h \in L^2(T)$ *the problem* (15) *has a unique solution* $x \in C^1(T)$ *for every* $\lambda > 0$.

Proof. Let $W^{1,2}_{\mathrm{per}}(T) = \{x \in W^{1,2}(T)\colon x(0) = x(b)\}$ and let $\alpha_1\colon W^{1,2}_{\mathrm{per}}(T) \to W^{1,2}_{\mathrm{per}}(T)^*$ be defined by

$$\langle \alpha_1(x), y\rangle = \int_0^b a(x(t))x'(t)y'(t)\,\mathrm{d}t \quad \text{for all } x, y \in W^{1,2}_{\mathrm{per}}(T).$$

Here by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $(W^{1,2}_{\mathrm{per}}(T), W^{1,2}_{\mathrm{per}}(T)^*)$. We will show that α_1 is pseudomonotone. Since α_1 is clearly bounded, it suffices to show that α_1 is generalized pseudomonotone. So we will show that if $x_n \xrightarrow{w} x$ in $W^{1,2}_{\mathrm{per}}(T)$, $\alpha_1(x_n) \xrightarrow{w} v$ in $W^{1,2}_{\mathrm{per}}(T)^*$ and $\overline{\lim}\,\langle\alpha_1(x_n), x_n - x\rangle \leqslant 0$, then $v = \alpha_1(x)$ and $\langle\alpha_1(x_n), x_n\rangle \to \langle\alpha_1(x), x\rangle$. To this end we have

$$\begin{aligned} \langle\alpha_1(x_n), x_n - x\rangle &= \int_0^b a(x_n(t))x_n'(t)(x_n - x)'(t)\,\mathrm{d}t \\ &= \int_0^b a(x_n(t))(x_n - x)'(t)^2\,\mathrm{d}t + \int_0^b a(x_n(t))x'(t)(x_n - x)'(t)\,\mathrm{d}t. \end{aligned} \tag{16}$$

Because $x_n \xrightarrow{w} x$ in $W^{1,2}_{\mathrm{per}}(T)$, we have $x_n \to x$ in $C(T)$ and so $a(x_n(\cdot)) \to a(x(\cdot))$ in $C(T)$. Therefore

$$\int_0^b a(x_n(t))x'(t)(x_n - x)'(t)\,\mathrm{d}t \to 0 \quad \text{as } n \to \infty.$$

Also by virtue of hypothesis $\mathrm{H(a)}_2$, we have

$$\int_0^b a(x_n(t))(x_n - x)'(t)^2\,\mathrm{d}t \geqslant c_1\|(x_n - x)'\|_2^2.$$

Since by hypothesis $\overline{\lim}\,\langle\alpha_1(x_n), x_n - x\rangle \leqslant 0$ and $\int_0^b a(x_n(t))x'(t)(x_n - x)'(t)\,\mathrm{d}t \to 0$, from (16) it follows that $\overline{\lim}\int_0^b a(x_n(t))(x_n - x)'(t)\,\mathrm{d}t \leqslant 0$. Therefore we have $x_n' \to x'$ in $L^2(T)$, hence $x_n \to x$ in $W^{1,2}_{\mathrm{per}}(T)$. It is clear from its definition that α_1 is continuous and so $\alpha_1(x_n) \to \alpha_1(x)$ in $W^{1,2}_{\mathrm{per}}(T)^*$. Hence $v = \alpha_1(x)$ and $\langle\alpha_1(x_n), x_n\rangle \to \langle\alpha_1(x), x\rangle$, which proves the pseudomonotonicity of α_1.

Next let $V_1\colon W^{1,2}_{\text{per}}(T)\to W^{1,2}_{\text{per}}(T)^*$ be defined by $V_1=\alpha_1+I+\hat{A}_\lambda$. Here $\hat{A}_\lambda\colon W^{1,p}_{\text{per}}(T)\to L^p(T)\subseteq W^{1,p}_{\text{per}}(T)^*$ is defined by $\hat{A}_\lambda(x)(\cdot)=A_\lambda(x(\cdot))$ (the Nemyckii operator corresponding to A_λ). We know that $\hat{A}_\lambda$ is monotone continuous, hence it is maximal monotone. Since the sum of pseudomonotone maps is pseudomonotone (see Theorem D (e)), we see at once that V_1 is pseudomonotone. In addition we have

$$\langle V_1(x),x\rangle\geqslant c_1\|x'\|_2^2+\|x\|_2^2\geqslant c_2\|x\|_{1,2}^2\quad\text{for some } c_2>0.$$

Hence V_1 is also coercive, thus it is surjective. So we can find $x\in W^{1,2}_{\text{per}}(T)$ such that $V_1(x)=h$. For all $\theta\in C_0^\infty(T)$, we have

$$\begin{gathered}\langle\alpha_1(x),\theta\rangle+(x,\theta)_{2,2}+(\hat{A}_\lambda(x),\theta)_{2,2}=(h,\theta)_{2,2},\\ \Longrightarrow -(a(x(t))x'(t))'=h(t)-x(t)-A_\lambda(x(t))\quad\text{a.e. on } T,\\ \Longrightarrow a(x(\cdot))x'(\cdot)\in W^{1,2}(T),\\ \Longrightarrow t\to\frac{a(x(t))x'(t)}{a(x(t))}=x'(t)\quad\text{is continuous, i.e. } x\in C^1(T).\end{gathered}$$

