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AN APPLICATION OF PÓLYA'S ENUMERATION THEOREM TO PARTITIONS OF SUBSETS OF POSITIVE INTEGERS

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Abstract. Let S be a non-empty subset of positive integers. A partition of a positive integer n into S is a finite nondecreasing sequence of positive integers $a_1, a_2, \ldots, a_r$ in S with repetitions allowed such that $\sum_{i=1}^{r} a_i = n$. Here we apply Pólya's enumeration theorem to find the number $\mathrm{P}(n;S)$ of partitions of n into S, and the number $\mathrm{DP}(n;S)$ of distinct partitions of n into S. We also present recursive formulas for computing $\mathrm{P}(n;S)$ and $\mathrm{DP}(n;S)$.

Keywords: Pólya's enumeration theorem, partitions of a positive integer into a non-empty subset of positive integers, distinct partitions of a positive integer into a non-empty subset of positive integers, recursive formulas and algorithms

1. Introduction

Let S be a non-empty subset of positive integers. A partition of a positive integer n into S is a finite nondecreasing sequence of positive integers $a_1, a_2, \ldots, a_r$ in S with repetitions allowed such that $\sum_{i=1}^{r} a_i = n$. The a_i's are called the parts of a partition of n.

Example 1. (a) Let S be the set of positive integers. Then the partition of positive integer n into S is the "usual" partition of n. For instance, there are 7 partitions of 5. Namely, 5, $1+4$, $2+3$, $1+1+3$, $1+2+2$, $1+1+1+2$ and $1+1+1+1+1$. (Usually, one writes the sequence as a series to indicate the sum is 5.) We note that each of the first three has distinct parts.

(b) Let S be the set of all odd positive integers. Then there are 3 odd partitions of 5. Namely, 5, $1+1+3$ and $1+1+1+1+1$.

(c) Let S be the set of all positive integers each of which is not a multiple of 3. Then there are 7 partitions of 6. Namely, $5+1$, $4+2$, $4+1+1$, $2+2+2$, $2+2+1+1$, $2+1+1+1+1$ and $1+1+1+1+1+1$.

(d) Let S be the set of all positive integers each of which is not a multiple of 3, 4 or 5. Then there are 4 partitions of 6. Namely, the last 4 partitions in (c).

(e) Let $S = \{1, 2, 4\}$. Then there are 4 partitions of 5. Namely, $1+4$, $1+2+2$, $1+1+1+2$ and $1+1+1+1+1$. There are 3 partitions of 9 with 1, 2, 3, 4 or 5 parts. Namely, $1+4+4$, $1+2+2+4$ and $1+2+2+2+2$.

There are many results on the partitions of positive integers. (See [1].) Here we will apply Pólya's enumeration theorem ([5], [2], [4], [6]) to the partitions of positive integers into S for any non-empty subset S of positive integers. Based on this application, we obtain a recursive formula for the number $\mathrm{P}(n; S)$ of partitions of a positive integer n into S and a recursive formula for the number $\mathrm{DP}(n; S)$ of partitions of a positive integer n into S with distinct parts. Based on these recursive formulas, we present computer programs for computing $\mathrm{P}(n; S)$ and $\mathrm{DP}(n; S)$; in particular, $\mathrm{P}(n; I)$, $\mathrm{P}(n; 0)$, $\mathrm{DP}(n; I)$ and $\mathrm{DP}(n; 0)$ as well as some subsets of positive integers where I is the set of positive integers and O is the set of positive odd integers.

2. Pólya's Enumeration Theorem

We shall state Pólya's Enumeration Theorem. Let G be a permutation group acting on a set $\{1, 2, \ldots, n\}$. Since every permutation can be uniquely written as a product of disjoint cycles, the cycle index G is defined as the following polynomial in $Q[x_1, x_2, \ldots, x_n]$ where Q is the field of rational numbers and $x_i x_j = x_j x_i$ for $i, j = 1, 2, \ldots, n$:

$$Z_G(x_1, x_2, \ldots, x_n) = \frac{1}{|G|} \sum_{\sigma \in G} x_1^{b_1} x_2^{b_2} \ldots x_n^{b_n}$$

where $|G|$ is the order of G and b_i is the number of cycles of length i in the disjoint cycle decomposition of σ for $i = 1, 2, \ldots, n$.

