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$w^*$-basic sequences and reflexivity of Banach spaces


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Abstract. We observe that a separable Banach space $X$ is reflexive iff each of its quotients with Schauder basis is reflexive. Similarly if $\mathcal{L}(X, Y)$ is not reflexive for reflexive $X$ and $Y$ then $\mathcal{L}(X_1, Y)$ is is not reflexive for some $X_1 \subset X$, $X_1$ having a basis.

Keywords: reflexive Banach space, Schauder basis, quotient space, $w^*$-basic sequence, tensor product

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Pelczyński [10] proved that Banach space $X$ is reflexive if each subspace with Schauder basis is reflexive. Actually this result stems from the work of [13] which in turn was inspired by the work of [11]. Here we add simple statements which may be considered as natural complements to the results of [11], [13] and [10]. The first one is a statement similar to that of Pelczyński for separable $X$ and quotients instead of subspaces. Namely we observe that a separable Banach space $X$ is reflexive if each of its quotients with Schauder basis is reflexive. From [7] we know that duals of quotient spaces with basis correspond to subspaces of the dual $X^*$ spanned by $w^*$-basic sequences. Thus our statement reads: A separable Banach space is reflexive if every $w^*$-basic sequence in $X^*$ spans a reflexive subspace. We may proceed similarly as in [10] but we use the tools of $w^*$-basic sequences which were not at hand for the authors of [11], [13] and [10]. Similarly we will consider also reflexivity of spaces of bounded operators or equivalently of $\pi$-tensor products of reflexive Banach spaces.
Following [7] we will denote by \(|A|\) the norm closed linear span of a set \(A\) and by \(\tilde{A}\) its \(w^*\) closed linear span if \(A \subset X^*\). By \(A_0\) we denote the polar set in \(X\) of a set \(A \subset X^*\). By a space having a basis we mean a Banach space with a Schauder basis.

A sequence \(\{x_n^*\}\) is called \(w^*\) basic [7], [8], [2] or [3] provided that there is a sequence \(\{x_n\} \subset X\) so that \(\{x_n, x_n^*\}\) is biorthogonal and for each \(x^* \in [\tilde{x}_n^*]\) we have

\[
\sum_{i=1}^{n} x^*(x_i)x_i^* \xrightarrow{w^*} x^*.
\]

From [7] we shall use the following two facts:

A) If \(\{x_n^*\}\) is \(w^*\) basic sequence then the factor space \(X/[x_n^*]_0\) has a basis and \([\tilde{x}_n^*]\) can be identified with \((X/[x_n^*]_0)^*\).

B) If \(X\) is separable then every \(w^*\) null sequence \(\{x_n^*\} \subset X^*\) which is not norm null has a \(w^*\) basic subsequence \(\{x_{nk}\}\).

Finally we recall two results of Holub and Heinrich [4], [5] (and slightly more restrictive [12]) on the reflexivity of the space \(\mathcal{L}(X,Y)\):

C) The space of bounded linear operators \(\mathcal{L}(X,Y)\) is reflexive if \(\mathcal{L}(X,Y) = \mathcal{H}(X,Y)\) and if \(X\) and \(Y\) are reflexive Banach spaces.

Conversely,

D) If \(\mathcal{L}(X,Y)\) is reflexive and if \(X\) or \(Y\) has the approximation property then \(\mathcal{L}(X,Y) = \mathcal{H}(X,Y)\). Of course \(X\) and \(Y\) are then reflexive spaces.

The statement C) was proved under more restrictive assumptions by Ruckle [12] and in the approximation property free form by [4], [5]. This approximation property free form seems not to be generally known as e.g. the recent paper [9] shows.

**Proposition 1.** Let \(X\) be a separable Banach space. Then \(X\) is reflexive iff each of its quotients which has a basis is reflexive.

