

Jaroslav Ježek; M. Maróti; R. McKenzie  
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## QUASIEQUATIONAL THEORIES OF FLAT ALGEBRAS

J. JEŽEK, Praha, M. MARÓTI and R. MCKENZIE, Nashville

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*Abstract.* We prove that finite flat digraph algebras and, more generally, finite compatible flat algebras satisfying a certain condition are finitely  $q$ -based (possess a finite basis for their quasiequations). We also exhibit an example of a twelve-element compatible flat algebra that is not finitely  $q$ -based.

*Keywords:* quasiequation, flat algebra

*MSC 2000:* 08C15, 08B05

### 1. INTRODUCTION

For a finite directed graph  $(V, E)$  one can define an algebra with the underlying set  $V \cup E \cup \{0\}$ , one constant  $0$  and two binary operations  $\wedge, \cdot$  in this way:  $a \wedge a = a$  and  $a \wedge b = 0$  whenever  $a \neq b$ ;  $ab = c$  whenever  $a, c \in V$  and  $b = (a, c) \in E$ ;  $ab = 0$  in all other cases. Algebras obtained from finite directed graphs in this way are called finite *flat digraph algebras*. One particular six-element flat digraph algebra (inherently non-finitely based for equations) played a significant role in the proof of undecidability of the existence of a finite basis for the equational theory of a finite algebra ([2], [3] and [4]). It was plausible to expect that it could serve a similar purpose in an attempt to prove that also the existence of a finite basis for the quasiequations of a finite algebra is undecidable. However, in this paper we are going to show that all finite flat digraph algebras are finitely  $q$ -based (possess a finite basis for their uasiequations), which makes them unsuitable. We will investigate a more

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general class of finite compatible flat algebras, in which (under a modest assumption on the signature) every algebra can be embedded both into a finitely  $q$ -based and into a non-finitely  $q$ -based algebra.

For the terminology and basic concepts of universal algebra the reader is referred to the monograph [5]. For the literature on quasiequational theories see, e.g., [1] and [6].

## 2. COMPATIBLE 0-SEMILATTICE ALGEBRAS

Let  $\sigma$  be a finite signature containing (among other symbols) a binary symbol  $\wedge$  (the meet) and a nullary symbol  $0$ .

By a *0-semilattice algebra* we mean a  $\sigma$ -algebra satisfying the equations

- (1)  $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ ,
- (2)  $x \wedge y = y \wedge x$ ,
- (3)  $x \wedge x = x$ ,
- (4)  $f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) = 0$  for every  $n$ -ary operation  $f$  of  $\sigma$  and every  $i \in \{1, \dots, n\}$ .

A 0-semilattice algebra is said to be *compatible* if it satisfies the equations

- (5)  $f(z_1, \dots, z_{i-1}, x \wedge y, z_{i+1}, \dots, z_n) = f(z_1, \dots, z_{i-1}, x, z_{i+1}, \dots, z_n) \wedge f(z_1, \dots, z_{i-1}, y, z_{i+1}, \dots, z_n)$  for every  $n$ -ary operation  $f$  of  $\sigma$  and every  $i \in \{1, \dots, n\}$ .

So, the class of compatible 0-semilattice  $\sigma$ -algebras is a variety.

For a variable  $x$ , *basic  $x$ -terms* of depth  $n$  are defined as follows. The term  $x$  is the only basic  $x$ -term of depth 0. For  $n > 0$ , basic  $x$ -terms of depth  $n$  are the terms  $f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)$  such that  $f$  is an  $n$ -ary operation of  $\sigma$ ,  $1 \leq i \leq n$ ,  $t$  is a basic  $x$ -term of depth  $n - 1$  and  $x_1, \dots$  are variables different from  $x$ . A basic  $x$ -term  $t$  will be usually denoted by  $t(x)$ , in which case  $t(u)$  stands for the term resulting from  $t$  by substituting  $u$  for  $x$  (where  $u$  is any term).

For a  $\sigma$ -algebra  $B$  and a basic  $x$ -term  $t$  of depth  $n$ , any interpretation of the variables different from  $x$  by elements of  $B$  gives rise to a unary polynomial of  $B$ . The unary polynomials obtained in this way will be called the *basic polynomials* of  $B$  of depth  $n$ .