Pólya's Enumeration Theorem. Let D be a finite set and S a countable set, S^D the set of all functions from the domain D into the codomain S, G a permutation group acting on D, w a function, called the weight function, from S into R where R is a commutative ring with an identity containing the field of rational numbers Q, and let a relation be defined on S^D such that for $f, g \in S^D$, $f \sim g$ if and only if there exists a $\sigma \in G$ with

$$f(\sigma d) = g(d) \quad \text{for every } d \in D.$$

(Since G is a group, the relation $\sim$ is an equivalence relation. Consequently, S^D is partitioned into disjoint equivalence classes each of which is called a pattern.) Then the total pattern or the counting series is

$$(1)\qquad Z_G\Bigl(\sum_{s\in S} w(s), \sum_{s\in S}(w(s))^2, \ldots, \sum_{s\in S}(w(s))^t, \ldots\Bigr).$$

3. Counting partitions of a positive integer

Theorem 1. (a) *For any positive integer k, let $D_k = \{1, 2, \ldots, k\}$, let S be a non-empty subset of positive integers, S^{D_k} the set of all functions from D_k into S, let the symmetric group S_k act on D_k, let the weight function $w\colon S \to Q[x]$ be defined as $w(i) = x^i$ for all i in S, and for $f, g \in S^{D_k}$, $f \sim g$ if and only if there exists a $\sigma \in S_k$ such that $f(\sigma d) = g(d)$ for every d in D_k. Then the number of partitions of a positive integer n with k parts into S is the coefficient of x^n in the counting series*

$$(2)\qquad Z_{S_k}\Bigl(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \ldots, \sum_{i\in S} x^{ki}\Bigr);$$

(b) *the number $\mathrm{P}(n; S)$ of partitions of n into S is the coefficient of x^n in the counting series*

$$(3)\qquad \sum_{k=1}^{\infty} Z_{S_k}\Bigl(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \ldots, \sum_{i\in S} x^{ki}\Bigr).$$

Proof. (a) We claim that each equivalence class in S^{D_k} with weight n (i.e., every function in the equivalence class has weight n) determines a partition of n into S with k parts. Let E be an equivalence class with weight n in S^{D_k}, and let f be a function in E. Then f has k values with repetition allowed in S such that $w(f) = x^n$. Since S is a subset of positive integers, we may arrange the k values of f in a nondecreasing order, say, $j_1 \leqslant j_2 \leqslant \ldots \leqslant j_k$. Since $w(f) = x^n$ and $w(f) = \prod_{i+1}^{k} w(f(i)) = x^{j_1+j_2+\ldots+j_k}$, we have $j_1 + j_2 + \ldots + j_k = n$. Thus, f corresponds to a partition of n into S with k parts. Since S_k acts on D_k and $f(\sigma d) = g(d)$ for some $\sigma \in S_k$ and all $d \in D_k$, the equivalence class containing f consists of all functions in S^{D_k} such that each has the function values $\{j_1, j_2, \ldots, j_k\}$. Thus, each equivalence class with weight n corresponds to a partition of n into S with k parts.

Conversely, each partition $t_1 \leqslant t_2 \leqslant \ldots \leqslant t_k$ of n into S with k parts determines an equivalence class with weight n in S^{D_k}. Clearly, $h(i) = t_i$ for $i = 1, 2, \ldots, k$ is a function in S^{D_k}, and $w(h) = \prod_{i=1}^{k} x^{t_i} = x^{t_1+t_2+\ldots+t_k} = x^n$. Thus, the partition of n into S with k parts determines the equivalence class containing h in S^{D_k}.

By Pólya's enumeration theorem, the coefficient of x^n in the counting series

$$Z_{S_k}\Big(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \ldots, \sum_{i\in S} x^{ki}\Big) \tag{2}$$

is the number of partitions of n into S with k parts.

(b) Summing over $k = 1, 2, \ldots$, we obtain

$$\sum_{k=1}^{\infty} Z_{S_k}\Big(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \ldots, \sum_{i\in S} x^{ki}\Big). \tag{3}$$

In (3), the coefficient of x^n is the number of partitions of n into S with k parts for $k = 1, 2, \ldots$, i.e., the coefficient of x^n is the number of partitions of n into S. □

