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\( w^*- \)basic sequences and reflexivity of Banach spaces


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Abstract. We observe that a separable Banach space $X$ is reflexive iff each of its quotients with Schauder basis is reflexive. Similarly if $\mathcal{L}(X, Y)$ is not reflexive for reflexive $X$ and $Y$ then $\mathcal{L}(X_1, Y)$ is not reflexive for some $X_1 \subset X$, $X_1$ having a basis.

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Pełczyński [10] proved that Banach space $X$ is reflexive if each subspace with Schauder basis is reflexive. Actually this result stems from the work of [13] which in turn was inspired by the work of [11]. Here we add simple statements which may be considered as natural complements to the results of [11], [13] and [10]. The first one is a statement similar to that of Pełczyński for separable $X$ and quotients instead of subspaces. Namely we observe that a separable Banach space $X$ is reflexive if each of its quotients with Schauder basis is reflexive. From [7] we know that duals of quotient spaces with basis correspond to subspaces of the dual $X^*$ spanned by $w^*$-basic sequences. Thus our statement reads: A separable Banach space is reflexive if every $w^*$-basic sequence in $X^*$ spans a reflexive subspace. We may proceed similarly as in [10] but we use the tools of $w^*$-basic sequences which were not at hand for the authors of [11], [13] and [10]. Similarly we will consider also reflexivity of spaces of bounded operators or equivalently of $\pi$-tensor products of reflexive Banach spaces.

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Following [7] we will denote by $[A]$ the norm closed linear span of a set $A$ and by $\widetilde{A}$ its $w^*$ closed linear span if $A \subset X^*$. By $A_0$ we denote the polar set in $X$ of a set $A \subset X^*$. By a space having a basis we mean a Banach space with a Schauder basis.

A sequence $\{x_n^*\}$ is called $w^*$ basic [7], [8], [2] or [3] provided that there is a sequence $\{x_n\} \subset X$ so that $\{x_n, x_n^*\}$ is biorthogonal and for each $x^* \in [x_n^*]$ we have

$$\sum_{i=1}^{n} x^*(x_i)x_i^* \xrightarrow{w} x^*.$$  

From [7] we shall use the following two facts:

A) If $\{x_n^*\}$ is $w^*$ basic sequence then the factor space $X/[x_n^*]_0$ has a basis and $[\widetilde{x_n^*}]$ can be identified with $(X/[x_n^*]_0)^*$. 

B) If $X$ is separable then every $w^*$ null sequence $\{x_n^*\} \subset X^*$ which is not norm null has a $w^*$ basic subsequence $\{x_{n_k}\}$. 

Finally we recall two results of Holub and Heinrich [4], [5] (and slightly more restrictive [12]) on the reflexivity of the space $L(X, Y)$:

C) The space of bounded linear operators $L(X, Y)$ is reflexive if $L(X, Y) = \mathcal{K}(X, Y)$ and if $X$ and $Y$ are reflexive Banach spaces. Conversely,

D) If $L(X, Y)$ is reflexive and if $X$ or $Y$ has the approximation property then $L(X, Y) = \mathcal{K}(X, Y)$. Of course $X$ and $Y$ are then reflexive spaces.

The statement C) was proved under more restrictive assumptions by Ruckle [12] and in the approximation property free form by [4], [5]. This approximation property free form seems not to be generally known as e.g. the recent paper [9] shows.

**Proposition 1.** Let $X$ be a separable Banach space. Then $X$ is reflexive iff each of its quotients which has a basis is reflexive.