Then by integration by parts, for every $y\in W^{1,2}_{\text{per}}(T)$, we have

$$\begin{aligned}\int_0^b(a(x(t))x'(t))'y(t)\,\mathrm{d}t&=a(x(b))x'(b)y(b)-a(x(0))x'(0)y(0)\\&\quad-\int_0^b a(x(t))x'(t)y'(t)\,\mathrm{d}t.\end{aligned}\tag{17}$$

Recall that

$$\begin{aligned}\int_0^b a(x(t))x'(t)y'(t)\,\mathrm{d}t=\langle\alpha_1(x),y\rangle&=\int_0^b(h(t)-x(t)-A_\lambda(x(t)))y(t)\,\mathrm{d}t\\&=-\int_0^b(a(x(t))x'(t))'y(t)\,\mathrm{d}t.\end{aligned}$$

Combining this equality with (17), we obtain

$$\begin{gathered}a(x(0))x'(0)y(0)=a(x(b))x'(b)y(b)\quad\text{for all } y\in W^{1,2}_{per}(T),\\ \Longrightarrow x'(0)y(0)=x'(b)y(b)\quad\text{for all } y\in W^{1,2}_{\text{per}}(T)\quad(\text{since } x(0)=x(b)),\\ \Longrightarrow x'(0)=x'(b).\end{gathered}$$

Therefore $x\in C^1(T)$ is a solution of (15).

Next we will show the uniqueness of this solution. Suppose $x, y \in C^1(T)$ are two solutions of (15). We have

$$(18)\qquad \begin{gathered} -(a(x(t))x'(t))' + x(t) + A_\lambda(x(t)) = h(t) \quad \text{a.e. on } T, \\ x(0) = x(b), \quad x'(0) = x'(b), \end{gathered}$$

$$(19)\qquad \begin{gathered} -(a(y(t))y'(t))' + y(t) + A_\lambda(y(t)) = h(t) \quad \text{a.e. on } T, \\ y(0) = y(b), \quad y'(0) = y'(b). \end{gathered}$$

With $k > 0$ as in hypothesis $\mathrm{H(a)}_2$, let

$$\xi_\varepsilon(r) = \begin{cases} \displaystyle\int_\varepsilon^r \frac{\mathrm{d}s}{k^2 s^2} & \text{if } r \geqslant \varepsilon, \\ 0 & \text{if } r < \varepsilon, \end{cases} \qquad \varepsilon > 0.$$

From Marcus-Mizel [30], we have $\xi_\varepsilon((x-y)(\cdot)) \in W^{1,2}(T)$. We subtract (19) from (18), multiply by $\xi_\varepsilon((x-y)(t))$ and then integrate over T. After an integration by parts on the first integral, we obtain

$$\begin{aligned} \int_0^b (a(x)x' - a(y)y')\frac{\mathrm{d}}{\mathrm{d}t}\xi_\varepsilon(x-y)\,\mathrm{d}t + \int_0^b (x-y)\xi_\varepsilon(x-y)\,\mathrm{d}t \\ + \int_0^b (A_\lambda(x) - A_\lambda(y))\xi_\varepsilon(x-y)\,\mathrm{d}t = 0. \end{aligned}$$

Note that

$$\int_0^b (x-y)\xi_\varepsilon(x-y)\,\mathrm{d}t = \int_{\{x-y\geqslant\varepsilon\}} (x-y)\xi_\varepsilon(x-y)\,\mathrm{d}t \geqslant 0$$

and

$$\int_0^b (A_\lambda(x) - A_\lambda(y))\xi_\varepsilon(x-y)\,\mathrm{d}t \geqslant 0$$

(since both $A_\lambda(\cdot)$ and $\xi_\varepsilon(\cdot)$ are monotone). So we obtain

$$(20)\qquad \begin{aligned} &\int_0^b (a(x)x' - a(y)y')\frac{\mathrm{d}}{\mathrm{d}t}\xi_\varepsilon(x-y)\,\mathrm{d}t \leqslant 0 \\ &\Rightarrow \int_0^b a(x)(x'-y')\frac{\mathrm{d}}{\mathrm{d}t}\xi_\varepsilon(x-y)\,\mathrm{d}t \\ &\leqslant -\int_0^b (a(x) - a(y))y'\frac{\mathrm{d}}{\mathrm{d}t}\xi_\varepsilon(x-y)\,\mathrm{d}t. \end{aligned}$$

We examine the integral on the left-hand side of (20). Using the chain rule of Marcus-Mizel [30] for $T_\varepsilon = \{t \in T\colon\ (x-y)(t) \geqslant \varepsilon\}$, we have

$$\begin{aligned}
(21)\qquad \int_0^b a(x)(x'-y')\frac{\mathrm{d}}{\mathrm{d}t}\xi_\varepsilon(x-y)\,\mathrm{d}t &= \int_{T_\varepsilon} a(x)(x'-y')^2\xi'_\varepsilon(x-y)\,\mathrm{d}t\\
&\geqslant c_1\int_{T_\varepsilon}(x'-y')^2\frac{1}{k^2(x-y)^2}\,\mathrm{d}t.
\end{aligned}$$