**Lemma 2.1.** *Let  $A$  be a compatible 0-semilattice algebra. Then  $p(a \wedge b) = p(a) \wedge p(b)$  for all basic polynomials  $p$  of  $A$  and all elements  $a, b \in A$ .*

*Proof.* It is easy. (Observe that the statement is not true for all unary polynomials  $p$ .) □

**Lemma 2.2.** *Let  $A$  be a compatible 0-semilattice algebra and  $F$  be a proper filter of  $A$  (i.e., a nonempty subset closed under meet, not containing 0 and such that  $b \in F$  whenever  $a \in F$  and  $a \leq b$ ). Then for every basic polynomial  $p$  of  $A$ ,  $p^{-1}(F)$  is either empty or a proper filter of  $A$ .*

*Proof.* It follows easily from Lemma 2.1. □

By a *flat* algebra we mean a 0-semilattice algebra  $A$  such that  $a \wedge b = 0$  for all pairs of distinct elements  $a, b \in A$ . Observe that a flat algebra is monotonic, i.e., satisfies  $x \leq y \rightarrow f(z_1, \dots, z_{i-1}, x, z_{i+1}, \dots, z_n) \leq f(z_1, \dots, z_{i-1}, y, z_{i+1}, \dots, z_n)$  for every  $n$ -ary operation  $f$  of  $\sigma$  and every  $i \in \{1, \dots, n\}$ .

One can easily see that a flat algebra is compatible if and only if

$$(5') \quad f(c_1, \dots, c_{i-1}, a, c_{i+1}, \dots, c_n) = f(c_1, \dots, c_{i-1}, b, c_{i+1}, \dots, c_n) \neq 0 \text{ implies } a = b$$

for every  $n$ -ary operation  $f$  of  $\sigma$  and every  $i \in \{1, \dots, n\}$ .

For every partial algebra  $G$  of a signature  $\tau$  not containing  $\wedge$  and 0 we can define a flat  $\tau \cup \{\wedge, 0\}$ -algebra  $A$ , called the *flat algebra over  $G$* , by  $A = G \cup \{0\}$ ,  $f(a_1, \dots, a_n) = a$  in  $A$  whenever  $f(a_1, \dots, a_n) = a$  in  $G$ , and  $f(a_1, \dots, a_n) = 0$  otherwise. This flat algebra is not necessarily compatible. For example, if  $G$  is a finite groupoid, then the flat algebra over  $G$  is compatible if and only if  $G$  is a quasigroup. Finite flat digraph algebras are all compatible.

**Observation 2.3.** *For every finite compatible flat algebra  $A$  there exists a first-order sentence  $\Phi$  such that the finite models of  $\Phi$  are precisely the finite algebras belonging to the quasivariety generated by  $A$ .*

*Proof.* Put  $K = |A|$ . It is easy to see that the following are equivalent for a finite compatible 0-semilattice algebra  $B$ :

- (e1)  $B$  belongs to the quasivariety generated by  $A$ ;
- (e2) every two elements  $b_0, b_1$  of  $B$  such that  $b_0 < b_1$  can be separated by a congruence of  $B$ , the factor by which is isomorphic to a subalgebra of  $A$ ;
- (e3) for every  $b_0, b_1 \in B$  with  $b_0 < b_1$  there exist elements  $c_1, \dots, c_r \in B$  for some  $r < K$  such that the principal filters  $F_1, \dots, F_r$  generated by  $c_1, \dots, c_r$  are pairwise disjoint,  $b_1 \in F_1$ ,  $b_0$  belongs to the complement  $O$  of  $F_1 \cup \dots \cup F_r$  in  $B$ , the equivalence  $R$  with blocks  $O, F_1, \dots, F_r$  is a congruence of  $B$  and the factor  $B/R$  is isomorphic to a subalgebra of  $A$ .

Clearly, the condition (e3) can be rewritten as a first-order sentence. □

### 3. THE QUASIVARIETY $Q'_A$

In the following let  $A$  be a finite compatible, flat algebra. Put  $K = |A|$ .

Denote by  $Q'_A$  the quasivariety determined by the equations (1)–(5) and the following quasiequations:

- (6)  $x_0 \leq x_1 \ \& \ t(x) \geq x_1 \ \& \ u(x) \geq x_1 \ \& \ t(y) \geq x_1 \ \& \ u(y) \wedge x_1 \leq x_0 \rightarrow x_0 = x_1$  for every pair of basic  $x$ -terms  $t, u$  of depth  $\leq K$ ;
- (7)  $x_0 \leq x_1 \ \& \ H_{t_1, \dots, t_K} \rightarrow x_0 = x_1$  for every  $K$ -tuple of basic  $x$ -terms  $t_1, \dots, t_K$  of depth  $\leq K$ , where  $H_{t_1, \dots, t_K}$  is the conjunction of the following equations:

$$\begin{aligned}
 t_i(x_i) &\geq x_1 \quad (i = 1, \dots, K), \\
 t_i(x_j) \wedge x_1 &\leq x_0 \quad (i, j = 1, \dots, K \text{ and } i \neq j).
 \end{aligned}$$