Example 2. Let $D_3 = \{1, 2, 3\}$, let S be the set of all positive integers $= \{1, 2, \ldots, n, \ldots\}$, S^{D_3} the set of all functions from D_3 into S, let

$$S_3 = \{1, (123), (132), (12), (13), (23)\}$$

act on D_3, and let let $w\colon S \to Q[x]$ be defined as $w(i) = x^i$ for $i = 1, 2, 3, \ldots$ Then the cycle index is

$$Z_{S_3}(x_1, x_2, x_3) = \tfrac{1}{6}(x_1^3 + 2x_3 + 3x_1x_2),$$

and $\sum_{i\in S} w(i) = \sum_{i=1}^{\infty} x^i$ (a formal power series), $\sum_{i\in S}(w(i))^2 = \sum_{i=1}^{\infty} x^{2i}$, $\sum_{i\in S}(w(i))^3 = \sum_{i=1}^{\infty} x^{3i}$. By (2), we have

$$\begin{aligned}
Z_{S_3}\Big(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \sum_{i\in S} x^{3i}\Big) &= Z_{S_3}\Big(\sum_{i=1}^{\infty} x^i, \sum_{i=1}^{\infty} x^{2i}, \sum_{i=1}^{\infty} x^{3i}\Big)\\
&= \tfrac{1}{6}\big((x^1 + x^2 + x^3 + \ldots + x^m + \ldots)^3 + 2(x^3 + x^6 + x^9 + \ldots + x^{3m} + \ldots)\\
&\quad + 3(x^1 + x^2 + x^3 + \ldots + x^m + \ldots)(x^2 + x^4 + x^6 + \ldots + x^{2m} + \ldots)\big)\\
&= \tfrac{1}{6}\big((x^3 + 3x^4 + 6x^5 + 10x^6 + 15x^7 + 21x^8 + \ldots) + (2x^3 + 2x^6 + \ldots)\\
&\quad + (3x^3 + 3x^4 + 6x^5 + 6x^6 + 9x^7 + 9x^8 + \ldots)\big)\\
&= x^3 + x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + \ldots,
\end{aligned}$$

which means:

For $n = 1$ or 2, there is no partition of n with 3 parts.

For $n = 3$, there is 1 partition of 3 with 3 parts. (Namely, $1 + 1 + 1$.)

For $n = 4$, there is 1 partition of 4 with 3 parts. (Namely, $1 + 1 + 2$.)

For $n = 5$, there are 2 partitions of 5 with 3 parts. (Namely, $1+1+3$ and $1+1+2$.)

For $n = 6$, there are 3 partitions of 6 with 3 parts. (Namely, $1 + 1 + 4, 1 + 2 + 3$ and $2 + 2 + 2$.)

For $n = 7$, there are 4 partitions of 7 with 3 parts. (Namely, $1 + 1 + 5, 1 + 2 + 4$, $1 + 3 + 3$ and $2 + 2 + 3$.)

For $n = 8$, there are 5 partitions of 8 with 3 parts. (Namely, $1 + 1 + 6, 1 + 2 + 5$, $1 + 3 + 4$, $2 + 2 + 4$ and $2 + 3 + 3$.)

Example 3. Let $S = \{1, 2, 4\}$. Let $D_t = \{1, 2, \ldots, t\}$ for $t = 1, 2, 3, 4$, let S^{D_t}, w and S_t be defined similarly to Example 2. We know the following cycle indices:

$$\begin{aligned} Z_{S_1}(x_1) &= x_1, \\ Z_{S_2}(x_1, x_2) &= \tfrac{1}{2}(x_1^2 + x_2), \\ Z_{S_3}(x_1, x_2, x_3) &= \tfrac{1}{6}(x_1^3 + 3x_1x_2 + 2x_3), \end{aligned}$$

and

$$Z_{S_4}(x_1, x_2, x_3, x_4) = \tfrac{1}{24}(x_1^4 + 6x_1^2x_2 + 8x_1x_3 + 3x_2^2 + 6x_4).$$

Then

$$\begin{aligned} &\sum_{k=1}^{4} Z_{S_k}\left(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \ldots\right) \\ &\quad = [(x + x^2 + x^4)] + \frac{1}{2}[(x + x^2 + x^4)^2 + (x^2 + x^4 + x^8)] \\ &\qquad + \frac{1}{6}[(x + x^2 + x^4)^3 + 3(x + x^2 + x^4)(x^2 + x^4 + x^8) + 2(x^3 + x^6 + x^{12})] \\ &\qquad + \frac{1}{24}[(x + x^2 + x^4)^4 + 6(x + x^2 + x^4)^2 + 8(x + x^2 + x^4)(x^3 + x^6 + x^{12}) \\ &\qquad + 3(x^2 + x^4 + x^8)^2 + 6(x^4 + x^8 + x^{16})] \\ &\quad = x + 2x^2 + 2x^3 + 4x^4 + 3x^5 + 4x^6 + 3x^7 + 4x^8 + 2x^9 + 3x^{10} + x^{11} \\ &\qquad + 2x^{12} + x^{13} + x^{14} + x^{16}, \end{aligned}$$

which means, for instance, for $n = 6$, there are 4 partitions of 6 with 1, 2, 3 or 4 parts. (Namely, $2 + 4$, $1 + 1 + 4$, $2 + 2 + 2$ and $1 + 1 + 2 + 2$.)

