

Marián Trenkler

On orthogonal Latin p -dimensional cubes

Czechoslovak Mathematical Journal, Vol. 55 (2005), No. 3, 725–728

Persistent URL: <http://dml.cz/dmlcz/128017>

Terms of use:

© Institute of Mathematics AS CR, 2005

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

ON ORTHOGONAL LATIN p -DIMENSIONAL CUBES

MARIÁN TRENKLER, Ružomberok

(Received November 4, 2002)

Abstract. We give a construction of p orthogonal Latin p -dimensional cubes (or Latin hypercubes) of order n for every natural number $n \neq 2, 6$ and $p \geq 2$. Our result generalizes the well known result about orthogonal Latin squares published in 1960 by R. C. Bose, S. S. Shikhande and E. T. Parker.

Keywords: Latin p -dimensional cube, Latin hypercube, Latin squares, orthogonal

MSC 2000: 05B15

In 1960, R. C. Bose, S. S. Shikhande and E. T. Parker [1] proved that two orthogonal Latin squares of order n exist if and only if $n \neq 2, 6$. (For more information about these topics see [2] and [3].)

A generalization of Latin squares are Latin p -dimensional cubes (sometimes called *Latin hypercubes*). In this paper we generalize the well know result from [1] into p -dimensional space for every natural number p .

Definition. A *Latin p -dimensional cube* of order n is a p -dimensional matrix

$$\mathbf{Q}^{p,n} = |\mathbf{q}(i_1, i_2, \dots, i_p); 1 \leq i_1, i_2, \dots, i_p \leq n|,$$

such that every row is a permutation of the set of natural numbers $1, 2, \dots, n$. By a *row* of $\mathbf{Q}^{p,n}$ we mean an n -tuple of elements $\mathbf{q}(i_1, i_2, \dots, i_p)$ which have identical coordinates at $p - 1$ places.

Definition. A p -tuple of Latin p -dimensional cubes

$$[\mathbf{Q}_k^{p,n} = |\mathbf{q}_k(i_1, i_2, \dots, i_p); 1 \leq i_1, i_2, \dots, i_p \leq n|, k = 1, 2, \dots, p]$$

of order n is called *orthogonal*, if whenever $i_1, i_2, \dots, i_p, i'_1, \dots, i'_p \in \{1, 2, \dots, n\}$ are such that

$$\mathbf{q}_k(i_1, i_2, \dots, i_p) = \mathbf{q}_k(i'_1, i'_2, \dots, i'_p) \quad \text{for all } k = 1, 2, \dots, p,$$

then we must have $i_k = i'_k$ for all $k = 1, 2, \dots, p$.

The construction of a p -tuple of orthogonal Latin p -dimensional cubes is contained in the proof of the following theorem.

Theorem. *A p -tuple of orthogonal Latin p -dimensional cubes $\mathbf{Q}_k^{p,n}$ of order n exists for every natural number $n \neq 2, 6$ and every natural number $p \geq 2$.*

Proof. Let $\mathbf{R}^n = |\mathbf{r}(i_1, i_2); 1 \leq i_1, i_2 \leq n|$ and $\mathbf{S}^n = |\mathbf{s}(i_1, i_2)|$ be two orthogonal Latin squares of order n . They will have a crucial role in our construction of p orthogonal Latin p -dimensional cubes $\mathbf{Q}_k^{p,n}$, $k = 1, 2, \dots, p$. The k -th cube arises using the square \mathbf{R}^n ($k - 1$)-times and the square \mathbf{S}^n ($p - k$)-times.

We define the k -th Latin p -dimensional cube

$$\mathbf{Q}_k^{p,n} = |\mathbf{q}_k(i_1, i_2, \dots, i_p)|$$

of order n by the following relation

$$\begin{aligned} & \mathbf{q}_k(i_1, \dots, i_p) \\ &= \mathbf{r}(i_1, \mathbf{r}(i_2, \mathbf{r}(i_3, \dots, \mathbf{r}(i_{k-1}, \mathbf{s}(i_k, \mathbf{s}(i_{k+1}, \dots, \mathbf{s}(i_{p-2}, \mathbf{s}(i_{p-1}, i_p)) \dots)) \dots))) \dots) \end{aligned}$$

for every $1 \leq i_1, i_2, \dots, i_p \leq n$.

1. Evidently, for every $k = 1, 2, \dots, p$, the set

$$\{\mathbf{q}_k(i_1, i_2, \dots, i_{j-1}, i_j, i_{j+1}, \dots, i_p); i_j = 1, 2, \dots, n\}$$

is equal to the set $\{1, 2, \dots, n\}$. From this it follows that $\mathbf{Q}_k^{p,n}$ is a Latin p -dimensional cube for all k .

