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STIELTJES PERFECT SEMIGROUPS ARE PERFECT

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Abstract. An abelian $*$-semigroup S is *perfect* (resp. *Stieltjes perfect*) if every positive definite (resp. completely so) function on S admits a unique disintegration as an integral of hermitian multiplicative functions (resp. nonnegative such). We prove that every Stieltjes perfect semigroup is perfect. The converse has been known for semigroups with neutral element, but is here shown to be not true in general. We prove that an abelian $*$-semigroup S is perfect if for each $s \in S$ there exist $t \in S$ and $m, n \in \mathbb{N}_0$ such that $m + n \geqslant 2$ and $s + s^* = s^* + mt + nt^*$. This was known only with $s = mt + nt^*$ instead. The equality cannot be replaced by $s + s^* + s = s + s^* + mt + nt^*$ in general, but for semigroups with neutral element it can be replaced by $s + p(s + s^*) = p(s + s^*) + mt + nt^*$ for arbitrary $p \in \mathbb{N}$ (allowed to depend on s).

Keywords: perfect, Stieltjes perfect, moment, positive definite, conelike, semi-$*$-divisible, $*$-semigroup

MSC 2000: 43A35, 44A60

0. Introduction

Perfect semigroups are abelian involution semigroups on which every positive definite function admits a unique disintegration as an integral of hermitian multiplicative functions; such as abelian groups with the inverse involution (the discrete version of the Bochner-Weil Theorem [23], [24], [30], of which the first instance was Herglotz' Theorem [20] of 1911). The simplest example of a perfect semigroup admitting unbounded positive definite functions is the additive semigroup of nonnegative rationals with its unique involution, the identity [2]. In [8] it was shown that with an abelian involution semigroup S with zero one can associate a family of subsemigroups of rational vector spaces with the identical involution such that S is perfect if and only if each of these semigroups is 'Stieltjes perfect', and that every perfect semigroup with zero is Stieltjes perfect. We shall show that every Stieltjes perfect semigroup (even without zero) is perfect. The example of $]1,\infty[\,\cap\,\mathbb{Q}$ will show that a perfect semigroup without zero need not be Stieltjes perfect. We shall show that an abelian involution semigroup is Stieltjes perfect if and only if it is perfect and 'Stieltjes flat'.

Quasi-perfect semigroups are abelian involution semigroups S that enjoy a similar unique disintegration of arbitrary positive definite functions but required only on the subset $S + S + S$. An abelian involution semigroup is perfect if and only if it is quasi-perfect and 'flat' [11]. An abelian involution semigroup is quasi-perfect if and only if the semigroup obtained by adjoining a zero is perfect [16]. The class of quasi-perfect semigroups has some stability properties that are superior to those of the class of perfect semigroups [16].

For every abelian involution semigroup S we will define a quotient semigroup $\varrho(S)$ such that S is perfect if and only if S is flat and $\varrho(S)$ is quasi-perfect. Moreover, $\varrho(S)$ carries the identical involution and is a disjoint union of subsemigroups of rational vector spaces. It is quasi-perfect if and only if each of these subsemigroups is quasi-perfect.

Since perfectness and Stieltjes perfectness are equivalent for semigroups with zero it may be of interest to note that Stieltjes perfectness is sometimes easier to establish directly. For example, completely positive definite functions on subgroups of rational vector spaces carrying the identical involution are automatically continuous on every finite-dimensional linear subspace and even extend to completely positive definite functions on the closure of the group with respect to the 'topology of finitely open sets' [5].

Let us call an abelian involution semigroup S *densely cosetlike* if the nonnegative characters on S separate points in S and for each archimedean component H of S the enveloping rational vector space of H is the sum of those of its linear subspaces U such that H contains a nonempty open subset of a coset (in U) of a dense subgroup of U. Every densely cosetlike semigroup is quasi-perfect [12]. It follows that a $*$-semigroup S is perfect if it is flat and $\varrho(S)$ is densely cosetlike. The result covers all known examples of perfect semigroups, or at least the classical ones (Herglotz' Theorem, etc.) and those published by the school following Berg, Christensen and Ressel [4].

1. Preliminaries

Let $(S, +, *)$ be an abelian $*$-semigroup, i.e., an abelian semigroup equipped with an involution $*$, so $(s + t)^* = s^* + t^*$ and $(s^*)^* = s$ for all $s, t \in S$. We call S simply a *$*$-semigroup*, even abbreviated to 'semigroup', e.g., in 'perfect semigroup'.

Let $S+S=\{s+t\colon\ s,t\in S\}$ and define $\overbrace{S+\ldots+S}^{N}$ similarly for arbitrary $N\in\mathbb{N}$. A function $\varphi\colon\ S+S\to\mathbb{C}$ is *positive definite* if $\sum\limits_{j,k=1}^{n} c_j\overline{c_k}\varphi(s_j+s_k^*)\geqslant 0$ for all $n\in\mathbb{N}$, $s_1,\ldots,s_n\in S$, and $c_1,\ldots,c_n\in\mathbb{C}$. The set of all such functions is denoted by $\mathscr{P}(S)$. Each $\varphi\in\mathscr{P}(S)$ satisfies the *Cauchy-Schwarz inequality*, cf. [4], Chapter 3, which has the special case $\Big|\sum\limits_{j=1}^{n} c_j\varphi(e+s_j)\Big|^2\leqslant\varphi(e+e^*)\sum\limits_{j,k=1}^{n} c_j\overline{c_k}\varphi(s_j+s_k^*)$ for $e\in S$, $n\in\mathbb{N}$, $s_i\in S$, and $c_i\in\mathbb{C}$. A function $\varphi\colon\ S\to\mathbb{C}$ is *hermitian* if $\varphi(s^*)=\overline{\varphi(s)}$ for all $s\in S$. Each $\varphi\in\mathscr{P}(S)$ is hermitian, cf. [4], Chapter 3.

Let S^* (resp. S^*_+) be the set of all *characters* on S, i.e., the hermitian complex-valued function σ on S, not identically zero, such that $\sigma(s+t)=\sigma(s)\sigma(t)$ for all $s,t\in S$ (resp. nonnegative such). Let Γ be any subset of S^* (having in mind S^* or S^*_+). Let $\mathscr{A}(\Gamma)$ be the least σ-ring of subsets of Γ rendering measurable (in the sense of Halmos [18]) for each $s\in S$ the function $\hat{s}$ on Γ defined by $\hat{s}(\sigma)=\sigma(s)$ for $\sigma\in\Gamma$. Since Γ can be either S^* or S^*_+ then the symbol '$\hat{s}$' has (at least) two senses but when writing an integral, the measure determines which space is meant. Clearly, $\mathscr{A}(S^*_+)=\{A\cap S^*_+\colon\ A\in\mathscr{A}(S^*)\}$. Denote by $F_+(\Gamma)$ (resp. $G_+(\Gamma)$) the set of all measures on $\mathscr{A}(\Gamma)$ integrating $|\hat{s}|^2$ (resp. $|\hat{s}|$) for all $s\in S$. Then $F_+(S^*_+)$ (resp. $G_+(S^*_+)$) is the set of those measures μ on $\mathscr{A}(S^*_+)$ for which the measure $A\mapsto\mu(A\cap S^*_+)$ on $\mathscr{A}(S^*)$ is in $F_+(S^*)$ (resp. $G_+(S^*)$). (We shall explain presently why it is essential to use a σ-ring that need not be a σ-algebra.)

For $s\in S$ and $n\in\mathbb{N}$ let $G_{s,n}$ (resp. $G^+_{s,n}$) be the set of those σ in S^* (resp. S^*_+) for which $|\sigma(s)|>1/n$. Clearly, $G^+_{s,n}=G_{s,n}\cap S^*_+$. Let $\mathscr{A}_0(\Gamma)$ be the subring of $\mathscr{A}(\Gamma)$ consisting of those measurable sets which are contained in the union of finitely many $G_{s,n}$. Clearly, $\mathscr{A}_0(S^*_+)=\{A\cap S^*_+\colon\ A\in\mathscr{A}_0(S^*)\}$. As in [9], p. 57, the subring $\mathscr{A}_0(\Gamma)$ generates $\mathscr{A}(\Gamma)$ as a σ-ring, so every measure on $\mathscr{A}_0(\Gamma)$ which is *finite* (on each set in $A\in\mathscr{A}_0(\Gamma)$) extends to a unique measure on $\mathscr{A}(\Gamma)$ (a *measure* is positive by definition except in 'complex measure'). If $\mu\in F_+(\Gamma)$ (in particular, if $\mu\in G_+(\Gamma)$) then $\mu\big|\mathscr{A}_0(\Gamma)$ is finite since $\mu(\Gamma\cap G_{s,n})/n^2\leqslant\int|\hat{s}|^2\,\mathrm{d}\mu<\infty$ for $s\in S$ and $n\in\mathbb{N}$. The linear hull of $F_+(\Gamma)$ (resp. $G_+(\Gamma)$) is denoted by $F(\Gamma)$ (resp. $G(\Gamma)$) and consists of complex measures defined on $\mathscr{A}_0(\Gamma)$. For $\mu\in F(\Gamma)$ (resp. $G(\Gamma)$) we define $\mathscr{L}\mu\colon\ S+S\to\mathbb{C}$ (resp. $S\to\mathbb{C}$) by $\mathscr{L}\mu(s)=\int_\Gamma\sigma(s)\,\mathrm{d}\mu(\sigma)$ for $s\in S+S$ (resp. S).