For the integral on the right-hand side of (20), we have

$$\begin{aligned}
(22)\qquad -\int_0^b (a(x)-a(y))y'\frac{\mathrm{d}}{\mathrm{d}t}\xi_\varepsilon(x-y)\,\mathrm{d}t &\leqslant \int_{T_\varepsilon} k|x-y|y'\frac{x'-y'}{k^2(x-y)^2}\,\mathrm{d}t\\
= \int_{T_\varepsilon} y'\frac{x'-y'}{k(x-y)}\,\mathrm{d}t &\leqslant \|y'\|_2\Big(\int_{T_\varepsilon}\frac{(x'-y')^2}{k^2(x-y)^2}\,\mathrm{d}t\Big)^{1/2}.
\end{aligned}$$

Using (21) and (22) in (20), we have

$$\begin{aligned}
c_1\int_{T_\varepsilon}(x'-y')^2\frac{1}{k^2(x-y)^2}\,\mathrm{d}t &\leqslant \|y'\|_2\Big(\int_{T_\varepsilon}(x'-y')^2\frac{1}{k^2(x-y)^2}\,\mathrm{d}t\Big)^{1/2}\\
\Longrightarrow \int_{T_\varepsilon}(x'-y')^2\frac{1}{k^2(x-y)^2}\,\mathrm{d}t &\leqslant \frac{1}{c_1^2}\|y'\|_2^2\\
\Longrightarrow \int_{T_\varepsilon}(x'-y')^2\frac{1}{(x-y)^2}\,\mathrm{d}t &\leqslant \frac{k^2}{c_1^2}\|y'\|_2^2.
\end{aligned}$$

Thus if $\theta(t) = (x-y)(t)$ for all $t \in T$, we have

$$\int_{T_\varepsilon}\frac{\theta'(t)^2}{\theta(t)^2}\,\mathrm{d}t \leqslant \frac{k^2}{c_1^2}\|y'\|_2^2.$$

Set

$$\eta_\varepsilon(r) = \begin{cases} \displaystyle\int_\varepsilon^r \frac{\mathrm{d}s}{s} & \text{if } r \geqslant \varepsilon,\\ 0 & \text{if } r < \varepsilon.\end{cases}$$

Then the last inequality reads

$$\int_0^b \Big(\frac{\mathrm{d}}{\mathrm{d}t}\eta_\varepsilon((x-y)(t))\Big)^2\,\mathrm{d}t \leqslant \frac{k^2}{c_1^2}\|y'\|_2^2.$$

Note that $\eta_\varepsilon((x-y)(\cdot)) \in W_0^{1,2}(0,b)$ (see for example Hu-Papageorgiou [24, p. 865]). So invoking Poincare's inequality, we infer that

$$\int_0^b |\eta_\varepsilon((x-y)(t))|^2\,\mathrm{d}t \leqslant \hat{c}\|y'\|_2^2 \quad \text{for some } \hat{c} > 0.$$

Letting $\varepsilon \downarrow 0$, we see that the left-hand side goes to $+\infty$, a contradiction. So for all $\varepsilon > 0$ we must have $|T_\varepsilon| = 0$ which means that $y \geqslant x$. In a similar fashion, we can show that $x \geqslant y$, therefore $x = y$, i.e. the solution of (15) is unique. $\square$

Remark. An attempt to extend this proof to the vector problem fails. Moreover, even in the scalar case the result fails if $a(x)x'$ is replaced by a fully nonlinear operator $a(x,x')$ as in Section 3. It would be very interesting to have a vectorial extension of Proposition 8.

Now consider the quasilinear operator

$$\alpha_2\colon\ D \subseteq L^2(T) \to L^2(T)$$

which is defined by $\alpha_2(x) = -(a(x)x')'$ for all $x \in D$ where

$$D = \{x \in C^1(T)\colon\ a(x)x' \in W^{1,2}(T)\colon\ x(0) = x(b),\ x'(0) = x'(b)\}.$$

Let $K_\lambda = \alpha_2 + I + \hat{A}_\lambda\colon\ D \subseteq L^2(T) \to L^2(T)$. From Proposition 8 we know that $R(K_\lambda) = L^2(T)$ and by virtue of the uniqueness of the solution of (15), K_λ is one-to-one and so $K_\lambda^{-1}\colon\ L^2(T) \to D \subseteq W^{1,2}_{\mathrm{per}}(T)$ is well-defined.

Proposition 9. *If hypotheses* $\mathrm{H(a)}_2$, $\mathrm{H(A)}_2$ *hold, then* $K_\lambda^{-1}\colon\ L^2(T) \to D \subseteq W^{1,2}_{\mathrm{per}}(T)$ *is completely continuous for every* $\lambda > 0$.