Using Theorem 1 we can obtain a recursive formula for $\mathrm{P}(n; S)$ with $n > 1$. Clearly, $\mathrm{P}(1; S) = 1$ if $1 \in S$, and $\mathrm{P}(1; S) = 0$ if $1 \notin S$.

Corollary 1.1.

$$\mathrm{P}(n;S)=\frac{1}{n}\Bigg(\sum_{i|n,i\in S} i+\sum_{k=1}^{n-1}\Bigg(\sum_{i|k,\,i\in S} i\Bigg)\mathrm{P}(n-k;S)\Bigg)\quad \textit{for } n>1. \tag{4}$$

Remark. If there exists no positive integer i such that $i \mid k$ and $i \in S$, then $\sum\limits_{i|k,\,i\in S} i=0$.

In order to prove Corollary 1.1, we need two identities which can be found in [4]. First,

$$1+\sum_{k=1}^{\infty} Z_{S_k}(f(x),f(x^2),\dots,f(x^k))=\exp\Bigg(\sum_{k=1}^{\infty}\frac{f(x^k)}{k}\Bigg) \tag{5}$$

where $f(x)$ is a function of x or a series of x.

Second, if

$$\sum_{m=0}^{\infty} A_m x^m=\exp\Bigg(\sum_{m=1}^{\infty} a_m x^m\Bigg), \tag{6}$$

then, for $m \geqslant 1$,

$$a_m=A_m-m^{-1}\Bigg(\sum_{k=1}^{m-1} k a_k A_{m-k}\Bigg).$$

P r o o f of Corollary 1.1. By Theorem 1, we have

$$1+\sum_{n=1}^{\infty}\mathrm{P}(n;S)x^n=1+\sum_{k=1}^{\infty} Z_{S_k}\Bigg(\sum_{i\in S} x^i,\sum_{i\in S} x^{2i},\dots,\sum_{i\in S} x^{ki}\Bigg). \tag{7}$$

By (5), the right-hand side of (7) is $\exp\Big(\sum\limits_{k=1}^{\infty}\Big(\sum\limits_{i\in S} x^{ki}/k\Big)\Big)$. So,

$$\begin{aligned}1+\sum_{n=1}^{\infty}\mathrm{P}(n;S)x^n &= \exp\Bigg(\sum_{k=1}^{\infty}\Bigg(\sum_{i\in S}\frac{x^{ki}}{k}\Bigg)\Bigg)=\exp\Bigg(\sum_{k=1}^{\infty}\Bigg(\sum_{i\in S}\frac{i x^{ki}}{ki}\Bigg)\Bigg)\\ &= \exp\Bigg(\sum_{n=1}^{\infty}\frac{1}{n}\Bigg(\sum_{i|n,\,i\in S} i\Bigg)x^n\Bigg).\end{aligned}$$

By (6), we have

$$\frac{1}{n}\sum_{i|n,\,i\in S} i=\mathrm{P}(n;S)-\frac{1}{n}\Bigg(\sum_{k=1}^{n-1} k\cdot\frac{1}{k}\Bigg(\sum_{i|k,\,i\in S} i\Bigg)\mathrm{P}(n-k;S)\Bigg).$$

Hence,

$$\mathrm{P}(n;S)=\frac{1}{n}\Bigl(\sum_{i|n,\,i\in S} i+\sum_{k=1}^{n-1}\Bigl(\sum_{i|k,\,i\in S} i\Bigr)\mathrm{P}(n-k,S)\Bigr).$$

Now we consider the set of positive integers $I=\{1,2,\ldots,n,\ldots\}$ and the set of positive odd integers $O=\{1,3,\ldots,2n-1,\ldots\}$. By using Corollary 1.1, we obtain recursive formulas $\mathrm{P}(n;I)$ and $\mathrm{P}(n;O)$ where n is a positive integer.