**Lemma 3.1.**  $Q'_A$  is a finitely  $q$ -based quasivariety containing  $A$ .

*Proof.* The set of quasiequations (6)–(7) is essentially finite, as it contains only finitely many quasiequations that differ by not only renaming their variables. Consequently,  $Q'_A$  is finitely  $q$ -based. It remains to prove that (6) and (7) are satisfied in  $A$ . Suppose that (6) fails in  $A$  by some interpretation  $v \mapsto v'$  of variables. Then  $x'_0 < x'_1$ , so that  $x'_0 = 0$ ; now  $t(x') \geq x'_1$  implies  $t(x') = x'_1$ . Similarly we get  $u(x') = x'_1$  and  $t(y') = x'_1$ . But  $A$  satisfies (5'), so  $t(x') = t(y') \neq 0$  implies  $x' = y'$ ; hence  $x'_1 = u(x') \wedge x'_1 = u(y') \wedge x'_1 = 0$ , a contradiction. Using the fact that  $A$  cannot contain  $K$  nonzero, pairwise distinct elements, one can similarly prove that  $A$  satisfies the quasiequations (7). □

**Lemma 3.2.** Let  $B \in Q'(A)$  and  $b_0, b_1 \in B$  be two elements such that  $b_1 \not\leq b_0$ ; let  $F$  be a maximal filter of  $B$  such that  $b_1 \in F$  and  $b_0 \notin F$ . For any two basic polynomials  $p, q$  of  $B$  of depth  $\leq K$ , the sets  $p^{-1}(F)$  and  $q^{-1}(F)$  are either disjoint or equal.

*Proof.* The two basic polynomials  $p$  and  $q$  correspond to two basic terms  $t$  and  $u$  of depth  $\leq K$ . Suppose that there exist elements  $x', y'$  such that  $p(x') \in F$ ,  $p(y') \in F$ ,  $q(x') \in F$  and  $q(y') \notin F$ . It follows from the maximality of  $F$  that there exists an element  $e \in F$  with  $q(y') \wedge e \leq b_0$ . Put  $x'_1 = p(x') \wedge p(y') \wedge q(x') \wedge e$ , so that  $x'_1 \in F$ . Put  $x'_0 = b_0 \wedge x'_1$ , so that  $x'_0 < x'_1$ . But the quasiequation (e6) interpreted by  $x \mapsto x', y \mapsto y', x_0 \mapsto x'_0, x_1 \mapsto x'_1$  gives  $x'_0 = x'_1$ , a contradiction. □

**Lemma 3.3.** Let  $B \in Q'(A)$  and  $b_0, b_1 \in B$  be two elements such that  $b_1 \notin b_0$ ; let  $F$  be a maximal filter of  $B$  such that  $b_1 \in F$  and  $b_0 \notin F$ . There are at most  $K - 1$  nonempty subsets of  $B$  that can be expressed as  $q^{-1}(F)$  for a basic polynomial  $q$  of  $B$ , and they can be arranged into a sequence  $F_1, \dots, F_r$  (for some  $r < K$ ) in such a way that  $F_1 = F$  and for every  $i \in \{2, \dots, r\}$  there are an index  $j \in \{1, \dots, i-1\}$  and a basic polynomial  $p_i$  of  $B$  of depth 1 with  $F_i = p_i^{-1}(F_j)$ . The collection  $F_1, \dots, F_r$ , together with the complement of their union, is a partition and the corresponding equivalence is a congruence of  $B$ .

*Proof.* Let us define a (finite or infinite) sequence  $F_1, p_1, F_2, p_2, \dots$  of filters  $F_i$  and basic polynomials  $p_i$  of depth  $\leq 1$  by induction in this way:  $F_1 = F$  and  $p_1$  is the identity on  $B$ ; if  $F_i, p_i$  have been defined and if there exist an element  $a \notin F_1 \cup \dots \cup F_i$  and a basic polynomial  $p$  of depth 1 such that  $p(a) \in F_j$  for some  $j \leq i$ , take one such pair  $a, p$  and put  $p_{i+1} = p$  and  $F_{i+1} = p_{i+1}^{-1}(F_j)$ ; if there is no such pair  $a, p$ , the sequence already constructed will have no continuation. Clearly (by induction on  $i$ ),  $F_i = q_i^{-1}(F)$  for a basic polynomial  $q_i$  of  $B$  of depth  $< i$ . The sets  $F_i$  are pairwise disjoint filters according to Lemmas 2.2 and 3.2.