**Proof.** Only the if part of the proposition is to be established. Thus we shall suppose that $X^*$ is not reflexive i.e. that the closed unit ball $B_{X^*}$ is not weakly compact. The Eberlein-Šmulian theorem yields a sequence $\{x_n^*\}$ in the unit ball $B_{X^*}$ no subsequence of which is weakly converging. Due to the separability of $X$ the closed unit ball $B_{X^*}$ is metrizable in the $w^*$ topology and thus the sequence $\{x_n^*\} \subset B_{X^*}$ has a $w^*$ converging subsequence. For simplicity we will denote this subsequence by $\{x_n^*\}$ again. We may suppose that $x_n^* \xrightarrow{w^*} 0$ (otherwise we take $x_n^* - w^* \lim x_n^*$). By our assumptions the sequence $\{x_n^*\}$ is not norm converging (to zero). The above mentioned result B) of [7] yields a $w^*$ basic subsequence which we shall call $\{x_n^*\}$ again. Having in mind the identification mentioned in A) we see that $\{x_n^*\}$ is in the unit ball of $(X/[x_n^*]_0)^* = [\widetilde{x_n^*}]$. Because $\{x_n^*\}$ has no weakly convergent subsequence we conclude that the dual unit ball of $X/[x_n^*]_0$ is not weakly compact and thus $X/[x_n^*]_0$ is not reflexive. From A) we also know that $X/[x_n^*]_0$ has a basis. $\square$

**Remark 1.** Note that actually we have proved slightly more, namely:
Let $X$ be a separable Banach space. Then $X$ is reflexive iff every $w^*$ basic sequence $\{x^*_n\}$ spans a normed closed reflexive subspace $[x^*_n] \subset X^*$.

**Remark 2.** We do not know if Proposition 1 holds also without the separability assumption. This general statement would then imply (similarly as also the statement mentioned in A) and B) does) a positive answer to the following question which is still not settled: Has every Banach space a separable quotient space?

Similarly we may consider quotients of the space $X$ by subspaces $A \subset X$ such that $A$ has a basis and the quotient space $X/A$ is not reflexive:

**Proposition 2.** Let $X$ be a nonreflexive Banach space. Then there is a subspace $A \subset X$ such that $A$ has a basis and the quotient space $X/A$ is not reflexive.

**Proof.** is contained in the proof of Lemma 2 in [1] and for the sake of completeness we will list it here: Suppose that $X$ is not reflexive. From the results of Singer [13] and from the above cited result of Pełczyński we conclude that there is a basic sequence $\{x_n\} \subset X$ with $\|x_n\| \geq 1$ such that $\left\{ \sum_{i=1}^p x_n \right\}_p$ is bounded. We put $A = [x_{2n-1}]$ and let $P$ be the quotient map of $X$ onto $X/A$. Then evidently $\{x_{2n-1}\}$ and $\{P(x_{2n})\}$ are basic sequences, $\{P(x_{2n})\}$ is not a norm null sequence and $\left\{ \sum_{i=1}^p P(x_{2n}) \right\}_p = \left\{ \sum_{i=1}^{2p} P(x_n) \right\}_p$ is bounded (in $p$). We conclude [13] that the sequence $\{P(x_{2n})\}$ spans a non reflexive subspace of $X/A$. □

Next we will consider the reflexivity of the space of bounded operators $\mathcal{L}(X,Y)$:

**Proposition 3.** Let $X, Y$ be reflexive Banach spaces and suppose that $\mathcal{L}(X,Y)$ is not reflexive. Then there is a subspace $X_1 \subset X$ such that $X_1$ has Schauder basis and such that $\mathcal{L}(X_1,Y)$ is not reflexive.