A function $\varphi\colon\ S+S\to\mathbb{C}$ (resp. $S\to\mathbb{C}$) is a *moment function* (resp. *Stieltjes moment function*) if $\varphi=\mathscr{L}\mu$ for some $\mu\in F_+(S^*)$ (resp. $\mu\in G_+(S^*_+)$), and a moment function (resp. Stieltjes moment function) φ is *determinate* (resp. *Stieltjes determinate*) if there is only one such μ. The set of all moment functions (resp. determinate moment functions, Stieltjes moment functions, Stieltjes determinate Stieltjes moment functions) is denoted by $\mathscr{H}(S)$ (resp. $\mathscr{H}_{\det(S)}$, $\mathscr{H}_S(S)$, $\mathscr{H}_{S,\det}(S)$). We have $\mathscr{H}_{\det}(S) \subset \mathscr{H}(S) \subset \mathscr{P}(S)$, as a simple computation shows. The $*$-semigroup S is *semiperfect* if $\mathscr{H}(S) = \mathscr{P}(S)$, and *perfect* if $\mathscr{H}_{\det}(S) = \mathscr{P}(S)$.

(The reason why we use the σ-ring $\mathscr{A}(S^*)$ rather than the similarly defined σ-algebra, call it $\tilde{\mathscr{A}}$, is that we want the $*$-semigroup $S = (\mathbb{Q}_+^{(\mathbb{R})} \setminus \{0\}, +, s^* = s)$ to be perfect (as it provably is with the definitions we use), which would not be the case if in the definition of perfectness we had used $\tilde{\mathscr{A}}$ instead of $\mathscr{A}(S^*)$ since then the zero measure(!) on $\mathscr{A}(S^*)$ would have had two extensions to a measure on $\tilde{\mathscr{A}}$, viz., the zero measure and the one which is the constant ∞ on $\tilde{\mathscr{A}} \setminus \mathscr{A}(S^*)$. It would not do to require all measures to be finite-valued since there are perfectly respectable moment functions that require for their integral representation measures that do assume the value ∞; such as the moment function $n \mapsto n^{-1}$ on the $*$-semigroup $(\mathbb{N}, +, n^* = n)$.)

A $*$-semigroup H is *$*$-archimedean* (resp. *archimedean*) if for all $x, y \in H$ there exist $n \in \mathbb{N}$ and $z \in H$ such that $n(x + x^*) = y + z$ (resp. $nx = y + z$). If H is $*$-archimedean then every character on H is nowhere zero, so if H^* separates points in H then H is cancellative [7]. A *$*$-subsemigroup* of a $*$-semigroup is a subsemigroup stable under the involution. A *$*$-archimedean component* of a $*$-semigroup is a maximal $*$-archimedean $*$-subsemigroup. Every $*$-semigroup is the disjoint union of its $*$-archimedean components, cf. [17], Section 4.3. If the involution is the identity then we (consistently) drop the '$*$-' of '$*$-archimedean'. The decomposition of an abelian semigroup into its archimedean components was first established by Tamura and Kimura, see the bibliography of [17].

A *face* of a $*$-semigroup S is a $*$-subsemigroup X of S such that if $x, y \in S$ and $x + y \in X$ then $x, y \in X$. If H is a $*$-archimedean component of S and if X is the least face of S containing H then $X + H \subset H$ (see, e.g., [9]).

A $*$-semigroup S is perfect if it is an *abelian inverse semigroup*, i.e., $s = s + s^* + s$ for each $s \in S$. This is probably an old result, but we cannot give a reference (however, see Paterson [22]). More generally, a $*$-semigroup S is perfect if it is *$*$-divisible*, i.e., for each $s \in S$ there exist $t \in S$ and $m, n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ such that $m + n \geqslant 2$ and $s = mt + nt^*$ (see the papers by Ressel and the first-mentioned author [15], and by the present authors [16]).

We shall show that a $*$-semigroup S is perfect if it is *semi-$*$-divisible*, i.e., for each $s \in S$ there exist $t \in S$ and $m, n \in \mathbb{N}_0$ such that $m + n \geqslant 2$ and $s + s^* = s^* + mt + nt^*$. The next step would be to replace the equality with $s + s^* + s = s + s^* + mt + nt^*$. The condition so obtained, however, is not sufficient even for semiperfectness. Indeed, the condition is clearly satisfied if S carries the identical involution and $3s = 5s$ for all $s \in S$. Let S be the abelian semigroup $\{1, 2, 3\}$ with $s \oplus t := (s + t) \wedge 3$, $s, t \in S$. Clearly, $3s = 5s$ for all $s \in S$. Note: $S \oplus S = \{2, 3\}$. Let φ be the indicator function of the set $\{2\}$ as a subset of $S + S$. Then $\varphi \in \mathscr{P}(S)$. Indeed, if $c_1, c_2, c_3 \in \mathbb{C}$ then

$\sum\limits_{j,k=1}^{3} c_j\overline{c_k}\varphi(j\oplus k) = |c_1|^2 \geqslant 0$. If $\sigma \in S^*$ then $\sigma(1)^3 = \sigma(3) = \sigma(4) = \sigma(1)^4$, so $\sigma(1) \in \{0,1\}$. Since $\sigma \neq 0$ and since 1 generates S we have $\sigma(1) = 1$, so σ is the constant 1. Thus every moment function on S is a constant. Since the function φ is not constant then it is not a moment function. Thus S is not semiperfect.

However, if S is a $*$-semigroup with zero such that for each $s \in S$ there exist $t \in S$, $p \in \mathbb{N}$, and $m, n \in \mathbb{N}_0$ such that $m + n \geqslant 2$ and

$$(*) \qquad s + p(s+s^*) = p(s+s^*) + mt + nt^*$$

then S is perfect. Indeed, suppose $\varphi \in \mathscr{P}(S)$. If $s, t \in S$ and $\sigma(s) = \sigma(t)$ for all $\sigma \in S^*$ then $\varphi(s) = \varphi(t)$ [6]. Thus we can pass to a quotient semigroup by collapsing s and t for all such (s,t). In other words, we may assume that S^* separates points in S. Given $s \in S$, let H be the $*$-archimedean component of S containing s. Since S^* separates points in S, the set H^* separates points in H, so H is cancellative. Let X be the least face of S containing H. Then $X + H \subset H$. In the identity $(*)$, since the left-hand side is in H, hence in particular in X, so is the right-hand side. By the definition of a face it follows that $mt + nt^* \in X$. Hence $s^* + mt + nt^* \in H + X \subset H$. Adding s^* to both sides of $(*)$, we obtain $s+s^*+p(s+s^*) = p(s+s^*)+(s^*+mt+nt^*)$. Since H is cancellative it follows that $s + s^* = s^* + mt + nt^*$. Since this argument applies to all $s \in S$, S is semi-$*$-divisible, hence perfect.

Instead of S having a zero it suffices that S is 'flat' as defined below. Indeed, as shown in [11] it then suffices to show that S is 'quasi-perfect' as defined in [16], which is equivalent to the $*$-semigroup $S \cup \{0\}$ being perfect. Obviously, this semigroup satisfies the hypothesis on S, so it is perfect by the preceding.

The semigroup $(\mathbb{N}_0, +)$ with its unique involution, the identity, is semiperfect by Hamburger's Theorem ([19], see [27] or [1]) but is not perfect since there exist indeterminate moment sequences, such as the example $n \mapsto (4n+3)!$ given by Stieltjes [28] in 1894. The semigroup $\mathbb{N}_0^2$ is non-semiperfect as shown by Berg, Christensen and Jensen [3] and independently by Schmüdgen [26], see [4, Chapter 6].

An *ideal* of a $*$-semigroup S is a nonempty $*$-stable subset T of S such that $S + T \subset T$. Every ideal of S is a $*$-subsemigroup of S. Let $\tilde{S}$ be the set $S \cup \{0\}$ where 0 is some element outside S, made into a $*$-semigroup by setting $0 + s = s + 0 = s$ for all $s \in S \cup \{0\}$ and $0^* = 0$. Then S is an ideal of $\tilde{S}$. For $\varphi \in \mathbb{C}^S$ and $r \in S$ we define $E_r\varphi \in \mathbb{C}^{\tilde{S}}$ by $E_r\varphi(x) = \varphi(r + x)$ for $x \in \tilde{S}$. A function $\varphi\colon S \to \mathbb{C}$ is *completely positive definite* if $E_r\varphi \in \mathscr{P}(\tilde{S})$ for all $r \in S$. Denote by $\mathscr{P}_c(S)$ the set of all such functions. Note that each $\varphi \in \mathscr{P}_c(S)$ is nonnegative. Now $\mathscr{H}_{S,\mathrm{det}}(S) \subset \mathscr{H}_S(S) \subset \mathscr{P}_c(S)$, as a simple computation shows. The $*$-semigroup S is *Stieltjes semiperfect* if $\mathscr{H}_S(S) = \mathscr{P}_c(S)$, and *Stieltjes perfect* if $\mathscr{H}_{S,\mathrm{det}}(S) = \mathscr{P}_c(S)$.

