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LACUNARY STRONG (A_{σ}, p) -CONVERGENCE

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Abstract. The definition of lacunary strongly convergence is extended to the definition of lacunary strong (A_{σ}, p) -convergence with respect to invariant mean when A is an infinite matrix and $p = (p_i)$ is a strictly positive sequence. We study some properties and inclusion relations.

Keywords: lacunary sequence, invariant convergence, infinite matrix

MSC 2000: 40A05, 40F05

1. INTRODUCTION

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional φ on l_{∞} , the space of real bounded sequences, is said to be an invariant mean or a σ -mean if and only if (i) $\varphi(x) \geq 0$ when the sequence $x = (x_n)$ satisfies $x_n \geq 0$ for all n , (ii) $\varphi(e) = 1$ where $e = (1, 1, 1, \dots)$, and (iii) $\varphi(x_{\sigma(n)}) = \varphi(x)$ for all $x \in l_{\infty}$. The mapping φ is assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m , where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n . Thus φ extends the limit functional to c , the space of convergent sequences, in the sense that $\varphi(x) = \lim x$ for all $x \in c$.

Consequently, $c \subset V_{\sigma}$ where V_{σ} is the set of bounded sequence all of whose σ -means are equal. If σ is the translation mapping $n \rightarrow n + 1$, a σ -mean is often called a Banach limit (see [1]) and V_{σ} is the set of almost convergent sequences (see [7]). $[V_{\sigma}]$ denotes the set of all strongly σ -convergent sequences (see [8]). It is easy to see that $[V_{\sigma}]$ is a proper closed linear subspace of V_{σ} . Furthermore, $[V_{\sigma}]$ contains c .

Several authors (e.g. in [2], [8], [11], [12]) have studied invariant convergent sequences.

By a lacunary sequence $\theta = (k_r)$ where $k_0 = 0$ we mean an increasing sequence of positive integers with $h_r = k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. The intervals determined

by θ will be denoted by $I_r = (k_{r-1}, k_r]$. The sequence space of lacunary strongly convergent sequences N_θ was defined in [6] as follows:

$$N_\theta = \left\{ x = (x_i) : \lim_{r \rightarrow \infty} h_r^{-1} \sum_{i \in I_r} |x_i - s| = 0 \text{ for some } s \right\}.$$

Lacunary convergent sequences have been studied most recently in [3], [4], [5], [10], and [11].

The purpose of this paper is to introduce and study a concept of lacunary strong (A_σ, p) -convergence.

Let $A = (a_{ik})$ be an infinite matrix of complex numbers. We write $Ax_{\sigma^i(n)} = (Ax_{\sigma^i(n)}(x))$ if $Ax_{\sigma^i(n)}(x) = \sum_{k=1}^{\infty} a_{ik}x_{\sigma^k(n)}$ converges for all n and i .

For $\sigma(n) = n + 1$ we write $Ax_{i+n} = (A_{i+n}(x))$ and $A_{i+n}(x) = \sum_{k=1}^{\infty} a_{ik}x_{n+k}$ for $Ax_{\sigma^i(n)} = (A_{\sigma^i(n)}(x))$ and $A_{\sigma^i(n)}(x) = \sum_{k=1}^{\infty} a_{ik}x_{\sigma^k(n)}$, respectively.

2. LACUNARY STRONGLY (A_σ, p) -CONVERGENT SEQUENCES

We now introduce the generalizations of lacunary strongly convergent sequences and investigate some inclusion relations.

Definition 1. Let $A = (a_{ik})$ be an infinite matrix of complex numbers and $p = (p_i)$ a sequence of strictly positive real numbers. We define spaces

$$\sigma_\theta[A, p] = \left\{ x = (x_i) : \lim_{r \rightarrow \infty} h_r^{-1} \sum_{i \in I_r} |Ax_{\sigma^i(n)} - s|^{p_i} = 0, \text{ uniformly in } n, \text{ for some } s \right\},$$

$$\sigma_\theta^0[A, p] = \left\{ x = (x_i) : \lim_{r \rightarrow \infty} h_r^{-1} \sum_{i \in I_r} |Ax_{\sigma^i(n)}|^{p_i} = 0, \text{ uniformly in } n \right\}.$$

A sequence x of real or complex numbers is said to be lacunary strongly (A_σ, p) -convergent to the value s if $x \in \sigma_\theta[A, p]$. Then we write $x_k \rightarrow s[\sigma_\theta[A, p]]$.