**Proof.** Suppose that $\mathcal{L}(X,Y)$ is not reflexive. The result C) mentioned in the introduction yields a noncompact operator $f \in \mathcal{L}(X,Y)$. Let $\{x_n\}$ be a bounded sequence in $\mathcal{L}(X,Y)$ such that $\{f(x_n)\}$ has no norm convergent subsequence. Then $\{x_n\}$ also has no norm convergent subsequence. The reflexivity of the space $X$ implies that there is a subsequence of the sequence $\{x_n\}$ weakly convergent to $x \in X$. Let us denote this subsequence again by $\{x_n\}$ and put $z_n = x_n - x$. Then $z_n \underset{w}{\to} 0$. The classical theorem of Pełczyński mentioned in the introduction yields a basic subsequence of the sequence $\{z_n\}$. As above we call this subsequence again $\{z_n\}$ and put $X_1 = [\{z_n\} \cup \{x\}]$. Then $X_1$ has a basis. Indeed, if $x \in [z_n]$ then $\{x_n\} = [z_n]$ and $\{z_n\}$ is a basis of $X_1$. If $x \notin [z_n]$ then $X_1$ is the direct sum of $[z_n]$ and the one dimensional subspace spanned by $x$ and thus $X_1$ again has a basis. In any case $\{x_n\} \subset X_1$. This last inclusion evidently implies that the restriction
$f|_{X_1} \in \mathcal{L}(X_1, Y)$ is not a compact operator. We note that $X_1$ has the approximation property. Again by the result D) of [4] and [5] mentioned in the introduction we conclude that $\mathcal{L}(X_1, Y)$ is not reflexive. \hfill \Box

**Remark 3.** Note that we have actually observed the following:

Let $X, Y$ be any Banach spaces and suppose that there is noncompact operator $f : X \to Y$. Then there is a subspace $X_1 \subset X$ such that $X_1$ has Schauder basis and such that the restriction $f|_{X_1}$ is a noncompact operator.

**Remark 4.** Dually Proposition 3 can be formulated as follows:

Let $X, Y$ be reflexive Banach spaces and suppose that $\mathcal{L}(X, Y)$ is not reflexive. Then there is a subspace $Y_1 \subset Y$ such that the quotient space $Y/Y_1$ has Schauder basis and such that $\mathcal{L}(X, Y/Y_1)$ is not reflexive.

Indeed, if there is noncompact operator $f : X \to Y$ then $f^* \in \mathcal{L}(Y^*, X^*)$ is also noncompact and thus $\mathcal{L}(Y^*, X^*)$ is nonreflexive. Using now Proposition 3 for $\mathcal{L}(Y^*, X^*)$ we get a subspace $Z \subset Y^*$ having a basis such that $\mathcal{L}(Z, X^*)$ is not reflexive. We put now $Y_1 = Z_0$. Evidently $Y/Y_1 = Z^*$ has a basis. Proceeding as above and using now the duality of subspaces and quotients we get our claim.

**Remark 5.** A slightly more general result than stated in the above remark may also be formulated:

Let $X, Y$ be Banach spaces, let $Y$ be separable and suppose that $\mathcal{L}(X, Y) \neq \mathcal{K}(X, Y)$. Then there is a subspace $Y_1 \subset Y$ such that the factor space $Y/Y_1$ has Schauder basis and such that $\mathcal{L}(X, Y/Y_1) \neq \mathcal{K}(X, Y/Y_1)$.

Indeed, let $f : X \to Y$ be a noncompact operator. Proceeding as in the proofs of Propositions 3 and 1 we find a $w^*$ basic sequence $\{y_n^*\} \subset Y^*$ such that the restriction $f^*|_{[y_n^*]}$ is noncompact. Let now $Y_1 = [y_n^*]_0$ and let $P$ be the projection of $Y$ onto $Y/Y_1$. Then evidently $Pf : X \to Y/Y_1$ is a noncompact operator whose dual is $f^*|_{[y_n^*]}$.

**Remark 6.** Having in mind the basic relation of $\mathcal{L}(X, Y)$ to tensor products, namely $(X \tilde{\otimes}_\pi Y)^* = \mathcal{L}(X, Y)$ we can reformulate Proposition 3:

Let $X, Y$ be reflexive Banach spaces and suppose that $X \tilde{\otimes}_\pi Y$ is not reflexive. Then there is a subspace $X_1 \subset X$ such that $X_1$ has Schauder basis and such that $X_1 \tilde{\otimes}_\pi Y$ is not reflexive.

**Question.** The statement listed in Remark 5 suggests the following question:

Suppose that there is a noncompact operator $f : X \to Y$. Does there exist a noncompact operator $g : X \to Y_1$ and a subspace $Y_1 \subset Y$, $Y_1$ having a basis?

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References


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