Proof. To prove the complete continuity of K_λ^{-1} we need to show that if $u_n \xrightarrow{w} u$ in $L^2(T)$, then $x_n = K_\lambda^{-1}(u_n) \to x = K_\lambda^{-1}(u)$ in $W^{1,2}_{\mathrm{per}}(T)$. For every $n \geqslant 1$ we have

$$\alpha_2(x_n) + x_n + \hat{A}_\lambda(x_n) = u_n$$
$$\Longrightarrow (\alpha_2(x_n), x_n)_{2,2} + \|x_n\|_2^2 \leqslant (u_n, x_n)_{2,2}\ (\text{since } (\hat{A}_\lambda(x_n), x_n)_{2,2} \geqslant 0).$$

Using Green's identity, we obtain

$$(\alpha_2(x_n), x_n)_{2,2} = \langle \alpha_1(x_n), x_n \rangle \geqslant c_1 \|x_n'\|_2^2, n \geqslant 1,$$

and so

$$c_1\|x_n'\|_2^2 + \|x_n\|_2^2 \leqslant \|u_n\|_2 \|x_n\|_{1,2}$$
$$\Longrightarrow c_2\|x_n\| \leqslant \|u_n\|_2 \quad \text{with } c_2 = \min\{c_1, 1\},$$
$$\Longrightarrow \{x_n\}_{n\geqslant 1} \subseteq W^{1,2}_{\mathrm{per}}(T) \text{ is bounded.}$$

So by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} y$ in $W^{1,2}_{\mathrm{per}}(T)$ and $x_n \to y$ in $C(T)$. We have

$$0 = \lim(\alpha_2(x_n), x_n - y)_{2,2} = \lim \langle \alpha_1(x_n), x_n - y \rangle$$

and from this, as in the proof of Proposition 8, it follows that $x_n \to y$ in $W^{1,2}_{\text{per}}(T)$. Then $a(x_n)x'_n \to a(y)y'$ in $L^2(T)$ and $\{a(x_n)x'_n\}_{n\geqslant 1} \subseteq W^{1,2}(T)$ is bounded. So $a(x_n)x'_n \xrightarrow{w} a(y)y'$ in $W^{1,2}(T)$ and thus in the limit as $n \to \infty$, we have

$$\begin{cases} -(a(y(t))y'(t))' + y(t) + A_\lambda(y(t)) = u(t) \quad \text{a.e. on } T, \\ y(0) = y(b), \quad y'(0) = y'(b). \end{cases}$$

Therefore $K_\lambda(y) = u$ and by Proposition 8, we have $y = x$. Hence $K_\lambda^{-1}(u_n) \to K_\lambda^{-1}(u)$ in $W^{1,2}_{\text{per}}(T)$ which proves the complete continuity of K_λ^{-1}. □

As we did in Section 3, first we will consider the following approximation to problem (2).

$$\text{(23)} \qquad \begin{cases} (a(x(t))x'(t))' \in A_\lambda(x(t)) + F(t, x(t), x'(t)) \quad \text{a.e. on } T, \\ x(0) = x(b), \quad x'(0) = x'(b). \end{cases}$$

Our hypotheses on the multifunction $F(t, x, y)$ are the following:

H(F)_3: $F\colon T \times \mathbb{R} \times \mathbb{R} \to P_{fc}(\mathbb{R})$ is a multifunction such that

(i) for all $x, y \in \mathbb{R}$, $t \to F(t, x, y)$ is measurable;

(ii) for almost all $t \in T$, $(x, y) \to F(t, x, y)$ has a closed graph;

(iii) for almost all $t \in T$, all $x, y \in \mathbb{R}$ and all $v \in F(t, x, y)$

$$vx \geqslant -\gamma_1|x|^2 - \gamma_2|x|\,|y| - c(t)|x|$$

with $\gamma_1, \gamma_2 > 0$ and $c \in L^1(T)$;

(iv) there exist $M > 0$ such that if $|x_0| > M$, then we can find $\delta > 0$ and $\xi > 0$ such that for almost all $t \in T$

$$\inf[vx + c_1|y|^2\colon\ |x - x_0| + |y| < \delta, \quad v \in F(t, x, y)] \geqslant \xi > 0$$

($c_1 > 0$ as in H(a)_1);

(v) for almost all $t \in T$, all $x, y \in \mathbb{R}$ and all $v \in F(t, x, y)$ we have

$$|v| \leqslant \gamma_3(t, |x|) + \gamma_4(t, |x|)|y|$$

with $\sup\limits_{0\leqslant r\leqslant k} \gamma_3(t, r) \leqslant \eta_{1,k}(t)$ a.e. on T, $\eta_{1,k} \in L^2(T)$ and $\sup\limits_{0\leqslant r\leqslant k} \gamma_4(t, r) \leqslant \eta_{2,k}(t)$ a.e. on T with $\eta_{2,k} \in L^\infty(T)$, for all $k > 0$.

Remark. By virtue of hypotheses $\mathrm{H(F)}_3$ (i) and (ii), we have $F(t,x,y) = [f_1(t,x,y), f_2(t,x,y)]$, with $t \to f_1(t,x,y)$, $f_2(t,x,y)$ measurable and $(x,y) \to -f_1(t,x,y)$, $f_2(t,x,y)$ upper semicontinuous for almost all $t \in T$ (see Hu-Papageorgiou [23, Example I.2.8, p. 32] or Klein-Thompson [26, p. 74]). Hypothesis $\mathrm{H(F)}_3$ (iv) is an extension to the present Caratheodory and set-valued setting of the classical Nagumo-Hartman condition (see Hartman [22, p. 433] and Erbe-Krawcewicz [10], [11]).

Proposition 10. *If hypotheses* $\mathrm{H(a)}_2$, $\mathrm{H(F)}_3$, $\mathrm{H(A)}_2$ *hold, then the problem* (23) *has at least one solution* $x \in C^1(T)$ *for every* $\lambda > 0$.