**Proof.** Only the if part of the proposition is to be established. Thus we shall suppose that \(X^*\) is not reflexive i.e. that the closed unit ball \(B_{X^*}\) is not weakly compact. The Eberlein-Šmulian theorem yields a sequence \(\{x_n^*\}\) in the unit ball \(B_{X^*}\) no subsequence of which is weakly converging. Due to the separability of \(X\) the closed unit ball \(B_{X^*}\) is metrizable in the \(w^*\) topology and thus the sequence \(\{x_n^*\} \subset B_{X^*}\) has a \(w^*\) converging subsequence. For simplicity we will denote this subsequence by \(\{x_n^*\}\) again. We may suppose that \(x_n^* \xrightarrow{w^*} 0\) (otherwise we take \(x_n^* - w^* \lim x_n^*\)). By our assumptions the sequence \(\{x_n^*\}\) is not norm converging (to zero). The above mentioned result B) of [7] yields a \(w^*\) basic subsequence which we shall call \(\{x_n^*\}\) again. Having in mind the identification mentioned in A) we see that \(\{x_n^*\}\) is in the unit ball of \((X/[x_n^*]_0)^* = [\tilde{x}_n^*]\). Because \(\{x_n^*\}\) has no weakly convergent subsequence we conclude that the dual unit ball of \(X/[x_n^*]_0\) is not weakly compact and thus \(X/[x_n^*]_0\) is not reflexive. From A) we also know that \(X/[x_n^*]_0\) has a basis. \(\square\)

**Remark 1.** Note that actually we have proved slightly more, namely:
Let $X$ be a separable Banach space. Then $X$ is reflexive iff every $w^*$ basic sequence $\{x^*_n\}$ spans a normed closed reflexive subspace $[x^*_n] \subset X^*$.

**Remark 2.** We do not know if Proposition 1 holds also without the separability assumption. This general statement would then imply (similarly as also the statement mentioned in A) and B) does) a positive answer to the following question which is still not settled: Has every Banach space a separable quotient space?

Similarly we may consider quotients of the space $X$ by subspaces $A \subset X$ such that $A$ has a basis and the quotient space $X/A$ is not reflexive:

**Proposition 2.** Let $X$ be a nonreflexive Banach space. Then there is a subspace $A \subset X$ such that $A$ has a basis and the quotient space $X/A$ is not reflexive.

**Proof.** is contained in the proof of Lemma 2 in [1] and for the sake of completeness we will list it here: Suppose that $X$ is not reflexive. From the results of Singer [13] and from the above cited result of Pełczyński we conclude that there is a basic sequence $\{x_n\} \subset X$ with $\|x_n\| \geq 1$ such that $\left\{\frac{1}{p} \sum_{i=1}^{p} x_n\right\}_p$ is bounded. We put $A = [x_{2n-1}]$ and let $P$ be the quotient map of $X$ onto $X/A$. Then evidently $\{x_{2n-1}\}$ and $\{P(x_{2n})\}$ are basic sequences, $\{P(x_{2n})\}$ is not a norm null sequence and $\left\{\frac{1}{p} \sum_{i=1}^{p} P(x_{2n})\right\}_p = \left\{\frac{1}{2p} \sum_{i=1}^{2p} P(x_n)\right\}_p$ is bounded (in $p$). We conclude [13] that the sequence $\{P(x_{2n})\}$ spans a non reflexive subspace of $X/A$. \hfill $\Box$

Next we will consider the reflexivity of the space of bounded operators $\mathcal{L}(X,Y)$:

**Proposition 3.** Let $X, Y$ be reflexive Banach spaces and suppose that $\mathcal{L}(X,Y)$ is not reflexive. Then there is a subspace $X_1 \subset X$ such that $X_1$ has Schauder basis and such that $\mathcal{L}(X_1,Y)$ is not reflexive.