Corollary 1.2. (a) $\mathrm{P}(1;I)=1$ *and for* $n>1$,

$$\mathrm{P}(n;I)=\frac{1}{n}\Bigl(\sum_{i|n,\,i\in S} i+\sum_{k=1}^{n-1}\Bigl(\sum_{i|k,\,i\in S} i\Bigr)\mathrm{P}(n-k;I)\Bigr). \tag{8}$$

(b) $\mathrm{P}(I;O)=1$ *and for* $n>1$,

$$\mathrm{P}(n;O)=\frac{1}{n}\Bigl(\sum_{i|n,\,i\in O} I+\sum_{k=1}^{n-1}\Bigl(\sum_{i|k,\,i\in O} i\Bigr)\mathrm{P}(n-k;O)\Bigr). \tag{9}$$

P r o o f. (a) (8) is obtained by substituting I for S in (4) in Corollary 1.1 and (9) is obtained by substituting O for S in (4) in Corollary 1.1. □

Example 4. We use Corollary 1.2 to compute $\mathrm{P}(n;I)$ and $\mathrm{P}(n;O)$ for $n=1,2,3,4,5$.

$$\begin{aligned}
\mathrm{P}(1;I)&=1,\\
\mathrm{P}(2;I)&=\tfrac{1}{2}(3+\mathrm{P}(1;I))=\tfrac{1}{2}(3+1)=2,\\
\mathrm{P}(3;I)&=\tfrac{1}{3}(4+\mathrm{P}(2;I)+3\mathrm{P}(1;I)=\tfrac{1}{3}(4+2+3)=3,\\
\mathrm{P}(4;I)&=\tfrac{1}{4}(7+\mathrm{P}(3;I)+3\mathrm{P}(2;I)+4\mathrm{P}(1;I))=\tfrac{1}{4}(7+3+6+4)=5,\\
\mathrm{P}(5;I)&=\tfrac{1}{5}(6+\mathrm{P}(4;I)+3\mathrm{P}(3;I)+4\mathrm{P}(2;I)+7\mathrm{P}(1;I))\\
&=\tfrac{1}{5}(6+5+9+8+7)=7,\\
\mathrm{P}(1;O)&=1,\\
\mathrm{P}(2;O)&=\tfrac{1}{2}(1+\mathrm{P}(1;O))=\tfrac{1}{2}(1+1)=1,\\
\mathrm{P}(3;O)&=\tfrac{1}{3}(4+\mathrm{P}(2;O)+\mathrm{P}(1;O)=\tfrac{1}{3}(4+1+1)=2,\\
\mathrm{P}(4;O)&=\tfrac{1}{4}(1+\mathrm{P}(3;O)+\mathrm{P}(2;O)+4\mathrm{P}(1;O))=\tfrac{1}{4}(1+2+1+4)=2,\\
\mathrm{P}(5;O)&=\tfrac{1}{5}(6+\mathrm{P}(4;O)+\mathrm{P}(3;O)+4\mathrm{P}(2;O)+\mathrm{P}(1;O))\\
&=\tfrac{1}{5}(6+2+2+4+1)=3.
\end{aligned}$$

4. Counting partitions of a positive integer into distinct parts

Theorem 2. (a) *Let k, D_k, S, S_k, S^{D_k} and w be the same as in Theorem 1. Also, let F be a subset of all one-to-one functions in S^{D_k}, and for $f, g \in F, f \sim g$ if and only if there exists a $\sigma \in S_k$ such that $f(\sigma d) = g(d)$ for every d in D_k. Then the number of partitions of a positive integer n with k distinct parts into S is the coefficient of x^n in the counting series*

$$Z_{A_k}\Big(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \dots, \sum_{i\in S} x^{ki}\Big) - Z_{S_k}\Big(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \dots, \sum_{i\in S} x^{ki}\Big) \tag{10}$$

where A_k is the alternating subgroup of S_k, and $Z_{A_1}(x_1) - Z_{S_1}(x_1)$ is defined to be x_1.

(b) *The number $\mathrm{DP}(n;S)$ of partitions of n into S with distinct parts is the coefficient of x^n in the counting series*

$$\sum_{k=1}^{\infty}\bigg(Z_{A_k}\Big(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \dots, \sum_{i\in S} x^{ki}\Big) - Z_{S_k}\Big(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \dots, \sum_{x\in S} x^{ki}\Big)\bigg). \tag{11}$$

The proofs are similar to those of Theorem 1 using the cycle indices of A_k and S_k for one-to-one functions. (See p. 48 in [4].)