As shown in [8], every perfect semigroup with zero is Stieltjes perfect. We will show that conversely, every Stieltjes perfect semigroup (even without zero) is perfect. In contrast, among finitely generated abelian semigroups with zero and the identical involution, there are both one which is semiperfect but not Stieltjes semiperfect and one which is Stieltjes semiperfect but not semiperfect [13].

We will define 'Stieltjes flat' semigroups in such a way that every Stieltjes semiperfect semigroup is Stieltjes flat and every Stieltjes flat perfect semigroup is Stieltjes perfect. Thus, a $*$-semigroup is Stieltjes perfect if and only if it is perfect and Stieltjes flat. We will consider in detail the semigroup $(]1,\infty[\cap\mathbb{Q},+)$, which will turn out to be perfect but not Stieltjes flat (hence not Stieltjes perfect).

When a function φ and a measure μ are related by the equation $\varphi=\mathscr{L}\mu$, the measure μ is called a *representing measure* of φ. For background on positive definite and (Radon) moment functions on semigroups, see Berg, Christensen, and Ressel [4].

2. The main result

A complex-valued function on a $*$-semigroup S is *Stieltjes singular* if it vanishes on $S+S$. For $s\in S$ define $G_s=\{\sigma\in S^*\colon \sigma(s)\neq 0\}$ and $G_s^+=G_s\cap S_+^*$, so $G_s=\bigcup\limits_{n=1}^{\infty} G_{s,n}$. Note: if $s,t\in S$ then $G_{s^*}=G_s$ and $G_{s+t}=G_s\cap G_t$, hence $G_{s+s^*}=G_s$. A $*$-semigroup is *Stieltjes flat* if it admits no nonzero Stieltjes singular completely positive definite function.

Lemma 1. *A $*$-semigroup S is Stieltjes flat if and only if $S=S+S$.*

P r o o f. The 'if' part is trivial. For the converse, suppose $S\neq S+S$. Choose $a\in S\setminus(S+S)$ and let φ be the indicator function of the set $\{a\}$ as a subset of S. Then φ is completely positive definite ([13, Theorem 1]). Since φ vanishes on $S+S$ but does not vanish identically, this proves that S is not Stieltjes flat. $\square$

Lemma 2. *No nonzero Stieltjes singular function is a Stieltjes moment function.*

P r o o f. Suppose φ is a Stieltjes singular Stieltjes moment function on a $*$-semigroup S; we have to show $\varphi=0$. Choose $\mu\in G_+(S_+^*)$ such that $\varphi=\mathscr{L}\mu$. For $s\in S$, since φ is Stieltjes singular we have $0=\varphi(2s)=\int_{S_+^*}\sigma(2s)\,\mathrm{d}\mu(\sigma)=\int_{S_+^*}\sigma(s)^2\,\mathrm{d}\mu(\sigma)$, so $\mu(G_s^+)=0$, hence $0=\int_{S_+^*}\sigma(s)\,\mathrm{d}\mu(\sigma)=\varphi(s)$, as desired. $\square$

Corollary 1. *Every Stieltjes semiperfect semigroup (in particular, every Stieltjes perfect semigroup) is Stieltjes flat.*

If S is a perfect semigroup then so is $\tilde{S}$ [25]. If S is a $*$-semigroup having a zero 0 then $\sigma(0)=1$ for all $\sigma\in S^*$, as is readily seen. Let us call S *determinate* if for one, hence for all (see [16]), $N\in\mathbb{N}$, if μ and ν are measures on $\mathscr{A}(S^*)$ integrating $|\hat{s}|^N$ for all $s\in S$ and such that $\int\hat{s}\,\mathrm{d}\mu=\int\hat{s}\,\mathrm{d}\nu$ for all $s\in\overbrace{S+\ldots+S}^{N}$ then $\mu=\nu$. Let us call S *quasi-perfect* if S is determinate and for one, hence for all (see [16]), $N\geqslant 3$, if $\varphi\in\mathscr{P}(S)$ then there is a measure μ on $\mathscr{A}(S^*)$ such that $\varphi(s)=\int\hat{s}\,\mathrm{d}\mu$ for all $s\in\overbrace{S+\ldots+S}^{N}$. A function $\varphi\colon S\to\mathbb{C}$ is *singular* if $\varphi\big|S+S+S=0$. No nonzero singular function is a moment function (see the proof of Lemma 2). A $*$-semigroup is *flat* if it admits no nonzero singular positive definite function. By the preceding, every semiperfect semigroup (in particular, every perfect semigroup) is flat. In fact, a $*$-semigroup is perfect if and only if it is quasi-perfect and flat [11]. Every ideal of a quasi-perfect semigroup is quasi-perfect [16]. A measure μ is *concentrated* on a set X if $\mu(A)=0$ for every measurable set A disjoint with X.

Lemma 3. *Every Stieltjes flat perfect semigroup is Stieltjes perfect. The condition of Stieltjes flatness is strictly necessary.*

Proof. Suppose S is a Stieltjes flat perfect semigroup; we have to show that S is Stieltjes perfect. Suppose $\varphi\in\mathscr{P}_c(S)$; we have to show $\varphi\in\mathscr{H}_{S,\mathrm{det}}(S)$. If $r\in S$ then $E_r\varphi\in\mathscr{P}(\tilde{S})$. Since S is perfect, so is $\tilde{S}$. Thus there is a unique $\mu_r\in F_+(\tilde{S}^*)$ such that $E_r\varphi=\mathscr{L}\mu_r$, that is,

$$\varphi(r+x)=\int_{\tilde{S}^*}\xi(x)\,\mathrm{d}\mu_r(\xi) \tag{1}$$

for $x\in\tilde{S}$. If $r,s\in S$ then $\int_{\tilde{S}^*}\xi(x)\,\mathrm{d}\mu_{r+s}(\xi)=\varphi(r+s+x)=\int_{\tilde{S}^*}\xi(s+x)\,\mathrm{d}\mu_r(\xi)=\int_{\tilde{S}^*}\xi(x)\xi(s)\,\mathrm{d}\mu_r(\xi)$ for all $x\in\tilde{S}$. By the uniqueness of μ_{r+s} it follows that

$$\mu_{r+s}=\xi(s)\,\mathrm{d}\mu_r(\xi), \tag{2}$$

cf. [4, 6.5.2]. Applying this twice, we get

$$\xi(s)\,\mathrm{d}\mu_r(\xi)=\xi(r)\,\mathrm{d}\mu_s(\xi). \tag{3}$$

Hence, if we define $G_r=\{\xi\in\tilde{S}^*\colon\ \xi(r)\neq 0\}$ and $\lambda_r=\xi(r)^{-1}\,\mathrm{d}\mu_r(\xi)\big|G_r$ for $r\in S$ then

$$\lambda_r\big|(G_r\cap G_s)=\lambda_s\big|(G_r\cap G_s) \tag{4}$$

for all $r, s \in S$. Therefore, if we define $G = \bigcup_{r \in S} G_r$ and denote by $\mathscr{A}_*(G)$ the σ-ring of subsets of G consisting of those sets in $\mathscr{A}(\tilde{S}^*)$ which are contained in the union of countably many G_r then there is a unique measure μ on $\mathscr{A}_*(G)$ such that

$$\lambda_r = \mu \big| G_r \tag{5}$$

for each $r \in S$. We claim that

$$\mu_r \big| \mathscr{A}_*(G) = \xi(r)\,\mathrm{d}\mu(\xi) \tag{6}$$

for each $r \in S$. To see that this is so, since every set in $\mathscr{A}_*(G)$ is contained in the union of countably many G_s, it suffices to verify $\mu_r \big| G_s = \xi(r)\,\mathrm{d}\mu(\xi) \big| G_s$ for $s \in S$. By (5) this is equivalent to $\mu_r \big| G_s = \xi(r)\,\mathrm{d}\lambda_s(\xi) = \xi(r)\xi(s)^{-1}\mu_s \big| G_s$. But this follows from (3). This proves (6).