Suppose that the sequence has at least  $K$  members  $F_1, \dots, F_K$ . For any  $i = 1, \dots, K$  take an element  $x'_i \in F_i$ , so that  $q_i(x'_i) \in F$ . For every  $i \neq j$  we have  $x'_j \notin F_i$ , i.e.,  $q_i(x'_j) \notin F$ , so that there exists an element  $e_{i,j} \in F$  with  $q_i(x'_j) \wedge e_{i,j} \leq b_0$ . There is an element  $x'_1 \in F$  such that  $x'_1 \leq q_i(x'_i)$  for all  $i$  and  $x'_1 \leq e_{i,j}$  for all  $i \neq j$ . Put  $x'_0 = b_0 \wedge x'_1$ , so that  $x'_0 < x'_1$ . But the quasiequation (e7), interpreted in the obvious way, says that  $x'_0 = x'_1$ , a contradiction.

So, the sequence  $F_1, p_1, \dots$  ends with  $F_r, p_r$  for some  $r \leq K - 1$ . Clearly, every subset of the form  $q^{-1}(F)$  for a basic polynomial  $q$  can be found among  $F_1, \dots, F_r$ . Put  $O = B - (F_1 \cup \dots \cup F_r)$ , so that  $0 \in O$  and  $F_1, \dots, F_r, O$  is a partition of  $B$ . It remains to prove that the corresponding equivalence is a congruence of  $B$ .

Suppose that there exist an  $n$ -ary operation  $f$  in  $\sigma$  and an  $n$ -tuple  $a_1, \dots, a_n$  of elements of  $B$  such that  $a_j \in O$  for some  $j$  but  $f(a_1, \dots, a_n) \in F_i$  for some  $i$ . Then  $p(a_j) \in F_i$  where  $p(x) = f(a_1, \dots, a_{j-1}, x, a_{j+1}, \dots, a_n)$  is a basic polynomial of depth 1 and  $a_j \notin F_1 \cup \dots \cup F_r$ , so that  $(q_i p)^{-1}(F)$  is nonempty and different from all  $F_1, \dots, F_r$ , a contradiction. We have proved that if at least one of the elements  $a_1, \dots, a_n$  belongs to  $O$ , then  $f(a_1, \dots, a_n) \in O$ .

Now it remains to show that if  $f$  is  $n$ -ary,  $f(a_1, \dots, a_n) \in F_j$  and  $a_i, a'_i \in F_k$  for some  $j, k \in \{1, \dots, r\}$  and  $i \in \{1, \dots, n\}$ , then  $f(a_1, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_n) \in F_j$ . Put  $q(x) = q_j(f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n))$ , so that  $q$  is a basic polynomial of  $B$  of depth at most  $K$ . We have  $q(a_i) \in F$  and  $q_k(a_i) \in F$ , so that  $q^{-1}(F) = q_k^{-1}(F)$ . Since  $a'_i$  belongs to this set, we get  $q(a'_i) \in F$ , i.e.,  $f(a_1, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_n) \in F_j$ .  $\square$

**Theorem 3.4.** *Let  $A$  be a finite compatible, flat algebra with  $K$  elements. Then  $Q'_A$  is a finitely  $q$ -based and locally finite quasivariety containing  $A$ ; every algebra in  $Q'_A$  is isomorphic to a subdirect product of algebras of cardinality at most  $K$ . Consequently,  $A$  is not inherently nonfinitely  $q$ -based.*

*Proof.* Let  $B \in Q'_A$ . For every pair  $b_0, b_1$  of distinct elements of  $B$  (we can assume that  $b_1 \not\leq b_0$ ) there exists a maximal filter of  $B$  containing  $b_1$  but not  $b_0$ , so that by Lemma 3.3 these two elements can be separated by a congruence with at most  $K$  blocks. It follows that every algebra from  $B$  is isomorphic to a subdirect product of algebras of cardinality at most  $K$ . Thus  $Q'_A$  is contained in a finitely generated variety and hence it is locally finite. According to Lemma 3.1,  $Q'_A$  is finitely  $q$ -based and contains  $A$ .  $\square$

#### 4. FINITELY $q$ -BASED COMPATIBLE FLAT ALGEBRAS

Let  $A$  be a finite compatible flat algebra. By a *segment* of  $A$  we will mean a nonempty subset of  $A$ , the elements of which can be arranged into a finite sequence  $0, c_1, \dots, c_r$  in such a way that  $c_1 \neq 0$  and for every  $i = 2, \dots, r$  there exists a basic polynomial  $p$  of  $A$  of depth 1 with  $p(c_i) = c_j$  for some  $j \in \{1, \dots, i - 1\}$ .