Proof. Let $N\colon W^{1,2}_{\mathrm{per}} \to P_{fc}(L^2(T))$ be the multivalued Nemyckii operator corresponding to F (i.e. $N(x) = S^2_{F(\cdot,x(\cdot),x'(\cdot))} = \{f \in L^2(T)\colon f(t) \in F(t,x(t),x'(t))$ a.e. on $T\}$). We know that N is usc from $W^{1,2}_{\mathrm{per}}(T)$ into $L^2(T)_w$. Let $N_1(x) = -N(x) + x$. Also let $K_\lambda = \alpha_2 + I + \hat{A}_\lambda\colon D \subseteq L^2(T) \to L^2(T)$ be as in Proposition 9. We consider the following abstract multivalued fixed point problem

$$x \in K_\lambda^{-1} N_1(x). \tag{24}$$

Let $S = \{x \in W^{1,2}_{\mathrm{per}}(T)\colon x \in \beta K_\lambda^{-1} N_1(x),\ 0 < \beta < 1\}$.

Claim. *S is bounded in $W^{1,p}_{\mathrm{per}}(T)$.*

Let $x \in S$. We have

$$\begin{gathered} K_\lambda\Big(\frac{1}{\beta}x\Big) \in N_1(x) \qquad (25)\\ \Longrightarrow \alpha_2\Big(\frac{1}{\beta}x\Big) + \frac{1}{\beta}x + \hat{A}_\lambda\Big(\frac{1}{\beta}x\Big) = -f + x \ \text{ with } \ f \in N(x) = S^2_{F(\cdot,x(\cdot),x'(\cdot))},\\ \Longrightarrow \Big(\alpha_2\Big(\frac{1}{\beta}x\Big), x\Big)_{2,2} + \frac{1}{\beta}\|x\|_2^2 \leqslant -(f,x)_{2,2} + \|x\|_2^2\\ \text{(because } \Big(\hat{A}_\lambda\Big(\frac{1}{\beta}x\Big), x\Big)_{2,2} \geqslant 0). \end{gathered}$$

Note that

$$\Big(\alpha_2\Big(\frac{1}{\beta}x\Big), x\Big)_{2,2} = \Big\langle \alpha_1\Big(\frac{1}{\beta}x\Big), x\Big\rangle = \int_0^b a\Big(\frac{1}{\beta}x(t)\Big)\frac{1}{\beta}|x(t)|^2\,\mathrm{d}t \geqslant \frac{c_1}{\beta}\|x\|_2^2.$$

Using this inequality in (25), we obtain

$$\begin{gathered} \frac{c_1}{\beta}\|x'\|_2^2 + \frac{1}{\beta}\|x\|^2 \leqslant -(f,x)_{2,2} + \|x\|_2^2, \qquad (26)\\ \Longrightarrow c_1\|x'\|_2^2 \leqslant -\beta(f,x)_{2,2} + (\beta - 1)\|x\|_2^2 \leqslant -\beta(f,x)_{2,2}\\ \text{since } 0 < \beta < 1). \end{gathered}$$

By virtue of hypothesis $\mathrm{H(F)}_3$ (iii), we have

$$-\beta(f,x)_{2,2} \leqslant \beta\gamma_1\|x\|_2^2 + \beta\gamma_2\|x\|_2\|x'\|_2 + \beta\|c\|_1\|x\|_\infty. \tag{27}$$

We will show that $\|x\|_\infty \leqslant M$, with $M > 0$ as in hypothesis $\mathrm{H(F)}_3$ (iv) (Nagumo-Hartman condition). To this end let $r(t) = |x(t)|^2$ and let $t_0 \in T$ be the point where $r(\cdot)$ attains its maximum. Suppose $r(t_0) > M^2$ and first assume that $0 < t_0 < b$. Then $0 = r'(t_0) = 2x(t_0)x'(t_0) \Longrightarrow x'(t_0) = 0$. From hypothesis $\mathrm{H(F)}_3$ (iv) we can find $\delta, \xi > 0$ such that for all $v \in F(t,x,y)$

$$\inf[vx + c_1|y|^2\colon\ |x - x(t_0)| + |y| < \delta] \geqslant \xi > 0 \quad \text{a.e. on } T.$$

Since $x \in S$, we have $x \in D$ and so $x \in C^1(T)$. Thus we can find $\delta_1 > 0$ such that if $t_0 < t \leqslant t_0 + \delta_1$ then $|x(t) - x(t_0)| + |x'(t)| < \delta$ (because $x'(t_0) = 0$). Then for almost all $t \in (t_0, t_0 + \delta_1]$ we have

$$\beta f(t)x(t) + \beta c_1|x'(t)|^2 \geqslant \beta\xi > 0. \tag{28}$$

From the equation $\alpha_2\big(\beta^{-1}x\big) + \beta^{-1}x + \hat{A}_\lambda\big(\beta^{-1}x\big) = -f + x$ we have

$$\begin{gathered}
-\Big(a\Big(\frac{1}{\beta}x(t)\Big)\frac{1}{\beta}x'(t)\Big)' + A_\lambda\Big(\frac{1}{\beta}x(t)\Big) = -f(t) + \Big(1 - \frac{1}{\beta}\Big)x(t) \quad \text{a.e. on } T,\\
x(0) = x(b), \quad x'(0) = x'(b).
\end{gathered}$$