2. Suppose that

$$(E_k) \quad \mathbf{q}_k(i_1, i_2, \dots, i_p) = \mathbf{q}_k(i'_1, i'_2, \dots, i'_p) \quad \text{for all } k = 1, 2, \dots, p.$$

From (E_1) and (E_2) it follows that

$$\begin{aligned} \mathbf{s}(i_1, \mathbf{s}(i_2, \mathbf{s}(i_3, \dots, \mathbf{s}(i_{p-1}, i_p) \dots))) &= \mathbf{s}(i'_1, \mathbf{s}(i'_2, \mathbf{s}(i'_3, \dots, \mathbf{s}(i'_{p-1}, i'_p) \dots))), \\ \mathbf{r}(i_1, \mathbf{s}(i_2, \mathbf{s}(i_3, \dots, \mathbf{s}(i_{p-1}, i_p) \dots))) &= \mathbf{r}(i'_1, \mathbf{s}(i'_2, \mathbf{s}(i'_3, \dots, \mathbf{s}(i'_{p-1}, i'_p) \dots))). \end{aligned}$$

Because \mathbf{R}^n and \mathbf{S}^n are orthogonal Latin squares, we have

$$i_1 = i'_1$$

and

$$\mathbf{s}(i_2, \mathbf{s}(i_3, \dots, \mathbf{s}(i_{p-1}, i_p) \dots)) = \mathbf{s}(i'_2, \mathbf{s}(i'_3, \dots, \mathbf{s}(i'_{p-1}, i'_p) \dots)).$$

Replace i'_1 by i_1 in (E_k) , $k = 1, 2, \dots, p$. From (E_2) and (E_3) it follows that

$$\begin{aligned} \mathbf{s}(i_2, \mathbf{s}(i_3, \dots, \mathbf{s}(i_{p-1}, i_p) \dots)) &= \mathbf{s}(i'_2, \mathbf{s}(i'_3, \dots, \mathbf{s}(i'_{p-1}, i'_p) \dots)), \\ \mathbf{r}(i_2, \mathbf{s}(i_3, \dots, \mathbf{s}(i_{p-1}, i_p) \dots)) &= \mathbf{r}(i'_2, \mathbf{s}(i'_3, \dots, \mathbf{s}(i'_{p-1}, i'_p) \dots)), \end{aligned}$$

and so

$$i_2 = i'_2$$

and

$$\mathbf{s}(i_3, \mathbf{s}(i_4, \dots, \mathbf{s}(i_{p-1}, i_p) \dots)) = \mathbf{s}(i'_3, \mathbf{s}(i'_4, \dots, \mathbf{s}(i'_{p-1}, i'_p) \dots)).$$

Continuing in this manner, after $(p - 1)$ steps from (E_{p-1}) and (E_p) we get

$$\begin{aligned} \mathbf{s}(i_{p-1}, i_p) &= \mathbf{s}(i'_{p-1}, i'_p), \\ \mathbf{r}(i_{p-1}, i_p) &= \mathbf{r}(i'_{p-1}, i'_p). \end{aligned}$$

From the assumption that \mathbf{R}^n and \mathbf{S}^n are orthogonal we get

$$i'_{p-2} = i_{p-2} \quad \text{and} \quad i'_{p-1} = i_{p-1},$$

which completes the proof of orthogonality. □

Remark 1. Our construction is based on a pair of orthogonal Latin squares and so we give no information about Latin p -dimensional cubes of order 2 and 6.

Remark 2. If n is odd then $\mathbf{R}^n = |\mathbf{r}(i_1, i_2) = (i_1 + i_2) \pmod n|$; $1 \leq i_1, i_2 \leq n|$ and $\mathbf{S}^n = |\mathbf{s}(i_1, i_2) = (i_1 - i_2) \pmod n|$ are mutually orthogonal Latin squares. Using these two squares the formula for making a magic p -dimensional cube of odd order was derived. (See [4].)

References

- [1] *R. C. Bose, S. S. Shrikhande and E. T. Parker*: Further results on the construction of mutually orthogonal Latin squares and the falsity of Euler's conjecture. *Canad. J. Math.* *12* (1960), 189–203.
- [2] *J. Dénes and A. D. Keedwell*: *Latin Squares and Their Applications*. Akadémiai Kiadó, Budapest, 1974.
- [3] *G. L. Mullen*: Orthogonal hypercubes and related designs. *J. Stat. Plann. Inference* *73* (1998), 177–188.
- [4] *M. Trenkler*: Magic p -dimensional cubes of order $n \not\equiv 2 \pmod{4}$. *Acta Arithmetica* *92* (2000), 189–194.

Author's address: Pedagogical Faculty, Catholic University, Andreja Hlinku 56, 034 01 Ružomberok, Slovak Republic, e-mail: trenkler@fedu.ku.sk.