Let  $S$  be a segment of  $A$ . The algebra obtained from  $S$ , considered as a partial subalgebra of  $A$ , by setting all the undefined operations to 0 will be called the *0-completion* of  $S$ .

Let  $S$  be a segment of  $A$  and  $S'$  be the subalgebra of  $A$  generated by  $S$ . The segment  $S$  is said to be *regular* if the equivalence on  $S'$  with the only non-singleton block  $\{0\} \cup (S' - S)$  is a congruence of  $S'$ . In that case, the factor of  $S'$  by this congruence is isomorphic to the 0-completion of  $S$ .

**Theorem 4.1.** *Let  $A$  be a finite compatible flat algebra such that the 0-completion of every regular segment of  $A$  belongs to the quasivariety generated by  $A$ . Then  $A$  is finitely  $q$ -based.*

*Proof.* Denote by  $Q''_A$  the subquasivariety of  $Q'_A$  determined by the quasi-equations (1)–(7) and all quasiequations in at most  $K$  variables that are satisfied in  $A$ . (Here  $K = |A|$ .) Since  $Q'_A$  is locally finite by Theorem 3.4,  $Q''_A$  is locally finite. Since only finitely many equations are needed to reduce the terms in at most  $K$  variables to a finite set  $T_0$  of such terms, and then quasiequations in at most  $K$  variables correspond to subsets of  $T_0^2$  with distinguished elements,  $Q''_A$  is finitely  $q$ -based. Of course,  $A \in Q''_A$ . We are going to prove that  $Q''_A$  is the quasivariety generated by  $A$ . It is sufficient to show that every finite algebra from  $Q''_A$  belongs to the quasivariety generated by  $A$ .

Let  $B$  be a finite algebra from  $Q''_A$ ; let  $b_0, b_1 \in B$  be such that  $b_1 \not\leq b_0$ . By 3.3 there is a congruence with at most  $K$  blocks  $O, F_1, \dots, F_r$ , yielding a quotient algebra  $C$ , such that  $F_1, \dots, F_r$  are filters (now they are principal filters),  $F_1 = F$ ,  $b_1 \in F_1$ ,  $b_0 \in O$ , and for every  $i \in \{2, \dots, r\}$  there exist an index  $j < i$  and a basic polynomial  $p_i$  of length 1 with  $F_i = p_i^{-1}(F_j)$ . But all the coefficients occurring in  $p_i$  belong to  $F_1 \cup \dots \cup F_r$ , so there exists a basic  $x$ -term  $u_i(x, x_1, \dots, x_r)$  of depth 1 such that  $u_i(F_i, F_1, \dots, F_r) \subseteq F_j$ . Now we can combine these terms  $u_i$  together to obtain, for each  $i$ , a basic  $x$ -term  $t_i(x, x_1, \dots, x_r)$  such that  $t_i(F_i, F_1, \dots, F_r) \subseteq F$ , i.e.,  $t_i^C(F_i, F_1, \dots, F_r) = F_1$ . (We take  $t_1 = x$ .) For any term  $u$  denote by  $t_i(u)$  the term obtained from  $t_i$  by replacing the only occurrence of  $x$  with  $u$ . Now consider the quasiequation

$$x_0 \leq x_1 \ \& \ D \rightarrow x_0 = x_1$$

where  $D$  is the conjunction of all these equations:

- (i)  $t_i(x_i) \geq x_1$ , for any  $i = 1, \dots, r$ ;
- (ii)  $t_i(x_j) \wedge x_1 \leq x_0$ , for any  $i, j \in \{1, \dots, r\}$  with  $i \neq j$ ;
- (iii)  $t_i(f(x_{i_1}, \dots, x_{i_n})) \geq x_1$ , for any  $n$ -ary operation  $f$  of  $\sigma$  and any  $i, i_1, \dots, i_n$  with  $f^C(F_{i_1}, \dots, F_{i_n}) = F_i$ ;
- (iv)  $t_i(u) \wedge x_1 \leq x_0$ , for any  $i = 1, \dots, r$  and any term  $u$  in variables  $x_1, \dots, x_r$  containing a subterm  $f(x_{i_1}, \dots, x_{i_n})$  with  $f^C(F_{i_1}, \dots, F_{i_n}) = O$  (it is possible to consider only finitely many such terms  $u$ ).