Using this in (28), we obtain

$$\begin{gathered}
\Big(\Big(a\Big(\frac{1}{\beta}x(t)\Big)x'(t)\Big)' - \beta A_\lambda\Big(\frac{1}{\beta}x(t)\Big) + (\beta - 1)x(t)\Big)x(t) + \beta c_1|x'(t)|^2\\
\geqslant \beta\xi > 0 \quad \text{a.e. on } (t_0, t_0 + \delta_1],\\
\Longrightarrow \int_{t_0}^t \Big(a\Big(\frac{1}{\beta}x(s)\Big)x'(s)\Big)'x(s)\,\mathrm{d}s - \beta\int_{t_0}^t A_\lambda\Big(\frac{1}{\beta}x(s)\Big)x(s)\,\mathrm{d}s + \beta c_1\int_{t_0}^t |x'(s)|^2\,\mathrm{d}s\\
\geqslant \beta\xi(t - t_0) > 0, \quad \text{for all } t \in (t_0, t_0 + \delta_1],\\
\Longrightarrow \int_{t_0}^t \Big(a\Big(\frac{1}{\beta}x(s)\Big)x'(s)\Big)'x(s)\,\mathrm{d}s + \beta c_1\int_{t_0}^t |x'(s)|^2\,\mathrm{d}s > 0,\\
\text{for all } t \in (t_0, t_0 + \delta_1].
\end{gathered}$$

From integration by parts, we have for $t \in (t_0, t_0 + \delta_1]$

$$a\Big(\frac{1}{\beta}x(t)\Big)x'(t)x(t) - a\Big(\frac{1}{\beta}x(t_0)\Big)x'(t_0)x(t_0) - \int_{t_0}^{t} a\Big(\frac{1}{\beta}x(s)\Big)|x'(s)|^2\,\mathrm{d}s$$
$$+\beta c_1 \int_{t_0}^{t} |x'(s)|^2\,\mathrm{d}s > 0$$
$$\Longrightarrow a\Big(\frac{1}{\beta}x(t)\Big)x'(t)x(t) - \int_{t_0}^{t} c_1|x'(s)|^2\,\mathrm{d}s$$
$$+\beta c_1 \int_{t_0}^{t} |x'(s)|^2\,\mathrm{d}s > 0$$
$$\Longrightarrow x'(t)x(t) > 0 \quad \text{i.e. } r'(t) > 0 \quad \text{for } t \in (t_0, t_0 + \delta_1],$$

which contradicts the choice of t_0. If $t_0 = 0$, then $r'(t_0) = r'(0) \leqslant 0$ and $r(0) = r(b)$ (by periodicity) so $r'(b) \geqslant 0$. But $r'(0) = 2x(0)x'(0) = 2x(b)x'(b) = r'(b)$. Hence $r'(0) = 0$ and we proceed as above. Similarly if $t_0 = b$. So finally $\|x\|_\infty \leqslant M$ for all $x \in S$. Using this in (27), we have

$$-\beta(f,x)_{2,2} \leqslant \beta\gamma_1 M_1^2 + \beta\gamma_2 M_1\|x'\|_2 + \beta\|c\|_1 M \quad \text{for some } M_1 > 0.$$

Using this estimate in (26), we obtain an $M_2 > 0$ such that $\|x'\|_2 \leqslant M_2$ for all $x \in S$. Therefore $S \subseteq W^{1,2}_{\mathrm{per}}(T)$ is bounded.

The Claim, together with Proposition 9, permit the use of Proposition 1, which gives a solution for the problem (24), i.e. there exists $x \in D \subseteq C^1(T)$ such that $x \in K_\lambda^{-1}N_1(x)$. Evidently $x \in C^1(T)$ is the desired solution of (23). □

Using the continuous selection argument in the proof of Theorem 7, we can have a version of Proposition 10 in which F has nonconvex values (nonconvex problem). In this case the hypotheses on the multifunction $F(t,x,y)$ are the following:

$\mathrm{H(F)}_4$: $F\colon T \times \mathbb{R} \times \mathbb{R} \to P_f(\mathbb{R})$ is a multifunction such that
- (i) $(t,x,y) \to F(t,x,y)$ is graph measurable;
- (ii) for almost all $t \in T$, $(x,y) \to F(t,x,y)$ is lsc;

  and it satisfies hypotheses $\mathrm{H(F)}_3$ (iii)–(v).

Proposition 11. *If hypotheses* $\mathrm{H(a)}_2$, $\mathrm{H(F)}_4$, $\mathrm{H(A)}_2$ *hold, then the problem* (23) *has at least one solution* $x \in C^1(T)$ *for every* $\lambda > 0$.

Now we will pass to the limit as $\lambda \downarrow 0$ and obtain solutions for the problem (2) for both the convex and nonconvex problems.

Theorem 12. *If hypotheses* $\mathrm{H(a)}_2$, $\mathrm{H(F)}_2$, $\mathrm{H(A)}_2$ *hold, then the problem* (2) *has at least one solution* $x \in C^1(T)$.