Similarly to (5), we have

$$\begin{aligned} 1 + \sum_{k=1}^{\infty} Z_{A_k}(f(x), f(x^2), \dots, f(x^k)) - Z_{S_k}(f(x), f(x^2), \dots, f(x^k)) \\ = \exp\Big(\sum_{k=1}^{\infty}(-1)^{k+1}\frac{f(x^k)}{k}\Big) \end{aligned} \tag{12}$$

where $f(x)$ is a function of x or a series of x. By using Theorem 2, (12) and (6), we obtain a recursive formula for $\mathrm{DP}(n;S)$ with $n > 1$. Clearly, $\mathrm{DP}(1;S) = 1$ if $1 \in S$, and $\mathrm{DP}(1;S) = 0$ if $1 \notin S$.

Corollary 2.1.

$$\mathrm{DP}(n;S) = \frac{1}{n}\bigg(\sum_{i|n,\, i\in S}(-1)^{n/i+1}\, i + \sum_{k=1}^{n-1}\Big(\sum_{i|k,\, i\in S}((-1)^{k/i+1}\, i)\mathrm{DP}(n-k;S)\Big)\bigg) \tag{13}$$

for $n > 1$.

By using Corollary 1.2 and Corollary 2.1, we can prove a well-known result which can be found in [1].

Corollary 2.2 (Euler). $\mathrm{DP}(n;I) = \mathrm{P}(n;O)$, *i.e., the number of partitions of* n *into distinct parts is equal to the number of partitions of* n *into odd parts.*

Proof. For $n = 1$, $\mathrm{DP}(1;I) = \mathrm{P}(I;O) = 1$. For $n > 1$, comparing the formulas for $\mathrm{P}(n;O)$ in Corollary 1.2 with the formula for $\mathrm{DP}(n;S)$ with $S = I$ in Corollary 2.1, we need only to prove that

$$\sum_{i|n,\, i\in I} (-1)^{(n/i)+1}\, i + \sum_{i|n,\, i\in O} i. \tag{14}$$

There are two cases to be considered.

Case 1. n is odd. $i \mid n$ implies i and n/i are odd, so $i \in O$ and $(n/i)+1$ is even. Thus, (14) holds.

Case 2. n is even. n can be written as $n = 2^t d$ where $t \geqslant 1$ and d is odd. Thus, a factor of n must have the form $2^j h$ where $0 \leqslant j \leqslant t$ and $h \mid d$, and

$$\sum_{i|n,\, i\in I} (-1)^{(n/i)+1}\, i = \sum_{h|d,\, 0\leqslant j\leqslant t} (-1)^{(2^t d/(2^j h))+1}\, 2^j h = \sum_{h|d} h\biggl(\sum_{j=0}^{t} (-1)^{2^{t-j}(d/h)+1}\, 2^j\biggr).$$

Since

$$\begin{aligned}\sum_{j=0}^{t} (-1)^{2^{t-j}(d/h)+1}\, 2^j &= -1 - 2 - 2^2 - 2^3 - \ldots - 2^{t-1} + 2^t\\ &= -(2^t - 1) + 2^t = 1,\end{aligned}$$

we have

$$\sum_{i|n,\, i\in I} (-1)^{(n/i)+1}\, i = \sum_{h|d} h = \sum_{h|n,\, h\in O} h = \sum_{i|n,\, i\in O} i.$$

Thus, (14) again holds. □

Example 5. Let D_3, S, S^{D_3}, S_3 and w be the same as in Example 2. We know that $A_3 = \{1, (123), (132)\}$ and $Z_{A_3}(x_1, x_2, x_3) = \frac{1}{3}(x_1^3 + 2x_3)$.

From Example 2 we have

$$\begin{aligned} Z_{S_3}\biggl(\sum_{i=1}^{\infty} x^i, \sum_{i=1}^{\infty} x^{2i}, \sum_{i=1}^{\infty} x^{3i}\biggr) &= x^3 + x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + \ldots\\ Z_{A_3}\biggl(\sum_{i=1}^{\infty} x^i, \sum_{i=1}^{\infty} x^{2i}, \sum_{i=1}^{\infty} x^{3i}\biggr) &= \frac{1}{3}\bigl((x^3 + 3x^4 + 6x^5 + 10x^6 + 15x^7 + 21x^8 + \ldots)\\ &\qquad + 2(x^3 + x^6 + x^9 + \ldots)\bigr)\\ &= x^3 + x^4 + 2x^5 + 4x^6 + 5x^7 + 7x^8 + \ldots \end{aligned}$$