Clearly, this quasiequation fails in  $B$ ; since it is a quasiequation in at most  $K$  variables  $x_0, \dots, x_r$ , it must fail in  $A$  by some elements  $a_0, a_1, \dots, a_r$ . But then the subset  $\{a_0, a_1, \dots, a_r\}$  is a regular segment of  $A$ , and the 0-completion of this subset is isomorphic to  $C$ . Since  $C$  belongs to the quasivariety generated by  $A$ , the elements  $b_0, b_1$  were separated by a congruence, the factor by which belongs to the quasivariety. □

**Corollary 4.2.** *Every finite flat digraph algebra is finitely  $q$ -based.*

*Proof.* In this case, all segments are subalgebras. □

**Corollary 4.3.** *The flat algebra over any finite quasigroup (considered as a groupoid) is finitely  $q$ -based.*

*Proof.* In this case, all regular segments are subalgebras. □

**Corollary 4.4.** *If  $\sigma$  is the signature containing only one unary symbol in addition to  $\wedge$  and  $0$ , then every finite compatible flat  $\sigma$ -algebra is finitely  $q$ -based.*

*Proof.* In this case, the 0-completion of every segment is isomorphic to a subalgebra. □



5. THE EMBEDDING THEOREM

**Theorem 5.1.** *Let  $\sigma$  be a finite signature containing, in addition to  $\wedge$  and  $0$ , at least two unary symbols  $f$  and  $g$  (and, possibly, some other operation symbols). Then every finite compatible flat  $\sigma$ -algebra can be embedded into two finite compatible flat  $\sigma$ -algebras, one finitely  $q$ -based and the other one not finitely  $q$ -based.*

*Proof.* Let  $G$  be a finite compatible flat algebra.

Denote by  $S_1, \dots, S_r$  all the segments of  $G$ . (It would be sufficient to take just those with the  $0$ -completions not belonging to the quasivariety generated by  $G$ .) For every  $i = 1, \dots, r$  let us take an isomorphic copy  $T_i$  of the partial algebra  $S_i - \{0\}$ , in such a way that the sets  $G, T_1, \dots, T_r$  are pairwise disjoint. Denote by  $G'$  the flat algebra with the underlying set  $G \cup T_1 \cup \dots \cup T_r$ , with the operations evaluated to  $0$  in all cases except when needed to define them in such a way that  $G$  is a subalgebra and  $T_i$  are partial subalgebras. It follows from Theorem 4.1 that  $G'$  is finitely  $q$ -based.

Next we are going to construct a non-finitely  $q$ -based extension of  $G$ . Let us take one fixed positive integer  $k$  such that  $k \geq 2$  and there is no sequence  $u_0, u_1, \dots, u_k$  of pairwise distinct elements of  $G - \{0\}$  such that  $g(u_{i-1}) = u_i$  for  $i = 1, \dots, k$ . Denote by  $A$  the flat algebra, with  $G$  as a subalgebra, containing  $k + 10$  additional elements  $u_0, u_1, \dots, u_k, a, b, c, v_2, a_2, b_2, v_3, a_3, c_3$  with all operations not inside  $G$  evaluated to  $0$  except for

$$\begin{aligned}
 &g(u_{i-1}) = u_i \quad \text{for } i = 1, \dots, k, \\
 &f(u_0) = a, \quad f(a) = b, \quad g(a) = c, \\
 &f(v_2) = a_2, \quad f(a_2) = b_2, \quad f(v_3) = a_3, \quad g(a_3) = b_3.
 \end{aligned}$$

(Fig. 1, in which the elements not belonging to  $G$  are pictured for  $k = 2$ , may help to understand this definition.)

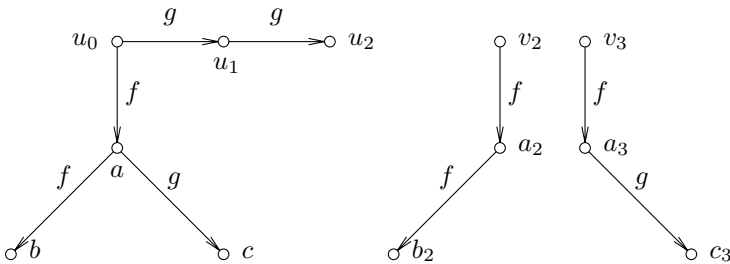


Fig. 1

Denote by  $Q$  the quasivariety generated by  $A$ . A  $\sigma$ -algebra  $B$  belongs to  $Q$  if and only if every two distinct elements of  $B$  can be separated by a homomorphism of  $B$  into  $A$ .