Proof. Let $\lambda_n \downarrow 0$, $\lambda_n > 0$ and let $x_n \in C^1(T)$, $n \geqslant 1$, be solutions of the problem (23) (Proposition 10). Arguing as in the proof of Proposition 10, we can show that $\{x_n\}_{n\geqslant 1} \subseteq W^{1,2}_{\mathrm{per}}(T)$ is bounded and so we may assume that $x_n \xrightarrow{w} x$ in $W^{1,2}_{\mathrm{per}}(T)$. Also for every $n \geqslant 1$, we have

$$\begin{aligned}(29)\qquad & \alpha_2(x_n) + \hat{A}_{\lambda_n}(x_n) = -f_n, \quad f_n \in N(x_n),\\ & \Longrightarrow (\alpha_2(x_n), \hat{A}_{\lambda_n}(x_n))_{2,2} + \|\hat{A}_{\lambda_n}(x_n)\|_2^2 = -(f_n, \hat{A}_{\lambda_n}(x_n))_{2,2}.\end{aligned}$$

Recall that $A_{\lambda_n}(\cdot)$ is Lipschitz continuous (see Proposition C (c)) and so $\hat{A}_{\lambda_n}(x_n) \in C(T)$ for all $n \geqslant 1$. Using integration by parts, we obtain

$$\begin{aligned}(\alpha_2(x_n), \hat{A}_{\lambda_n}(x_n))_{2,2} &= -\int_0^b (a(x_n(t))x_n'(t))'\hat{A}_{\lambda_n}(x_n)\,\mathrm{d}t\\ &= -a(x_n(b))x_n'(b)A_{\lambda_n}(x_n(b)) + a(x_n(0))x_n'(0)A_{\lambda_n}(x_n(0))\\ &\quad + \int_0^b a(x_n(t))x_n'(t)\frac{\mathrm{d}}{\mathrm{d}t}A_{\lambda_n}(x_n(t))\,\mathrm{d}t.\end{aligned}$$

Recall that $A_{\lambda_n}(\cdot)$ is Lipschitz continuous. So from the chain rule of Marcus-Mizel [30] we have $\frac{\mathrm{d}}{\mathrm{d}t}A_{\lambda_n}(x_n(t)) = A'_{\lambda_n}(x_n(t))x_n'(t)$ a.e. on T and from the monotonicity of $A_{\lambda_n}(\cdot)$ we have $A'_{\lambda_n}(x_n(t)) \geqslant 0$ a.e. on T. Since

$$a(x_n(0))x_n'(0)A_{\lambda_n}(x_n(0)) = a(x_n(b))x_n'(b)A_{\lambda_n}(x_n(b))$$

(periodic boundary conditions), we obtain

$$(\alpha_2(x_n), \hat{A}_{\lambda_n}(x_n))_{2,2} = \int_0^b a(x_n(t))A_{\lambda_n}(x_n(t))|x_n(t)|^2\,\mathrm{d}t \geqslant 0.$$

So using this in (29), we obtain

$$\begin{aligned}&\|\hat{A}_{\lambda_n}(x_n)\|_2^2 \leqslant \|f_n\|_2\|\hat{A}_{\lambda_n}(x_n)\|_2,\\ &\Longrightarrow \{\hat{A}_{\lambda_n}(x_n)\}_{n\geqslant 1} \subseteq L^2(T) \quad \text{is bounded (hypothesis } \mathrm{H(F)}_3\,\text{(v))}.\end{aligned}$$

We may assume that $\hat{A}_{\lambda_n}(x_n) \xrightarrow{w} u$, $f_n \xrightarrow{w} f$ in $L^2(T)$. Also as in the proof of Proposition 8, we can show that $x_n \to x$ in $W^{1,2}_{\mathrm{per}}(T)$. Hence, by virtue of Proposition B as before (see the proof of Proposition 4), we can check that $f \in N(x)$. In the limit as $n \to \infty$, we obtain

$$\begin{cases} (a(x(t))x'(t))' = u(t) + f(t) & \text{a.e. on } T,\\ x(0) = x(b), \quad x'(0) = x'(b).\end{cases}$$

We will finish the proof if we show that $u(t) \in A(x(t))$ a.e. on T. Let $\hat{J}_\lambda\colon\ L^2(T) \to L^2(T)$ be the Nemyckii operator corresponding to the resolvent function J_λ of the operator A, i.e. $\hat{J}_\lambda(x)(\cdot) = J_\lambda(x(\cdot))$ (see Section 2). Recall that J_λ is nonexpansive. So from Marcus-Mizel [30], we have $\hat{J}_\lambda(x_n) \in W^{1,2}(T)$, $\frac{\mathrm{d}}{\mathrm{d}t} J_{\lambda_n}(x_n(t)) = J'_{\lambda_n}(x_n(t))x'_n(t)$ and $|J'_{\lambda_n}(x_n(t))| \leqslant 1$ a.e. on T. Therefore $|J'_{\lambda_n}(x_n(t))x'_n(t)| \leqslant \|x'_n(t)\|$ a.e. on T and so $\{\hat{J}_{\lambda_n}(x_n)\}_{n\geqslant 1} \subseteq W^{1,2}_{\text{per}}(T)$ is bounded. Thus we may assume that $\hat{J}_{\lambda_n}(x_n) \xrightarrow{w} z$ in $W^{1,2}(T)$ and $\hat{J}_\lambda(x_n) \to z$ in $C(T)$. We know that

$$J_{\lambda_n}(x_n(t)) + \lambda_n A_{\lambda_n}(x_n(t)) = x_n(t) \quad \left(\text{since } A_{\lambda_n} = \frac{1}{\lambda_n}(I - J_{\lambda_n})\right)$$
$$\Longrightarrow \hat{J}_{\lambda_n}(x_n) + \lambda_n \hat{A}_{\lambda_n}(x_n) = x_n.$$

Passing to the limit as $n \to \infty$ and since $\lambda_n \to 0$, $\{\hat{A}_{\lambda_n}(x_n)\}_{n\geqslant 1} \subseteq L^2(T)$ is bounded, we obtain $z = x$. So $\hat{J}_{\lambda_n}(x_n) \to x$ in $C(T)$.