**Proof.** Suppose that $\mathcal{L}(X,Y)$ is not reflexive. The result C) mentioned in the introduction yields a noncompact operator $f \in \mathcal{L}(X,Y)$. Let $\{x_n\}$ be a bounded sequence in $\mathcal{L}(X,Y)$ such that $\{f(x_n)\}$ has no norm convergent subsequence. Then $\{x_n\}$ also has no norm convergent subsequence. The reflexivity of the space $X$ implies that there is a subsequence of the sequence $\{x_n\}$ weakly converging to $x \in X$. Let us denote this subsequence again by $\{x_n\}$ and put $z_n = x_n - x$. Then $z_n \overset{w}{\to} 0$. The classical theorem of Pełczyński mentioned in the introduction yields a basic subsequence of the sequence $\{z_n\}$. As above we call this subsequence again $\{z_n\}$ and put $X_1 = [\{z_n\} \cup \{x\}]$. Then $X_1$ has a basis. Indeed, if $x \in [z_n]$ then $[x_n] = [z_n]$ and $\{z_n\}$ is a basis of $X_1$. If $x \notin [z_n]$ then $X_1$ is the direct sum of $[z_n]$ and the one dimensional subspace spanned by $x$ and thus $X_1$ again has a basis. In any case $\{x_n\} \subset X_1$. This last inclusion evidently implies that the restriction
f|X_1 ∈ \mathcal{L}(X_1, Y) is not a compact operator. We note that X_1 has the approximation property. Again by the result D) of [4] and [5] mentioned in the introduction we conclude that \mathcal{L}(X_1, Y) is not reflexive.

Remark 3. Note that we have actually observed the following:

Let X, Y be any Banach spaces and suppose that there is noncompact operator f: X → Y. Then there is a subspace X_1 ⊂ X such that X_1 has Schauder basis and such that the restriction f|X_1 is a noncompact operator.

Remark 4. Dually Proposition 3 can be formulated as follows:

Let X, Y be reflexive Banach spaces and suppose that \mathcal{L}(X, Y) is not reflexive. Then there is a subspace Y_1 ⊂ Y such that the quotient space Y/Y_1 has Schauder basis and such that \mathcal{L}(X, Y/Y_1) is not reflexive.

Indeed, if there is noncompact operator f: X → Y then f* ∈ \mathcal{L}(Y^*, X^*) is also noncompact and thus \mathcal{L}(Y^*, X^*) is nonreflexive. Using now Proposition 3 for \mathcal{L}(Y^*, X^*) we get a subspace Z ⊂ Y* having a basis such that \mathcal{L}(Z, X^*) is not reflexive. We put now Y_1 = Z_o. Evidently Y/Y_1 = Z* has a basis. Proceeding as above and using now the duality of subspaces and quotients we get our claim.

Remark 5. A slightly more general result then stated in the above remark may also be formulated:

Let X, Y be Banach spaces, let Y be separable and suppose that \mathcal{L}(X, Y) ≠ \mathcal{K}(X, Y). Then there is a subspace Y_1 ⊂ Y such that the factor space Y/Y_1 has Schauder basis and such that \mathcal{L}(X, Y/Y_1) ≠ \mathcal{K}(X, Y/Y_1).

Indeed, let f: X → Y be a noncompact operator. Proceeding as in the proofs of Propositions 3 and 1 we find a w* basic sequence \{y_n^*\} ⊂ Y* such that the restriction f*|_[y_n] is noncompact. Let now Y_1 = \{y_n^*\}_o and let P be the projection of Y onto Y/Y_1. Then evidently Pf: X → Y/Y_1 is a noncompact operator whose dual is f*|_[y_n^*].

Remark 6. Having in mind the basic relation of \mathcal{L}(X, Y) to tensor products, namely (X\hat{⊗}_q Y)^* = \mathcal{L}(X, Y) we can reformulate Proposition 3:

Let X, Y be reflexive Banach spaces and suppose that X\hat{⊗}_q Y is not reflexive. Then there is a subspace X_1 ⊂ X such that X_1 has Schauder basis and such that X_1\hat{⊗}_q Y is not reflexive.

Question. The statement listed in Remark 5 suggests the following question:

Suppose that there is a noncompact operator f: X → Y. Does there exist a noncompact operator g: X → Y_1 and a subspace Y_1 ⊂ Y, Y_1 having a basis?

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References


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