By Theorem 2(a), the number of partitions of n with 3 distinct parts into the set of positive integers is the coefficient of x^n in the counting series

$$Z_{A_3}\left(\sum_{i=1}^{\infty} x^i, \sum_{i=1}^{\infty} x^{2i}, \sum_{i=1}^{\infty} x^{3i}\right) = Z_{S_3}\left(\sum_{i=1}^{\infty} x^i, \sum_{i=1}^{\infty} x^{2i}, \sum_{i=1}^{\infty} x^{3i}\right) = x^6 + x^7 + 2x^8 + \dots$$

For $n = 6$, $1 + 2 + 3$ is the only partition of 6 with 3 distinct parts.
For $n = 7$, $1 + 2 + 4$ is the only partition of 7 with 3 distinct parts.
For $n = 8$, $1+2+5$ and $1+3+4$ are the only partitions of 8 with 3 distinct parts.

Example 6. Let $D_k = \{1, 2, \dots, k\}$ and $S = \{1, 2, 4\}$. We want to compute $\mathrm{DP}(n; S)$. Since S contains only 3 positive integers, none of the functions from D_k, $k \geqslant 4$, into S can be one-to-one. Hence, to compute $\mathrm{DP}(n; S)$, we only have to compute the following with $Z_{A_1}(x_1) - Z_{S_1}(x_1)$ being defined to be x_1:

$$\begin{aligned}
&\sum_{k=1}^{3}\left(Z_{A_k}\left(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \dots, \sum_{i\in S} x^{ki}\right) - Z_{S_k}\left(\sum_{i\in S} x^i, \sum_{i\in S} x^{2i}, \dots, \sum_{i\in S} x^{ki}\right)\right)\\
&\quad = (x + x^2 + x^4) + (x + x^2 + x^4)^2 - \tfrac{1}{2}\big((x + x^2 + x^4)^2 + (x^2 + x^4 + x^8)\big)\\
&\qquad + \tfrac{1}{3}\big((x + x^2 + x^4)^3 + 2(x^3 + x^6 + x^{12})\big)\\
&\qquad - \tfrac{1}{6}\big((x + x^2 + x^4)^3 + 3(x + x^2 + x^4)(x^2 + x^4 + x^8) + 2(x^3 + x^6 + x^{12})\big)\\
&\quad = (x + x^2 + x^4) + (x^3 + x^5 + x^6) + x^7 = x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7.
\end{aligned}$$

Thus, the distinct partitions of n into $S = \{1, 2, 4\}$ are 1, 2, 4, $1+2$, $1+4$, $2+4$ and $1 + 2 + 4$.

5. Algorithms

(a) An algorithm for computing $\mathrm{P}(n; S)$ where n is a positive integer and S is a non-empty subset of positive integers.

Based on the recursive formula for $\mathrm{P}(n; S)$ from Corollary 1.1, an algorithm for computing $\mathrm{P}(n; S)$ can be given as follows:

Step 1: Determine $\mathrm{P}(1; S)$, $\mathrm{P}(1; S) = 1$ if $1 \in S$; otherwise, $\mathrm{P}(1; S) = 0$. This is the base case of the algorithm.

Step 2: Compute the sum of factors of a positive integer $k \leqslant n$. The factors should be in S. Use SumOfFactors (k, S) to denote the sum.

Step 3: Recursively compute $\mathrm{P}(n; S)$ using the formula. An implementation of the algorithm is described as follows:

Input n and S;
If ($n == 1$ and $1 \in S$) then
  P$(n; S) := 1$;
If ($n == 1$ and $1 \notin S$) then
  P$(n; S) := 0$;
Sum $:= 0$;
For $k := 1$ to $n - 1$ do
  Begin
  Sum := Sum + SumOfFactors$(k, S) *$ P$(n - k; S)$;
  End
Sum := Sum + SumOfFactors$(n; S)$;
P$(n; S) :=$ Sum$/n$;

Using the algorithm we can obtain P$(n; I)$ and P$(n; O)$ for all positive integers n where I is the set of all positive integers and O is the set of all positive odd integers. The following is a table of P$(n; I)$ and P$(n; O)$ for $n = 1, 2, \ldots, 20$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
P$(n; I)$	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	490	627
P$(n; O)$	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	38	46	54	64

(b) An algorithm for computing DP$(n; S)$ where n is a positive integer and S is a non-empty subset of positive integers.