Since S is Stieltjes flat we have by Lemma 1 that $S = S + S$, that is, an arbitrary $t \in S$ can be written $t = r + s$ with $r, s \in S$. Substituing $x = s$ in (1) (and using the fact that the function $\xi \mapsto \xi(s)$ lives on the set G_s which, together with all its measurable subsets, is in $\mathscr{A}_*(G)$), we get (using also (6))

$$\varphi(t) = \int_G \xi(t)\,\mathrm{d}\mu(\xi) \tag{7}$$

for $t \in S$. Let us show that μ is concentrated on $\tilde{S}^*_+$. As in [8] it suffices to verify that if $r \in \tilde{S}$ then μ is concentrated on the set $\{\xi \in \tilde{S}^* \colon \xi(r) \geqslant 0\}$. If $r = 0$ then this is so since every character on $\tilde{S}$ takes the value 1 at 0. Thus we may assume $r \in S$. We then have to show that the measure $\xi(r)\,\mathrm{d}\mu(\xi)$ is positive. But it is equal to $\mu_r \big| \mathscr{A}_*(G)$, which is positive. What we have just shown implies, by (7), that φ is a Stieltjes moment function. To see that it is Stieltjes determinate, it suffices to note that the semigroup S, being perfect, is determinate, hence trivially 'Stieltjes determinate'. (The converse implication will be shown in Lemma 10.)

We have used without proof the fact that the mapping $\sigma \mapsto \tilde{\sigma} \colon S^* \to \tilde{S}^*$ defined by $\tilde{\sigma} \big| S = \sigma (\sigma \in S^*)$ is an isomorphism between the measurable spaces $(S^*, \mathscr{A}(S^*))$ and $(G, \mathscr{A}_*(G))$.

To see that the condition of Stieltjes flatness cannot be omitted, it suffices to exhibit a $*$-semigroup which is perfect but not Stieltjes perfect. Consider the semigroup $S = \,]1, \infty[\, \cap \mathbb{Q}$, which is a subsemigroup of the additive group of nonnegative rationals. Let us first show that S is perfect. It suffices to show that S is quasi-perfect and flat. It was shown by Berg [2] that every countable 2-divisible abelian semigroup with zero and the identical involution is perfect. This applies to $\mathbb{Q}_+$. In particular, $\mathbb{Q}_+$ is quasi-perfect. Hence, its ideal S is quasi-perfect. To see that S is flat, suppose

φ is a singular positive definite function on S; we have to show that $\varphi = 0$. Since $]3, \infty[\, \cap\, \mathbb{Q} = S + S + S$, φ vanishes on $]3, \infty[\, \cap\, \mathbb{Q}$. Given $u \in S + S$, we can choose $s, t \in S$ such that $u = s + t$. Then $\varphi(u)^2 \leqslant \varphi(2s)\varphi(2t)$. Thus it suffices to show $\varphi(2s) = 0$ for all $s \in S$. Define sequences $(s_n)_{n=0}$ and $(t_n)_{n=1}^{\infty}$ of rationals by the conditions that $s_0 = s$ and that if $n \in \mathbb{N}$ then $t_n = \frac{1}{2}(1 + s_{n-1})$ and $2s_{n-1} = s_n + t_n$. Let us show by induction that these are sequences in S. First, $s_0 = s \in S$. Now suppose that $n \in \mathbb{N}$ and $s_{n-1} \in S$. Since $s_{n-1} > 1$, we have $t_n = \frac{1}{2}(1 + s_{n-1}) > 1$, so $t_n \in S$. Now $s_n = 2s_{n-1} - t_n = 2s_{n-1} - \frac{1}{2}(1 + s_{n-1}) > 2s_{n-1} - \frac{1}{2}(s_{n-1} + s_{n-1}) = s_{n-1} > 1$, so $s_n \in S$. For each n, we have $\varphi(2s_{n-1})^2 = \varphi(s_n + t_n)^2 \leqslant \varphi(2s_n)\varphi(2t_n)$. Thus, if $\varphi(2s_n) = 0$ for some n then also for $n - 1$ instead of n, leading by induction to the desired conclusion. Therefore, it suffices to show that $\varphi(2s_n) = 0$ for all sufficiently large n. Since we have already shown that φ vanishes on $]3, \infty[\, \cap\, \mathbb{Q}$ it suffices to show that $s_n \to \infty$ as $n \to \infty$. But $s_n = 2s_{n-1} - \frac{1}{2}(1 + s_{n-1}) = \frac{1}{2}(3s_{n-1} - 1)$, so $s_n - 1 = \frac{3}{2}(s_{n-1} - 1)$, whence by induction $s_n - 1 = \left(\frac{3}{2}\right)^n (s_0 - 1) \to \infty$. Thus S is perfect. It remains to be shown that S is not Stieltjes perfect. By Corollary 1 and Lemma 1, it suffices to show $S \neq S + S$, which is obvious. □

The Dirac measure at a point x is denoted by ε_x.

Theorem 1. *If S is a Stieltjes perfect semigroup, so is $\tilde{S}$.*

Proof. Since the paper is already quite long, we merely refer to the corresponding result for perfectness instead of Stieltjes perfectness [25]. □

A $*$-semigroup S is quasi-perfect if and only if $\tilde{S}$ is perfect [16].

A homomorphism h of one $*$-semigroup into another is a *$*$-homomorphism* if $h(s^*) = h(s)^*$ for all s in the domain.

Lemma 4. *Suppose h is a $*$-homomorphism of a $*$-semigroup S into a $*$-semigroup T. If $\varphi \in \mathscr{P}_c(T)$ then $\varphi \circ h \in \mathscr{P}_c(S)$.*

Proof. The zeros of $\tilde{S}$ and $\tilde{T}$ can both be denoted by 0 with no risk of confusion. Extend h to a $*$-homomorphism $\tilde{h}$ of $\tilde{S}$ into $\tilde{T}$ by setting $\tilde{h}(0) = 0$. If $r \in S$ then $E_r(\varphi \circ h) = (E_{h(r)}\varphi) \circ \tilde{h} \in \mathscr{P}(\tilde{S})$. We have used the well-known fact that positive definiteness is preserved under composition with $*$-homomorphisms. □

Theorem 2. *Every $*$-homomorphic image of a flat semigroup is flat. Similarly for Stieltjes flatness.*

Proof. Suppose h is a $*$-homomorphism of a flat semigroup S onto a $*$-semigroup T; we have to show that T is flat. Suppose $\varphi \in \mathscr{P}(T)$ and $\varphi\big|(T + T + T) = 0$; we have to show $\varphi = 0$ on $T + T$. We have $\varphi \circ h \in \mathscr{P}(S)$. Since $h(S + S + S) =$

$h(S)+h(S)+h(S)=T+T+T$ we have $(\varphi\circ h)\big|(S+S+S)=0$. Since S is flat it follows that $\varphi\circ h=0$ on $S+S$. Since $h(S)=T$ it follows that $\varphi=0$ on $T+T$. The case of Stieltjes flatness is obvious from Lemma 1. $\square$

For every $*$-semigroup S we define $S^{\#}=\{s+s^*\colon\ s\in S\}$. Note that $S^{\#}$ is the image of S under the $*$-homomorphism $s\mapsto s+s^*\colon\ S\to S$. In particular, $S^{\#}$ is a $*$-semigroup. Note that it carries the identical involution.

Lemma 5. *If S is a $*$-semigroup and $\varphi\in\mathscr{P}(S)$ then $\varphi\big|S^{\#}\in\mathscr{P}_c(S^{\#})$.*

Proof. For ease of notation, write $T=S^{\#}$. Since T is a $*$-subsemigroup of S, we can identify $\tilde T$ with a $*$-subsemigroup of $\tilde S$ by identifying the zeros of the two semigroups. Suppose $r\in T$; we have to show $E_r\varphi\in\mathscr{P}(\tilde T)$. Choose $a\in S$ such that $r=a+a^*$. Suppose $n\in\mathbb{N}$, $x_1,\dots,x_n\in\tilde T$, and $c_1,\dots,c_n\in\mathbb{C}$; we have to show $\sum\limits_{j,k=1}^{n}c_j\overline{c_k}\varphi(r+x_j+x_k^*)\geqslant 0$. But the left-hand side is equal to $\sum\limits_{j,k=1}^{n}c_j\overline{c_k}\varphi(s_j+s_k^*)$ where $s_j=a+x_j\in S$ for $j=1,\dots,n$; hence it is nonnegative by the positive definiteness of φ. $\square$

Theorem 3. *Every Stieltjes flat semigroup is flat. The converse is false.*

Proof. By Lemma 1, every Stieltjes flat semigroup S satisfies $S=S+S$, hence $S=S+S+S$, whence S is trivially flat. The converse is false by Lemma 3 and the fact that every perfect semigroup is flat. $\square$

Suppose S is a $*$-semigroup. A family $(\lambda_s)_{s\in S+S}$ of complex-valued functions on a set $\mathscr{A}$ is *positive definite* if for each $A\in\mathscr{A}$ the function $s\mapsto\lambda_s(A)\colon\ S+S\to\mathbb{C}$ is positive definite.