If $p_i = 1$ for all i , we write $\sigma_\theta[A]$ and $\sigma_\theta^0[A]$ for $\sigma_\theta[A, p]$ and $\sigma_\theta^0[A, p]$, respectively. If A is a unit matrix we write σ_θ and σ_θ^0 for $\sigma_\theta[A]$ and $\sigma_\theta^0[A]$, respectively. Hence σ_θ is the same as the space L_θ of [1]. For $\sigma(n) = n + 1$, the space σ_θ is the same as M_θ , the space of lacunary strongly almost convergent sequences (see [2]).

The following inequality will be used frequently throughout the paper:

$$(1) \quad |a_i + b_i|^{p_i} \leq \max(1, 2^{H-1})(|a_i|^{p_i} + |b_i|^{p_i})$$

where a_i, b_i are complex numbers and $H = \sup p_i < \infty$.

It is easy to see that $\sigma_\theta[A, p]$ and $\sigma_\theta^0[A, p]$ are linear spaces. We consider only $\sigma_\theta[A, p]$.

Suppose that $x_i \rightarrow s^1$ and $y_i \rightarrow s^2$ in $\sigma_\theta[A, p]$ and that a, b are in C , the set of complex numbers. Then there exist integers T_a and T_b such that $|a| \leq T_a$ and $|b| \leq T_b$. From (1) we have

$$h_r^{-1} \sum_{i \in I_r} |aAx_{\sigma^i(n)} + bAy_{\sigma^i(n)} - (as^1 + bs^2)|^{p_i} \leq \max\{1, 2^{H-1}\} \\ \times \left[(T_a)^H h_r^{-1} \sum_{i \in I_r} |Ax_{\sigma^i(n)} - s^1|^{p_i} + (T_b)^H h_r^{-1} \sum_{i \in I_r} |Ay_{\sigma^i(n)} - s^2|^{p_i} \right].$$

This implies that $ax + by \in \sigma_\theta[A, p]$.

The lacunary sequence $\theta^1 = (k_r^1)$ is called a lacunary refinement of $\theta = (k_r)$ if $(k_r) \subset (k_r^1)$ (see [4]).

Now we establish the inclusion relations among $\sigma_\theta[A, p]$ for different θ .

Theorem 2. (i) Let θ^1 be a refinement of θ . Then $\sigma_{\theta^1}[A, p] \subset \sigma_\theta[A, p]$.

(ii) Let L be a set of lacunary sequences closed under arbitrary union and intersection. Writing $\alpha = \bigcup_{\theta \in L} \theta$ and $\gamma = \bigcap_{\theta \in L} \theta$ we have $\sigma_\alpha[A, p] \subset \sigma_\theta[A, p] \subset \sigma_\gamma[A, p]$ for all $\theta \in L$.

Proof. Let $x \in \sigma_{\theta^1}[A, p]$ and let us suppose that there is a finite number of points $\theta^1 = (k_r^1)$ in the interval $I_r = (k_{r-1}, k_r]$. We assume for simplicity that there is exactly one point k_r^1 of θ^1 in the interval I_r , that is, $k_{r-1} = k_{j-1}^1 < k_j^1 < k_{j+1}^1 = k_r$. Now, from (1) we have

$$h_r^{-1} \sum_{i \in I_r} |Ax_{\sigma^i(n)} - s|^{p_i} \leq \max\{1, 2^{H-1}\} \\ \times \left[(h_r^{-1} h_r^1) h_r^{-1} \sum_{i \in I_r^1} |Ax_{\sigma^i(n)} - s|^{p_i} + (h_r^{-1} h_r^2) h_r^{-2} \sum_{i \in I_r^2} |Ax_{\sigma^i(n)} - s|^{p_i} \right],$$

where $I_r^1 = (k_{r-1}, k_j]$, $I_r^2 = (k_j, k_r]$, $h_r^1 = k_j - k_{r-1}$ and $h_r^2 = k_r - k_j$. Since $x \in \sigma_{\theta^1}(A, p)$, $0 < h_r^{-1} h_r^1 \leq 1$ and $0 < h_r^{-1} h_r^2 \leq 1$, we have $x \in \sigma_\theta[A, p]$. This proves (i).

(ii) follows from the fact that α is a refinement of θ and θ is a refinement of γ for every $\theta \in L$. \square

For $\sigma(n) = n + 1$ we write $M_\theta[A, p]$ for $\sigma_\theta[A, p]$. We now give a lemma to be used later.