Note that $|J_{\lambda_n}(x_n(t)) - J_{\lambda_n}(x(t))| \leqslant \|x_n - x\|_\infty \to 0$. So $\hat{J}_{\lambda_n}(x) \to x$ in $C(T)$ and of course in $L^2(T)$. Recall that $A = \partial\varphi$ for some $\varphi \in \Gamma_0(\mathbb{R})$ and so $\hat{A} = \partial\Phi$ with

$$\Phi(x) = \begin{cases} \displaystyle\int_0^b \varphi(x(t))\,\mathrm{d}t & \text{if } \varphi(x(\cdot)) \in L^1(T), \\ +\infty & \text{otherwise} \end{cases}$$

(see for example Brezis [3, p. 47] or Hu-Papageorgiou [23, p. 349]). Then $\hat{J}_{\lambda_n}$ is the resolvent operator of $\hat{A}$ and so $\hat{J}_{\lambda_n}(x) \to \operatorname{proj}(x; \overline{D})$, with D being the domain of $\hat{A} = \partial\Phi$ (i.e. $D = \{x \in L^2(T)\colon\ \hat{A}(x) = \partial\Phi(x) \neq \emptyset\}$). But we know that $\hat{J}_{\lambda_n}(x) \to x$ in $L^2(T)$. Hence $x = \operatorname{proj}(x; \overline{D})$, i.e. $x \in \overline{D}$. We have $\hat{A}_{\lambda_n}(x_n) \in \hat{A}(\hat{J}_{\lambda_n}(x_n))$ (see Theorem C (b)), $\hat{J}_{\lambda_n}(x_n) \to x \in \overline{D}$ in $L^2(T)$ and $\hat{A}_{\lambda_n}(x_n) \xrightarrow{w} u$ in $L^2(T)$. Invoking proposition 4.33 (b), p. 351, of Hu-Papageorgiou [23], we conclude that $u \in \hat{A}(x)$. Therefore finally we have

$$(a(x(t))x'(t))' \in A(x(t)) + F(t, x(t), x'(t)) \quad \text{a.e. on } T,$$
$$x(0) = x(b), \quad x'(0) = x'(b),$$

i.e. $x \in C^1(T)$ is the desired solution of (2). □

In a similar fashion using the continuous selection argument of the proof of Theorem 7 and Proposition 11, we can have the nonconvex variant of Theorem 12.

Theorem 13. *If hypotheses* $\mathrm{H(a)}_2$, $\mathrm{H(F)}_4$, $\mathrm{H(A)}_2$ *hold, then the problem* (2) *has at least one solution* $x \in C^1(T)$.

Application 2. In the past even for single-valued problems there were no existence theorems for periodic equations with unilateral constraints (differential variational inequalities). So as an illustration that our work provides solutions where the earlier literature fails, we consider a scalar differential variational inequality with periodic boundary conditions. To this end let

$$\varphi(x) = \delta_{\mathbb{R}_+}(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}_+, \\ +\infty & \text{otherwise} \end{cases}$$

and $A = \partial\varphi$. We know that $\operatorname{dom} A = \mathbb{R}_+$, $A(x) = \{0\}$ if $x > 0$ and $A(0) = \mathbb{R}_-$. Also let $F = f\colon T \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a single-valued Caratheodory function (i.e. measurable in $t \in T$, continuous in $x, y \in \mathbb{R}$), which satisfies hypotheses $\mathrm{H(F)}_3$ (iii)–(v) (with F replaced by f). Then the problem (2) becomes the following variational inequality:

$$\left\{\begin{array}{l} (a(x(t))x'(t)) \leqslant f(t, x(t), x'(t)) \quad \text{a.e. on } \{t \in T\colon x(t) = 0\}, \\ (a(x(t))x'(t)) = f(t, x(t), x'(t)) \quad \text{a.e. on } \{t \in T\colon x(t) > 0\}, \\ x(t) \geqslant 0 \quad \text{for all } t \in T, \ x(0) = x(b), \ x'(0) = x'(b). \end{array}\right\} \tag{30}$$

Corollary 14. *If* A, f *are as above and* $\mathrm{H(a)}_2$ *holds, then the problem* (30) *has at least one solution* $x \in C^1(T)$.

The following functions $f(t, x, y)$ satisfy the stated hypotheses:

$$f(t, x, y) = x^3 - \sqrt{|x|} - h(t) + y$$

and

$$f(t, x, y) = x^5 - \ln|x| - \sin y - h(t) \quad \text{with } h \in L^\infty(T).$$

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