If $\mathscr{A}$ and $\mathscr{B}$ are σ-rings then $\mathscr{A}\otimes\mathscr{B}$ denotes the σ-ring generated by the set of all sets of the form $A\times B$ with $A\in\mathscr{A}$ and $B\in\mathscr{B}$. A function $\theta\colon\ \mathscr{A}\times\mathscr{B}\to[0,\infty]$ is a *bimeasure* if for each $A\in\mathscr{A}$ the function $B\mapsto\theta(A,B)\colon\ \mathscr{B}\to[0,\infty]$ is a measure and for each $B\in\mathscr{B}$ the function $A\mapsto\theta(A,B)\colon\ \mathscr{A}\to[0,\infty]$ is a measure. A bimeasure θ is *induced* by a measure μ on $\mathscr{A}\otimes\mathscr{B}$ (or on some larger σ-ring) if $\theta(A,B)=\mu(A\times B)$ for all $A\in\mathscr{A}$ and $B\in\mathscr{B}$. Even a finite bimeasure on the product of two σ-fields is not always induced by a measure, cf. [4, 2.1.31]. However, if $\mathscr{A}$ and $\mathscr{B}$ are $\mathscr{A}(S^*)$ and $\mathscr{A}(T^*)$ for two $*$-semigroups S and T with zeros then the answer is affirmative for finite bimeasures, cf. [15, Lemma 1]. We shall need the fact that zeros are not needed and that the result remains true for bimeasures that are only assumed to be finite on $\mathscr{A}_0(S^*)\times\mathscr{A}_0(T^*)$. No new ideas are needed for the proof; see [15]. Also, one can replace S^* by S^*_+, or T^* by T^*_+, or both. In this case, if (say) S^* is replaced by S^*_+ then at the point in the proof in [15] where 'saturated subsemigroups' are used, apply Lemma 3.1 in [8]. (Or consider the following argument: if ω is a nonnegative character on U and if σ is a character on S that extends ω then $|\sigma|$ is a nonnegative character on S that extends ω.)

The indicator function of a set X is denoted by 1_X.

A $*$-semigroup S is *Stieltjes determinate of order* $N \in \mathbb{N}$ if whenever μ and ν are measures on $\mathscr{A}(S_+^*)$ integrating $\hat{s}^N$ for all $s \in S$ and such that $\int \hat{s}\,\mathrm{d}\mu = \int \hat{s}\,\mathrm{d}\nu$ for all $s \in \overbrace{S + \ldots + S}^{N}$ then $\mu = \nu$. As in the case of ordinary determinacy (see [16]), a $*$-semigroup is Stieltjes determinate of some order if and only if it is Stieltjes determinate of every order. It is then said to be *Stieltjes determinate*. Clearly, a $*$-semigroup is Stieltjes perfect if and only if it is Stieltjes semiperfect and Stieltjes determinate.

Given a subset M of $\mathbb{C}$, a $*$-semigroup S is said to be *M-separative* if the M-valued characters on S separate points in S. The *greatest M-separative $*$-homomorphic image* of S is the quotient $*$-semigroup $S/{\sim}$ where $\sim$ is the congruence relation in S defined by the condition that $s \sim t$ if and only if $\sigma(s) = \sigma(t)$ for every M-valued character σ on S. If f is a $*$-homomorphism of S into an M-separative semigroup T then there is a unique $*$-homomorphism g of $S/{\sim}$ into T such that $f = g \circ h$ where h is the quotient mapping of S onto $S/{\sim}$.

Let $\mathbb{T}$ be the complex unit circle and write $\mathbb{T}_0 = \mathbb{T} \cup \{0\}$ for brevity. A $*$-semigroup is $\mathbb{T}_0$-separative if and only if it is an abelian inverse semigroup (see Warne and Williams [29]). Note that every character on an abelian inverse semigroup is $\mathbb{T}_0$-valued. For every $*$-semigroup S we denote by π (or π_S, if S has to be specified) the quotient mapping of S onto its greatest $\mathbb{T}_0$-separative $*$-homomorphic image.

Lemma 6. *If S is a $*$-semigroup then $\pi(S)$ can be identified with the quotient $*$-semigroup $S/{\sim}$ where $\sim$ is the least equivalence relation containing the binary relation R in S defined by the condition that sRt if and only if there exist $a \in S$ and $b \in \tilde{S}$ such that $s = a + b$ and $t = a + a^* + a + b$.*

P r o o f. We first verify that $\sim$ is a congruence relation and $S/{\sim}$ is $\mathbb{T}_0$-separative. Since $\sim$ is an equivalence relation by definition, in order to verify that it is a congruence relation we need only verify that if $s, t \in S$ are such that $s \sim t$ then $s^* \sim t^*$ and $s + u \sim t + u$ for all $u \in S$. If sRt, choose $a \in S$ and $b \in \tilde{S}$ such that $s = a + b$ and $t = a + a^* + a + b$. Then $s^* = a^* + b^*$ and $t^* = a^* + (a^*)^* + a^* + b^*$, so $s^* R t^*$ and in particular, $s^* \sim t^*$. Since the binary relation M in S defined by the condition that sMt if and only if $s^* \sim t^*$ is thus an equivalence relation containing R, and since $\sim$ is the least such equivalence relation, we infer that M contains $\sim$, that is, if $s \sim t$ then $s^* \sim t^*$. By a similar argument, in order to show that if $s \sim t$ then $s + u \sim t + u$ for $u \in S$, it suffices to show the conclusion under the hypothesis sRt. Choosing $a \in S$ and $b \in \tilde{S}$ such that $s = a + b$ and $t = a + a^* + a + b$, we have $s + u = a + c$

and $t+u = a+a^*+a+c$ where $c = b+u$, hence $s+uRt+u$ and in particular, $s+u \sim t+u$. Thus $\sim$ is a congruence relation. Therefore $S/{\sim}$ is a $*$-semigroup. Denote the quotient mapping of S onto $S/{\sim}$ by h. If $a \in S$ then $aRa+a^*+a$ and in particular, $a \sim a+a^*+a$, that is, $h(a) = h(a+a^*+a) = h(a)+h(a)^*+h(a)$. This being so for every such a, since $S/{\sim} = h(S)$ we infer that $u = u+u^*+u$ for all $u \in S/{\sim}$, that is, $S/{\sim}$ is an abelian inverse semigroup. By the above, this means that $S/{\sim}$ is $\mathbb{T}_0$-separative. Since h is a $*$-homomorphism of S into the $\mathbb{T}_0$-separative semigroup $S/{\sim}$, there is a unique $*$-homomorphism g of $\pi(S)$ into $S/{\sim}$ such that $h = g \circ \pi$. Since $S/{\sim} = h(S)$ we have $S/{\sim} = g(\pi(S))$. To see that g is an isomorphism we only need to verify that it is one-to-one. Let us define a mapping f of $S/{\sim}$ into $\pi(S)$ by the condition that $f(h(s)) = \pi(s)$ for $s \in S$. To see that this is a proper definition, we have to verify that if $s, t \in S$ are such that $s \sim t$ then $\pi(s) = \pi(t)$. Since the binary relation L in S defined by the condition that sLt if and only if $\pi(s) = \pi(t)$ is an equivalence relation, we may assume sRt. Choose $a \in S$ and $b \in \tilde{S}$ such that $s = a+b$ and $t = a+a^*+a+b$. Suppose σ is a $\mathbb{T}_0$-valued character on S. Since $|\sigma(a)| \in \{0,1\}$ then $\sigma(a) = \sigma(a)|\sigma(a)|^2$, so $\sigma(s) = \sigma(a)\sigma(b) = \sigma(a)|\sigma(a)|^2\sigma(b) = \sigma(t)$. This being so for every such σ, we infer $\pi(s) = \pi(t)$, as desired. Thus f is well-defined. Since $h = g \circ \pi$ and $\pi = f \circ h$ we have $\pi = f \circ (g \circ \pi) = (f \circ g) \circ \pi$. Since π maps S onto $\pi(S)$, we conclude that $f \circ g$ is the identity on $\pi(S)$. Therefore g is one-to-one. This completes the proof. $\square$

If μ is a measure and f a mapping then we denote by μ^f the image measure of μ under f whenever this makes sense. If f and g are complex-valued functions on sets X and Y, respectively, then $f \otimes g$ denotes the function $(x,y) \mapsto f(x)g(y)$ on $X \times Y$. The absolute value of a measure μ is denoted by $|\mu|$. If X is a topological space then its Borel field is denoted by $\mathscr{B}(X)$.