For every positive integer  $n$  let  $A_n$  be the  $\sigma$ -algebra with elements  $0, u_0, \dots, u_k, \alpha_{i,j}, \beta_i, \gamma_j$  ( $0 \leq i \leq n, 0 \leq j \leq n-1, i-1 \leq j \leq i$ ) and with operations defined in this way:  $A_n$  is a semilattice with the only comparabilities  $0 < u_i$  ( $i = 0, \dots, k$ ),  $0 < \beta_n < \beta_{n-1} < \dots < \beta_0$ ,  $0 < \gamma_{n-1} < \gamma_{n-2} < \dots < \gamma_0$ ,  $0 < \alpha_{n,n-1} < \alpha_{n-1,n-1} < \alpha_{n-1,n-2} < \dots < \alpha_{1,0} < \alpha_{0,0}$ ; the other operations evaluate to 0 except that  $g(u_{i-1}) = u_i$  ( $i = 1, \dots, k$ ),  $f(u_0) = \alpha_{0,0}$ ,  $f(\alpha_{i,j}) = \beta_i$ ,  $g(\alpha_{i,j}) = \gamma_j$ . (Fig. 2, in which the situation is illustrated for  $k = 2$  and  $n = 3$ , may help to understand this definition. In the picture lines with arrows indicate unary operations, while the other lines represent coverings but the covers between 0 and the elements  $u_i$  are not indicated.)

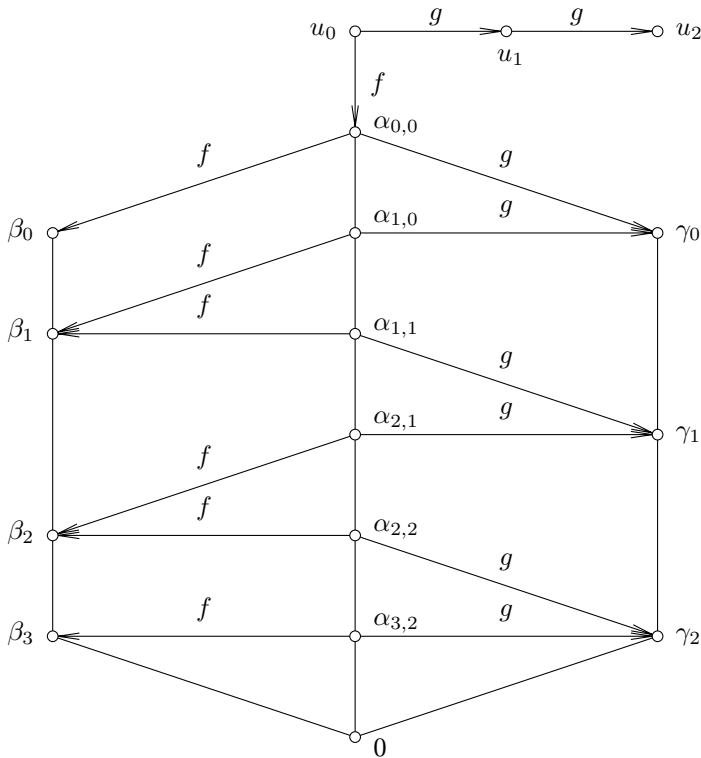


Fig. 2

Denote by  $r_n$  the equivalence on  $A_n$  with the only non-singleton block  $\{0, \beta_n\}$ . Clearly,  $r_n$  is a congruence of  $A_n$ . Denote the factor  $A_n/r_n$  by  $B_n$ . For  $a \in A_n - \{0, \beta_n\}$ , the element  $a/r_n$  will be identified with  $a$ .

Suppose that there exists a homomorphism  $H: B_n \rightarrow A$  such that  $H(u_k) \neq H(0/r_n)$ , i.e.,  $H(u_k) \neq 0$ . Since  $g^k(u_0) = u_k$  in  $B_n$  and there is no other element  $e$