Based on the recursive formula for DP$(n; S)$ from Corollary 2.1, an algorithm for computing DP$(n; S)$ can be given as follows:

Step 1: Determine DP$(1; S)$. DP$(1; S) = 1$ if $1 \in S$; otherwise, DP$(1; S) = 0$. This is the base case of the algorithm.

Step 2: Compute the sum of signed factors of a positive integer $k \leqslant n$. The factors should be in S. The sign of a factor i is $+$ (or $-$) if $k/i + 1$ is even (or odd). Use SumOfSignedFactors(k, S) to denote the sum.

Step 3: Recursively compute DP$(n; S)$ using the formula. An implementation of the algorithm is described as follows:

Input n and S;
If ($n == 1$ and $1 \in S$) then
  DP$(n; S) := 1$;
If ($n == 1$ and $\notin S$) then
  DP$(n; S) := 0$;
Sum$:= 0$;
For $k := 1$ to $n - 1$ do

Begin

Sum := Sum + SumOfSignedFactors(k, S) to denote the sum.

End

Sum := Sum + SumOfSignedFactors($n; S$);

DP($n; S$) := Sum/n;

Using the algorithm we can obtain DP($n; I$) and DP($n; O$) for all positive integers n where I is the set of all positive integers and O is the set of all positive odd integers. The following table shows DP($n; I$) and DP($n; O$) for $n = 1, 2, \ldots, 20$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
DP($n; I$)	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	38	46	54	64
DP($n; O$)	1	0	1	1	1	1	1	2	2	2	2	3	3	3	4	5	5	5	6	7

(c) Let E_i be the set of all positive integers each of which is not a multiple of the positive integer i for $i = 3, 4, 5, 6$. Using the first algorithm, we can obtain P($n; E_i$) for all positive integers n and for $i = 3, 4, 5, 6$. The following table gives (P($n; i$) for $n = 1, 2, \ldots, 20$ and $i = 3, 4, 5, 6$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
P($n; E_3$)	1	2	2	4	5	7	9	13	16	22	27	36	44	57	70	89	108	135	163	202
P($n; E_4$)	1	2	3	4	6	9	12	16	22	29	38	50	64	82	105	132	166	208	258	320
P($n; E_5$)	1	2	3	5	6	10	13	19	25	34	44	60	76	100	127	164	205	262	325	409
P($n; E_6$)	1	2	3	5	7	10	14	20	27	39	49	65	85	111	143	184	234	297	374	470

Using the second algorithm, we have the following table.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
DP($n; E_3$)	1	1	1	1	2	2	3	3	3	4	5	6	7	8	9	10	12	14	16	18
DP($n; E_4$)	1	1	2	1	2	3	3	4	5	6	7	8	9	11	13	16	18	21	24	27
DP($n; E_5$)	1	1	2	2	2	3	4	4	6	7	8	10	12	14	16	19	22	26	30	35
DP($n; E_6$)	1	1	2	2	3	3	4	5	6	8	9	11	13	16	19	22	26	30	35	41

(d) Using both algorithms, we have the following table for $S = \{1, 2, 4\}$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
P($n; S$)	1	2	2	4	4	6	6	9	9	12	12	16	16	20	20	25	25	30	30	36
DP($n; S$)	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0

(e) Let F be the set of all positive integers each of which is not a multiple of 3 or 4. Let G be the set of all positive integers each of which is not a multiple of 3, 4 or 5. Let H be the set of all positive integers each of which is not a multiple of 3, 4, 5 or 6. Using both algorithms, we have the following table:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\mathrm{P}(n;F)$	1	2	2	3	4	5	7	8	10	13	16	20	24	30	36	43	52	61	73	86
$\mathrm{DP}(n;F)$	1	1	1	0	1	1	2	2	1	2	2	3	4	4	4	4	5	6	7	7
$\mathrm{P}(n;G)$	1	2	2	3	3	4	5	6	7	8	10	11	14	17	20	23	27	31	36	41
$\mathrm{DP}(n;G)$	1	1	1	0	0	0	1	1	1	1	1	1	2	3	2	2	2	2	3	4
$\mathrm{P}(n;H)$	1	2	2	3	3	4	5	6	7	8	10	11	14	17	20	23	27	31	36	41
$\mathrm{DP}(n;H)$	1	1	1	0	0	0	1	1	1	1	1	1	2	3	2	2	2	2	3	4

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