Lemma 3. Suppose that for a given $\varepsilon > 0$ there exist k_0 and n_0 such that

$$k^{-1} \sum_{i=0}^{k-1} |Ax_{i+n} - s|^{p_i} < \varepsilon$$

for all $k \geq k_0$ and $n \geq n_0$. Then $x \in w[A, p]$, where

$$w[A, p] = \left\{ x = (x_i) : \lim_{k \rightarrow \infty} k^{-1} \sum_{i=0}^{k-1} |Ax_{i+n} - s|^{p_i} = 0 \text{ uniformly in } n, \text{ for some } s \right\}.$$

Proof. Let $\varepsilon > 0$ be given. Choose k_1 and n_0 such that

$$k^{-1} \sum_{i=0}^{k-1} |Ax_{i+n} - s|^{p_i} < \frac{\varepsilon}{2}$$

for all $k \geq k_1$, $n \geq n_0$. It is enough to prove that there is k_2 such that $k \geq k_2$, $0 \leq n \leq n_0$,

$$(2) \quad k^{-1} \sum_{i=0}^{k-1} |Ax_{i+n} - s|^{p_i} < \varepsilon.$$

Taking $k_0 = \max(k_1, k_2)$, (2) will hold for $k \geq k_0$ and for all n , which gives the result. Once n_0 has been chosen, n_0 is fixed, so

$$\sum_{i=0}^{n_0-1} |Ax_{i+n} - s|^{p_i} = R(n) \quad (\text{say}).$$

Now, taking $0 \leq n \leq n_0$ and $k > n_0$, we have

$$k^{-1} \sum_{i=0}^{k-1} |Ax_{i+n} - s|^{p_i} = k^{-1} \left(\sum_{i=0}^{n_0-1} + \sum_{i=n_0}^{k-1} \right) |Ax_{i+n} - s|^{p_i} \leq \frac{R(n)}{k} + \frac{\varepsilon}{2}.$$

Taking k sufficiently large, we have (2) and hence the result. □

Theorem 4. $w[A, p] = M_\theta[A, p]$ for every lacunary sequence θ , where

$$M_\theta[A, p] = \left\{ x = (x_i) : \lim_{r \rightarrow \infty} h_r^{-1} \sum_{i \in I_r} |Ax_{i+n} - s|^{p_i} = 0 \text{ uniformly in } n, \text{ for some } s \right\}.$$

Proof. Let $x \in M_\theta[A, p]$. Then given $\varepsilon > 0$, there exist r_0 and s such that

$$h_r^{-1} \sum_{i=0}^{h_r-1} |Ax_{i+n} - s|^{p_i} < \varepsilon$$

for $r \geq r_0$ and $n = k_{r-1} + 1 + u$, $u \geq 0$. Let $k \geq h_r$, write $k = mh_r + p$, where $0 \leq p \leq h_r$, m is an integer. Since $k \geq h_r$, $m \geq 1$, we have

$$\begin{aligned} k^{-1} \sum_{i=0}^{k-1} |Ax_{i+n} - s|^{p_i} &\leq k^{-1} \sum_{i=0}^{(m+1)h_r-1} |Ax_{i+n} - s|^{p_i} = k^{-1} \sum_{j=0}^m \sum_{i=jh_r}^{(j+1)h_r-1} |Ax_{i+n} - s|^{p_i} \\ &\leq (m+1)k^{-1}h_r\varepsilon \leq 2mk^{-1}h_r\varepsilon \end{aligned}$$

for $k^{-1}h_r \leq n$, and since $mk^{-1}h_r \leq 1$ we conclude

$$k^{-1} \sum_{i=0}^{k-1} |Ax_{i+n} - s|^{p_i} \leq 2\varepsilon.$$

Then by Lemma 3, $M_\theta[A, p] \subset w[A, p]$. It is easy to see that $w[A, p] \subset M_\theta[A, p]$ for every θ . □

Theorem 5. Let A be a strongly regular matrix and $0 < \inf p_i$, then $x_i \rightarrow s$ implies $x_i \rightarrow s[M_\theta[A, p]]$.

Proof. Suppose that $x_i \rightarrow s$ as $i \rightarrow \infty$. This implies $Ax_{i+n} \rightarrow s$ as $i \rightarrow \infty$ uniformly in n . Since $0 < h = \inf p_i$ we have $\lim_{i \rightarrow \infty} |Ax_{i+n} - s|^h = 0$ uniformly in n . So for $0 < \varepsilon < 1$, there is $i_0 \in \mathbb{N}$ such that for all $i > i_0$ and for all n , $|Ax_{i+n} - s|^h < \varepsilon < 1$ and since $p_i > h$ for all i ,

$$|Ax_{i+n} - s|^{p_i} \leq |Ax_{i+n} - s|^h < \varepsilon.$$

Then $\lim_{i \rightarrow \infty} |Ax_{i+n} - s|^{p_i} = 0$ uniformly in n and therefore $\lim_{k \rightarrow \infty} k^{-1} \sum_{i=0}^{k-1} |Ax_{i+n} - s|^{p_i} = 0$. From Theorem 4 we have $\lim_{r \rightarrow \infty} h_r^{-1} \sum_{i \in I_r} |Ax_{i+n} - s|^{p_i} = 0$ uniformly in n . So $x_i \rightarrow s[M_\theta[A, p]]$. □