Lemma 7. *Suppose S is a $*$-semigroup. For $\sigma \in S^*$ define a $\mathbb{T}_0$-valued character $\operatorname{sgn}\sigma$ on S by*

$$\operatorname{sgn}\sigma(s) = \begin{cases} \sigma(s)/|\sigma(s)| & \text{if } \sigma(s) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then there is a unique character $f(\sigma)$ on $\pi(S)$ such that $\operatorname{sgn}\sigma = f(\sigma) \circ \pi$. The mapping $f\colon S^* \to \pi(S)^*$ is measurable with respect to the σ-rings $\mathscr{A}(S^*)$ and $\mathscr{A}(\pi(S)^*)$. The mapping $\sigma \mapsto |\sigma|\colon S^* \to S_+^*$ is measurable with respect to the σ-rings $\mathscr{A}(S^*)$ and $\mathscr{A}(S_+^*)$, so the mapping $g\colon S^* \to \pi(S)^* \times S_+^*$ defined by $g(\sigma) = (f(\sigma), |\sigma|)$ for $\sigma \in S^*$ is measurable with respect to the σ-rings $\mathscr{A}(S^*)$ and $\mathscr{A}(\pi(S)^*) \otimes \mathscr{A}(S_+^*)$. The mapping $h\colon \pi(S)^* \times S_+^* \to S^*$ defined by $h(\omega, \varrho) = (\omega \circ \pi)\varrho$ for $\omega \in \pi(S)^*$ and $\varrho \in S_+^*$ is measurable with respect to the σ-rings mentioned (in reverse order). Moreover, $h \circ g$ is the identity on S^*. Hence, the mapping $\nu \mapsto \nu^g$ taking mea-

sures on $\mathscr{A}(S^*)$ to measures on $\mathscr{A}(\pi(S)^*)\otimes\mathscr{A}(S_+^*)$ is one-to-one. A measure μ on $\mathscr{A}(\pi(S)^*)\otimes\mathscr{A}(S_+^*)$ which is finite on sets of the form $A\times B$ with $A\in\mathscr{A}_0(\pi(S)^*)$ and $B\in\mathscr{A}_0(S_+^*)$ is the image under g of a measure on $\mathscr{A}(S^*)$ if and only if $\mu((G_{\pi(t)}\times G_s^+)\setminus(G_{\pi(s+t)}\times G_{s+t}^+))=0$ for all $s,t\in S$. When this is so, the unique measure ν on $\mathscr{A}(S^*)$ such that $\mu=\nu^g$ is given by $\nu=\mu^h$.

Now suppose $\nu\in F_+(S^*)$. Define $\mu=\nu^g$. For $B\in\mathscr{A}_0(S_+^*)$ define a measure μ^B on $\mathscr{A}(\pi(S)^*)$ by $\mu^B(A)=\mu(A\times B)$ for $A\in\mathscr{A}(\pi(S)^*)$. Then $\mu^B\in F_+(\pi(S)^*)$. For $u\in\pi(S)$ define a mapping $\mu_u\colon\ \mathscr{A}_0(S_+^*)\to\mathbb{C}$ by $\mu_u(B)=\mathscr{L}\mu^B(u)$ for $B\in\mathscr{A}_0(S_+^*)$. Then μ_u is a complex measure. For $s\in S$ the measure $|\mu_{\pi(s)}|$ is concentrated on the set G_s^+. For $s\in S$ define $\mu_s=\varrho(s)\,\mathrm{d}\mu_{\pi(s)}$. Then $\mu_s\in F(S_+^*)$ and $\mathscr{L}\nu(s+x)=\mathscr{L}\mu_s(x)$ for all $x\in S^\#$.

P r o o f. To see that $\operatorname{sgn}\sigma$ is a character if $\sigma\in S^*$, first note that since σ is nonzero, so is $\operatorname{sgn}\sigma$. Obviously $\operatorname{sgn}\sigma$ is hermitian. It remains to be shown that $\operatorname{sgn}\sigma(s+t)=\operatorname{sgn}\sigma(s)\operatorname{sgn}\sigma(t)$ for $s,t\in S$. If either $\sigma(s)$ or $\sigma(t)$ is zero then both sides are zero. In the complementary case, $\operatorname{sgn}\sigma(s+t)=\sigma(s+t)/|\sigma(s+t)|=\sigma(s)\sigma(t)/|\sigma(s)\sigma(t)|=\operatorname{sgn}\sigma(s)\operatorname{sgn}\sigma(t)$, as desired. Thus $\operatorname{sgn}\sigma$ is a character. The existence and uniqueness of $f(\sigma)$ follows by the definition of $\pi(S)$. Since we are considering only one σ-ring on each set, we shall just speak of 'measurability' of a mapping without specifying the σ-ring. To see that f is measurable, by the definition of $\mathscr{A}(\pi(S)^*)$ it suffices to verify that for $u\in\pi(S)$ the mapping $f_u\colon\ S^*\to\mathbb{C}$ defined by $f_u(\sigma)=f(\sigma)(u)$ for $\sigma\in S^*$ is measurable. Choosing $s\in S$ such that $u=\pi(s)$, we have $f_u(\sigma)=f(\sigma)(\pi(s))=\operatorname{sgn}\sigma(s)$. Since the mapping $\sigma\mapsto\sigma(s)$ is measurable by the definition of $\mathscr{A}(S^*)$, it now suffices to note that the mapping $z\mapsto z/|z|$ of $\mathbb{C}\setminus\{0\}$ into itself is measurable. Thus f is measurable. The measurability of the mapping $\sigma\mapsto|\sigma|$ follows by a similar argument. To see that h is measurable it suffices to verify that for $s\in S$ the function $(\omega,\varrho)\mapsto(\omega\circ\pi(s))\varrho(s)$ is measurable. But it is the product of two measurable functions. To see that $h\circ g$ is the identity on S^*, suppose $\sigma\in S^*$; we have to show $\sigma=h(g(\sigma))$, that is, $\sigma(s)=h(g(\sigma))(s)$ for $s\in S$. But $h(g(\sigma))(s)=h(f(\sigma),|\sigma|)(s)=(f(\sigma)\circ\pi)(s)|\sigma|(s)=\operatorname{sgn}\sigma(s)|\sigma(s)|=\sigma(s)$. Since $h\circ g$ is the identity on S^* we have $\nu=\nu^{h\circ g}=(\nu^g)^h$ for every measure ν on $\mathscr{A}(S^*)$, showing that the mapping $\nu\mapsto\nu^g$ is one-to-one.

Suppose μ is a measure on $\mathscr{A}(\pi(S)^*)\otimes\mathscr{A}(S_+^*)$ which is finite on sets of the form $A\times B$ with $A\in\mathscr{A}_0(\pi(S))$ and $B\in\mathscr{A}_0(S_+^*)$; we have to show that μ is the image under g of a measure on $\mathscr{A}(S^*)$ if and only if it satisfies the condition in the statement. If ν is any measure on $\mathscr{A}(S^*)$ such that $\mu=\nu^g$ then, since $h\circ g$ is the identity on S^*, it follows that $\nu=\nu^{h\circ g}=(\nu^g)^h=\mu^h$. Thus there is only one candidate for such a measure. We see that a necessary and sufficient condition is that $\mu=(\mu^h)^g=\mu^{g\circ h}$.

Since $\mathscr{A}(\pi(S)^*) \otimes \mathscr{A}(S_+^*)$ is generated (as a σ-ring) by the set of all sets of the form $A \times B$ with $A \in \mathscr{A}_0(\pi(S))$ and $B \in \mathscr{A}_0(S_+^*)$, a formally weaker but in fact equivalent condition is that $\mu(A \times B) = \mu((g \circ h)^{-1}(A \times B))$ for all $A \in \mathscr{A}_0(\pi(S)^*)$ and $B \in \mathscr{A}_0(S_+^*)$. (We are using the fact that a measure on a σ-ring is uniquely determined by its restriction to a generating subring on which it is finite, cf. [18, Theorem A, p. 53].)

To get further, we need to study the mapping $g \circ h$. For $\omega \in \pi(S)^*$ and $\varrho \in S_+^*$ we have $g \circ h(\omega, \varrho) = g(h(\omega, \varrho)) = g((\omega \circ \pi)\varrho) = (f((\omega \circ \pi)\varrho), |(\omega \circ \pi)\varrho|)$. Now $f((\omega \circ \pi)\varrho)$ is characterized by the fact that $f((\omega \circ \pi)\varrho) \circ \pi = \operatorname{sgn}((\omega \circ \pi)\varrho)$. It is a trivial exercise to verify that $\operatorname{sgn}(\sigma\tau) = (\operatorname{sgn}\sigma)\operatorname{sgn}\tau$ for all $\sigma, \tau \in S^*$. Thus $f((\omega \circ \pi)\varrho) \circ \pi = (\operatorname{sgn}(\omega \circ \pi))\operatorname{sgn}\varrho$. Since $\omega \circ \pi$ already takes values in $\mathbb{T}_0$ we have $\operatorname{sgn}(\omega \circ \pi) = \omega \circ \pi$, so $f((\omega \circ \pi)\varrho) \circ \pi = (\omega \circ \pi)\operatorname{sgn}\varrho = (\omega \circ \pi)(f(\varrho) \circ \pi) = (\omega f(\varrho)) \circ \pi$. Since the range of π is the domain of definition of $f((\omega \circ \pi)\varrho)$, it follows that $f((\omega \circ \pi)\varrho) = \omega f(\varrho)$. Thus $g \circ h(\omega, \varrho) = (\omega f(\varrho), |(\omega \circ \pi)\varrho|)$. But since ϱ is nonnegative we have $|(\omega \circ \pi)\varrho| = |\omega \circ \pi|\varrho = (|\omega| \circ \pi)\varrho$. Thus $g \circ h(\omega, \varrho) = (\omega f(\varrho), (|\omega| \circ \pi)\varrho)$.