in  $A$  with  $g^k(e) \neq 0$  and  $g^{k+1}(e) = 0$  other than  $u_0$ , we get  $H(u_0) = u_0$  and then  $H(u_i) = H(g^i(u_0)) = g^i(H(u_0)) = g^i(u_0) = u_i$  for all  $i$ . Now  $H(\alpha_{0,0}) = H(f(u_0)) = f(H(u_0)) = f(u_0) = a$ . Consequently,  $H(\beta_0) = b$  and  $H(\gamma_0) = c$ . Since  $g(\alpha_{1,0}) = \gamma_0$  and  $a$  is the only element of  $A$  with  $g(a) = c$ , it follows that  $H(\alpha_{1,0}) = a$ . If  $H(\alpha_{i,i-1}) = a$  for some  $i < n$ , then using  $f$  in a similar way we can show that  $H(\alpha_{i,i}) = a$ , and then using  $g$  to show that  $H(\alpha_{i+1,i}) = a$ . By induction we get  $H(\alpha_{n,n-1}) = a$ . But then  $H(0/r_n) = H(\beta_n/r_n) = H(f(\alpha_{n,n-1})) = f(a) = b$ , a contradiction.

Since the element  $u_k$  cannot be separated from  $0/r_n$  by a homomorphism of  $B_n$  into  $A$ , we conclude that  $B_n$  does not belong to  $Q$ .

Let  $\alpha_{m,m'}$  be an element of  $B_n$  such that  $0 < m < n$ . Clearly, the set  $C = B_n - \{\alpha_{m,m'}\}$  is a subalgebra of  $B_n$ . We are going to prove that  $C$  belongs to  $Q$ . For this purpose, it is sufficient to show that whenever  $e, e'$  are two elements of  $C$  such that  $e$  is covered by  $e'$ , then  $e, e'$  can be separated by a homomorphism of  $C$  into  $A$ .

For every  $i \leq n - 1$  define a mapping  $\psi_i$  of  $B_n$  into  $A$  by  $\psi_i(u_0) = v_2, \psi_i(e) = a_2$  for  $e \geq \alpha_{i,i}, \psi_i(e) = b_2$  for  $e \geq \beta_i$  and  $\psi_i(e) = 0$  for all other elements  $e$ . Also, for every  $i \leq n - 1$  define a mapping  $\chi_i$  of  $B_n$  into  $A$  by  $\chi_i(u_0) = v_3, \chi_i(e) = a_3$  for  $e \geq \alpha_{i+1,i}, \chi_i(e) = c_3$  for  $e \geq \gamma_i$  and  $\chi_i(e) = 0$  for all other elements  $e$ . It is easy to check that both  $\psi_i$  and  $\chi_i$  are homomorphisms. Consequently, their restrictions to  $C$  are homomorphisms of  $C$  into  $A$ . The only pairs of covers not separated by any of these homomorphisms are the pairs  $(0, u_1), \dots, (0, u_k)$ . So, it remains to separate these pairs of elements.

If  $m = m'$ , then these pairs are separated by the homomorphism  $\varphi$  defined in this way:  $\varphi(u_0) = u_0, \dots, \varphi(u_k) = u_k, \varphi(e) = a$  for  $e \geq \alpha_{m,m-1}, \varphi(e) = b$  for  $e \geq \beta_m, \varphi(e) = c$  for  $e \geq \gamma_{m-1}$  and  $\varphi(e) = 0$  for all other elements  $e$ . If  $m' = m - 1$ , then they are separated by the homomorphism  $\varphi'$  defined in this way:  $\varphi'(u_0) = u_0, \dots, \varphi'(u_k) = u_k, \varphi'(e) = a$  for  $e \geq \alpha_{m',m'}, \varphi'(e) = b$  for  $e \geq \beta_{m'}, \varphi'(e) = c$  for  $e \geq \gamma_{m'}$  and  $\varphi'(e) = 0$  for all other elements  $e$ .

We have proved that  $C$  belongs to  $Q$ . Since every subalgebra of  $B_n$  generated by at most  $n - k$  elements is contained in at least one such  $C$ , it follows that every subalgebra generated by at most  $n - k$  elements belongs to  $Q$ . Consequently, there is no base for the quasiequations of  $Q$  that would contain only quasiequations in at most  $n - k$  variables. Since  $k$  was fixed while  $n$  was arbitrary, there is no finite base at all.  $\square$

**Remark 5.2.** In the above construction of the algebra  $A$  it was not essential that the elements  $b_2$  and  $c_3$  are distinct.

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*Authors’ addresses:* J. Ježek, MFF UK, Sokolovská 83, 186 00 Praha 8, Czech Republic, e-mail: jezek@karlin.mff.cuni.cz; M. Maróti and R. McKenzie, Department of Mathematics, Vanderbilt University, 1326 Stevenson Center, Nashville, TN 37240, e-mails: miklos.maroti@vanderbilt.edu, ralph.n.mckenzie@vanderbilt.edu.