Theorem 6. Let A be a limitation method, $x \in \ell_\infty$ and $p_i = p$ for all i . Then $w(Ap) \equiv M_\theta(Ap)$ for every lacunary sequence θ , where

$$w(Ap) = \left\{ x = (x_i) : \lim_{k \rightarrow \infty} k^{-1} \sum_{i=0}^{k-1} (Ax_{i+n} - s)^p = 0 \text{ uniformly in } n, \text{ for some } s \right\}$$

and

$$M_\theta(Ap) = \left\{ x = (x_i) : \lim_{r \rightarrow \infty} h_r^{-1} \sum_{i \in I_r} (Ax_{i+n} - s)^p = 0 \text{ uniformly in } n, \text{ for some } s \right\}.$$

In order to prove this theorem, we need the following lemma.

Lemma 7. Suppose that for a given $\varepsilon > 0$ there exist k_0 and n_0 such that

$$k^{-1} \sum_{i=0}^{k-1} (Ax_{i+n} - s)^p < \varepsilon$$

for all $k \geq k_0$ and $n \geq n_0$. Then $x \in w(Ap)$.

Proof. The proof of Lemma 7 is similar to the proof of Lemma 3. □

Proof of Theorem 6. Let $x \in M_\theta(Ap)$. Then given $\varepsilon > 0$, there exist r_0 and n_0 such that

$$(3) \quad h_r^{-1} \left| \sum_{i=0}^{h_r-1} (Ax_{i+n} - s)^p \right| < \frac{\varepsilon}{2}$$

for $r \geq r_0$, $n \geq n_0$ and $n = k_{r-1} + 1 + u$, $u \geq 0$. Let $k \geq h_r$, m be an integer greater than or equal to 1. Then

$$(4) \quad k^{-1} \left| \sum_{i=0}^{k-1} (Ax_{i+n} - s)^p \right| \leq k^{-1} \sum_{j=0}^m \left| \sum_{i=jh_r}^{(j+1)h_r-1} (Ax_{i+n} - s)^p \right| + k^{-1} \sum_{i=mh_r}^{k-1} |Ax_{i+n} - s|^p.$$

Since A is a limitation method and $x \in \ell_\infty$, let for all i and n , $|Ax_{i+n} - s|^p \leq M$ (say). So (3) and (4) imply

$$k^{-1} \left| \sum_{i=0}^{k-1} (Ax_{i+n} - s)^p \right| \leq mk^{-1}h_r \frac{\varepsilon}{2} + Mk^{-1}h_r.$$

For $k^{-1}h_r \leq 1$, since $mk^{-1}h_r \leq 1$, we can make $Mk^{-1}h_r$ less than $\varepsilon/2$ by taking k sufficiently large. So

$$k^{-1} \left| \sum_{i=0}^{k-1} (Ax_{i+n} - s)^p \right| < \varepsilon$$

for all $k \geq k_0$ and $n \geq n_0$. Hence, by Lemma 7, $M_\theta(Ap) \subset w(Ap)$. It is trivial to see that $w(Ap) \subset M_\theta(Ap)$ for every θ . This completes the proof. □

Let $p_k = s$ for all k , $q_k = t$ for all k and $0 < s \leq t$. Then it follows from Hölder's inequality that

$$h_r^{-1} \sum_{i \in I_r} |Ax_{\sigma'(n)}|^s \leq \left(h_r^{-1} \sum_{i \in I_r} |Ax_{\sigma^i(n)}|^t \right)^{s/t}$$

and therefore $\sigma_\theta^0[A, q] \subseteq \sigma_\theta^0[A, p]$.

We now consider the case that (p_i) and (q_i) are not constant sequences. We are able to prove $\sigma_\theta^0[A, q] \subseteq \sigma_\theta^0[A, p]$ only under additional conditions.

Theorem 8. *Let $0 < p_i \leq q_i$ for all k and let (q_i/p_i) be bounded. Then*

$$\sigma_\theta^0[A, q] \subseteq \sigma_\theta^0[A, p] \quad \text{and} \quad \sigma_\theta[A, q] \subseteq \sigma_\theta[A, p].$$

Proof. If we take $t_i = |Ax_{\sigma^i(n)}|^{p_i}$ for all i , then using the same technique as in [9, Theorem 2], it is easy to prove the theorem. \square

Result.

- (i) *If $0 < \inf p_i \leq p_i \leq 1$ for all k , then $\sigma_\theta[A] \subseteq \sigma_\theta[A, p]$.*
- (ii) *$1 \leq p_i \leq \sup p_i = H < \infty$, then $\sigma_\theta[A, p] \subseteq \sigma_\theta[A]$.*

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