We see from this that a necessary and sufficent condition is that $\mu(A \times B) = \mu(\{(\omega, \varrho) \in \pi(S)^* \times S_+^* \colon \omega f(\varrho) \in A, (|\omega| \circ \pi)\varrho \in B\})$ for all $A \in \mathscr{A}_0(\pi(S)^*)$ and $B \in \mathscr{A}_0(S_+^*)$. By the definition of $\mathscr{A}(\pi(S)^*)$, this σ-ring is generated by sets of the form $\{\omega \in \pi(S)^* \colon \omega(u) \in X\}$ with $u \in \pi(S)$ and $X \in \mathscr{B}(\mathbb{T})$. Thus we may assume that A has this form. Similarly for B. In other words, it is necessary and sufficient that $\mu(\{(\omega, \varrho) \in \pi(S)^* \times S_+^* \colon \omega(u) \in X, \varrho(s) \in Y\}) = \mu(\{(\omega, \varrho) \in \pi(S)^* \times S_+^* \colon \omega(u)f(\varrho)(u) \in X, |\omega(\pi(s))|\varrho(s) \in Y\})$ for all $u \in \pi(S)$, $X \in \mathscr{B}(\mathbb{T})$, $s \in S$, and $Y \in \mathscr{B}([0, \infty[)$. Since $u = \pi(t)$ for some $t \in S$, an equivalent condition is that $\mu(\{(\omega, \varrho) \in \pi(S)^* \times S_+^* \colon \omega(\pi(t)) \in X, \varrho(s) \in Y\}) = \mu(\{(\omega, \varrho) \in \pi(S)^* \times S_+^* \colon \omega(\pi(t))\operatorname{sgn}\varrho(t) \in X, |\omega(\pi(s))|\varrho(s) \in Y\})$ for all $s, t \in S$, $X \in \mathscr{B}(\mathbb{T})$, and $Y \in \mathscr{B}(]0, \infty[)$. Now since ϱ is nonnegative, $\operatorname{sgn}\varrho$ is $\{0,1\}$-valued. Since $0 \notin X$ it follows that $\omega(\pi(t))\operatorname{sgn}\varrho(t) \in X$ if and only if $\omega(\pi(t)) \in X$ and $\varrho(t) > 0$. Similarly, $|\omega(\pi(s))|\varrho(s) \in Y$ if and only if $\varrho(s) \in Y$ and $\omega(\pi(s)) \neq 0$. Thus a necessary and sufficient condition is that $\mu(\{(\omega, \varrho) \in \pi(S)^* \times S_+^* \colon \omega(\pi(t)) \in X, \varrho(s) \in Y\}) = \mu(\{(\omega, \varrho) \in \pi(S)^* \times S_+^* \colon \omega(\pi(t)) \in X, \varrho(s) \in Y, \varrho(t) > 0, \omega(\pi(s)) \neq 0\})$ for all data as above.

Now the set on the right-hand side is a subset of the one in the left-hand side, so if the latter has finite measure then the condition is equivalent to the measure of the set difference being 0. By the hypothesis on μ, if the condition is satisfied then we can apply it with $X = \mathbb{T}$ and $Y = [1/n, \infty[$ for any $n \in \mathbb{N}$. Then the set difference mentioned has measure 0, and letting $n \to \infty$ we get $\mu((G_{\pi(t)} \times G_s^+) \setminus [(G_{\pi(t)} \cap G_{\pi(s)}) \times (G_s^+ \cap G_t^+)]) = 0$.

Conversely, if the last condition holds then it clearly follows that we have the desired identity for all X and Y. To get the condition in the statement it now suffices to note that $G_{\pi(s)} \cap G_{\pi(t)} = G_{\pi(s)+\pi(t)} = G_{\pi(s+t)}$ and similarly for the other set.

Now suppose $\nu \in F_+(S^*)$. To see that $\mu^B \in F_+(\pi(S)^*)$ for $B \in \mathscr{A}_0(S_+^*)$, we have to show $\int |\hat{u}| \, \mathrm{d}\mu^B < \infty$ for $u \in \pi(S)$. Since $\pi(S)$ is an abelian inverse semigroup we have $|\hat{u}| \leqslant 1$, so it suffices to show $\mu^B(G_u) < \infty$, that is, $\mu(G_u \times B) < \infty$. By the definition of $\mathscr{A}_0(S_+^*)$ it suffices to show $\mu(G_u \times G_{s,n}^+) < \infty$ for $s \in S$ and $n \in \mathbb{N}$. But since $\nu \in F_+(S^*)$ we have $\infty > \int |\sigma(s)| \, \mathrm{d}\nu(\sigma) = \int \varrho(s) \, \mathrm{d}\mu(\omega, \varrho) \geqslant \int_{G_u \times G_{s,n}^+} \varrho(s) \, \mathrm{d}\mu(\omega, \varrho) \geqslant \mu(G_u \times G_{s,n}^+)/n$. Thus $\mu^B \in F_+(\pi(S)^*)$. To see that μ_u is a complex measure for $u \in \pi(S)$, suppose $(B_n)_{n=1}^\infty$ is a sequence of pairwise disjoint sets in $\mathscr{A}_0(S_+^*)$ such that the set $B = \bigcup_{n=1}^\infty B_n$ is again in $\mathscr{A}_0(S_+^*)$; we have to show that $\mathscr{L}\mu^B(u) = \sum_{n=1}^\infty \mathscr{L}\mu^{B_n}(u)$, that is, $\int \hat{u} \otimes 1_B \, \mathrm{d}\mu = \sum_{n=1}^\infty \int \hat{u} \otimes 1_{B_n} \, \mathrm{d}\mu$. Since the function $\hat{u}$ is bounded (by 1), this follows by bounded convergence, recalling that $\mu(G_u \times B) < \infty$.

We next have to show that if $s \in S$ then the measure $|\mu_{\pi(s)}|$ is concentrated on the set G_s^+, or equivalently, on the set G_s. Suppose A is a measurable set disjoint with G_s; we have to show $|\mu_{\pi(s)}|(A) = 0$. There is a sequence $(H_n)_{n=1}^\infty$ of sets in $\{G_{t,n} \colon t \in S, n \in \mathbb{N}\}$ such that $A \subset \bigcup_{n=1}^\infty H_n$. Now $A = \bigcup_{n=1}^\infty (A \cap H_n)$. Moreover, for each n the set $A \cap H_n$ is in $\mathscr{A}_0(S_+^*)$ and is disjoint with G_s.

Thus it suffices to show that if $B \in \mathscr{A}_0(S_+^*)$ and $B \cap G_s = \emptyset$ then $|\mu_{\pi(s)}|(B) = 0$. By the definition of the absolute value of a measure, this amounts to showing that if $B \in \mathscr{A}_0(S_+^*)$ and $B \cap G_s = \emptyset$ then $\mu_{\pi(s)}(B) = 0$. (We are using the fact that $\mathscr{A}_0(S_+^*)$ is a hereditary subset of $\mathscr{A}(S_+^*)$, i.e., every measurable subset of a set in $\mathscr{A}_0(S_+^*)$ is again in $\mathscr{A}_0(S_+^*)$.) By the definition of μ_u for $u \in \pi(S)$, this amounts to showing that $\mathscr{L}\mu^B(\pi(s)) = 0$, that is, $\int_{\pi(S)^*} \omega(\pi(s)) \, \mathrm{d}\mu^B(\omega) = 0$. Since the function $\omega \mapsto \omega(\pi(s))$ vanishes off $G_{\pi(s)}$, it suffices to show $\mu^B(G_{\pi(s)}) = 0$. By the definition of μ^B, we have to show $\mu(G_{\pi(s)} \times B) = 0$. Since $\mu = \nu^g$, this is equivalent to showing $\nu(C) = 0$ where C is the set of those $\sigma \in S^*$ such that $f(\sigma) \in G_{\pi(s)}$ and $|\sigma| \in B$. Since ν is a measure it suffices to show $C = \emptyset$. So suppose $\sigma \in C$; we shall derive a contradiction. Since $f(\sigma) \in G_{\pi(s)}$ we have $0 \neq f(\sigma)(\pi(s)) = \operatorname{sgn} \sigma(s)$. This is equivalent to $\sigma(s) \neq 0$. It follows that $|\sigma(s)| \neq 0$, so $|\sigma| \in G_s$, contradicting the facts that $|\sigma| \in B$ and that B is disjoint with G_s.

For $A \in \mathscr{A}_0(\pi(S)^*)$ and $B \in \mathscr{A}_0(S_+^*)$ we have $\int 1_A \, \mathrm{d}\mu^B = \mu^B(A) = \mu(A \times B) = \int 1_A \otimes 1_B \, \mathrm{d}\mu$. The identity $\int f \, \mathrm{d}\mu^B = \int f \otimes 1_B \, \mathrm{d}\mu$, which thus holds if f is the indicator function of a measurable set, extends by standard measure-theoretic arguments first to simple measurable functions, then to nonnegative measurable functions, and finally to arbitrary measurable functions such that one of the integrals exists. Applying it to $f = \hat{u}$ with $u \in \pi(S)$, we obtain $\int \hat{u} \otimes 1_B \, \mathrm{d}\mu = \int \hat{u} \, \mathrm{d}\mu^B = \mathscr{L}\mu^B(u) =$

$\mu_u(B) = \int 1_B \,\mathrm{d}\mu_u$. The identity $\int \hat{u} \otimes g \,\mathrm{d}\mu = \int g \,\mathrm{d}\mu_u$, which thus holds if g is the indicator function of a measurable set, extends, etc. Applying it to $g = \hat{s}$ with $s \in \pi^{-1}(u)$, we obtain $\mathscr{L}\mu_s(x) = \int \hat{x} \,\mathrm{d}\mu_s = \int \varrho(x)\varrho(s) \,\mathrm{d}\mu_u(\varrho) = \int \varrho(s+x) \,\mathrm{d}\mu_u(\varrho) = \int \omega(u)\varrho(s+x) \,\mathrm{d}\mu(\omega, \varrho) = \int f(\sigma)(s)|\sigma(s+x)| \,\mathrm{d}\nu(\sigma) = \int \operatorname{sgn}\sigma(s)|\sigma(s)|\sigma(x) \,\mathrm{d}\nu(\sigma) = \int \sigma(s)\sigma(x) \,\mathrm{d}\nu(\sigma) = \int \sigma(s+x) \,\mathrm{d}\nu(\sigma) = \mathscr{L}\nu(s+x)$ for $x \in S^{\#}$, as desired. We have used the fact that $\sigma(x) \geqslant 0$ for $\sigma \in S^*$ since $x \in S^{\#}$. This completes the proof. □

If h is a $*$-homomorphism of a $*$-semigroup S into a $*$-semigroup T, we define the *dual mapping* $h^*\colon T^* \to S^* \cup \{0\}$ by $h^*(\tau) = \tau \circ h$ for $\tau \in T^*$. We cannot claim that h^* maps T^* into S^* in general, but note that if $h(S) = T$, the mapping $h^*(T^*) \subset S^*$. If μ is a measure on $\mathscr{A}(T^*)$, by abuse of notation we denote by μ^{h^*} the image measure of the measure $\mu\big|(h^*)^{-1}(S^*)$ under the mapping $h^*\big|(h^*)^{-1}(S^*)$. Clearly, if $\mu \in F_+(T^*)$ then $\mu^{h^*} \in F_+(S^*)$ and $\mathscr{L}(\mu^{h^*}) = (\mathscr{L}\mu) \circ h$. Similarly for measures on T^*_+.

If T is a $*$-subsemigroup of a $*$-semigroup S then we define $p_{S,T}\colon S^* \to T^* \cup \{0\}$ by $p_{S,T}(\sigma) = \sigma\big|T$ for $\sigma \in S^*$. If μ is a measure on $\mathscr{A}(S^*)$ then by abuse of notation we denote by $\mu^{p_{S,T}}$ the image measure of the measure $\mu\big|p_{S,T}^{-1}(T^*)$ under the mapping $p_{S,T}\big|p_{S,T}^{-1}(T^*)$. If $\mu \in F_+(S^*)$ then $\mu^{p_{S,T}} \in F_+(T^*)$ and $\mathscr{L}(\mu^{p_{S,T}}) = (\mathscr{L}\mu)\big|(T+T)$. Similarly for measures on S^*_+.

Lemma 8. *Every $*$-homomorphic image of a Stieltjes semiperfect semigroup is Stieltjes semiperfect.*

Proof. As the proof of the corresponding statement for semiperfectness ([15, Proposition 1]). □

Lemma 9. *Every $*$-homomorphic image of a Stieltjes determinate semigroup is Stieltjes determinate.*

Proof. Suppose h is a $*$-homomorphism of a Stieltjes determinate semigroup S onto a $*$-semigroup T; we have to show that T is Stieltjes determinate. Suppose $\mu, \nu \in G_+(T^*_+)$ and $\mathscr{L}\mu = \mathscr{L}\nu$; we have to show $\mu = \nu$. We have $\mu^{h^*}, \nu^{h^*} \in G_+(S^*_+)$ and $\mathscr{L}(\mu^{h^*}) = (\mathscr{L}\mu) \circ h = (\mathscr{L}\nu) \circ h = \mathscr{L}(\nu^{h^*})$. Since S is Stieltjes determinate it follows that $\mu^{h^*} = \nu^{h^*}$. Since $h(S) = T$ then the mapping h^* is one-to-one. It is tempting to conclude directly that $\mu = \nu$, but we think this is unsafe. Instead, note that (cf. [15]) $\mathscr{A}(S^*) = \bigcup\limits_{D \in \mathscr{D}(S)} p_{S,D}^{-1}(\mathscr{A}(D^*))$ where $\mathscr{D}(S)$ is the set of all countable $*$-subsemigroups of S. Moreover, for $D \in \mathscr{D}(S)$ the σ-ring $\mathscr{A}(D^*)$ is just the Borel field when D^* is considered with the topology of pointwise convergence (cf. [15]). Thus it suffices to show $\mu^{p_{T,E}} = \nu^{p_{T,E}}$ for each $E \in \mathscr{D}(T)$. Since E is countable and $h(S) = T$ we can choose $D \in \mathscr{D}(S)$ such that $E = h(D)$. Then $(\mu^{p_{T,E}})^{(h|D)^*} = (\mu^{h^*})^{p_{S,D}} =$

$(\nu^{h^*})^{p_{S,D}} = (\nu^{p_{T,E}})^{(h|D)^*}$. Since the mapping $(h\big|D)^*$ is a homeomorphism, we infer $\mu^{p_{T,E}} = \nu^{p_{T,E}}$, as desired. $\square$

Theorem 4. *Every $*$-homomorphic image of a Stieltjes perfect semigroup is Stieltjes perfect.*

P r o o f. A $*$-semigroup is Stieltjes perfect if and only if it is Stieltjes semiperfect and Stieltjes determinate. Use Lemmas 8 and 9. $\square$

Lemma 10. *A $*$-semigroup is Stieltjes determinate if and only if it is determinate.*

P r o o f. The 'if' part is trivial. Suppose S is a Stieltjes determinate semigroup; we have to show that S is determinate. The $*$-semigroup $S^\#$, being a $*$-homomorphic image of S, is Stieltjes determinate (Lemma 9), and that is all that we shall use. Suppose $\nu_1, \nu_2 \in F_+(S^*)$ and $\mathscr{L}\nu_1 = \mathscr{L}\nu_2$; we have to show $\nu_1 = \nu_2$. With g as in Lemma 7, define $\mu_i = \nu_i^g$ for $i = 1, 2$. By the same lemma it suffices to show $\mu_1 = \mu_2$. Since the σ-ring $\mathscr{A}(\pi(S)^*) \otimes \mathscr{A}(S_+^*)$ is generated by the set of all sets of the form $A \times B$ with $A \in \mathscr{A}_0(\pi(S)^*)$ and $B \in \mathscr{A}_0(S_+^*)$ it suffices to show $\mu_1(A \times B) = \mu_2(A \times B)$ for such A and B. For $i = 1, 2$ and $B \in \mathscr{A}_0(S_+^*)$, define a measure μ_i^B on $\mathscr{A}(\pi(S)^*)$ by $\mu_i^B(A) = \mu_i(A \times B)$ for $A \in \mathscr{A}(\pi(S)^*)$. We then have to show $\mu_1^B = \mu_2^B$. By the same lemma, $\mu_i^B \in F_+(\pi(S)^*)$ for $i = 1, 2$. Since $\pi(S)$, being an abelian inverse semigroup, is perfect it suffices to show $\mathscr{L}\mu_1^B = \mathscr{L}\mu_2^B$. For $i = 1, 2$ and $u \in \pi(S)$ define a complex measure $\mu_{i,u}$ on $\mathscr{A}_0(S_+^*)$ by $\mu_{i,u}(B) = \mathscr{L}\mu_i^B(u)$ for $B \in \mathscr{A}_0(S_+^*)$, cf. the same lemma. We now have to show $\mu_{1,u} = \mu_{2,u}$. For $i = 1, 2$ and $s \in S$ define $\mu_{i,s} = \varrho(s)\,\mathrm{d}\mu_{i,\pi(s)}(\varrho)$. By the same lemma, $\mu_{i,s} \in F(S_+^*)$ and $\mathscr{L}\mu_{1,s}(x) = \mathscr{L}\nu_1(s+x) = \mathscr{L}\nu_2(s+x) = \mathscr{L}\mu_{2,s}(x)$ for $x \in S^\#$. Since $S^\#$ is Stieltjes determinate it follows that $\mu_{1,s} = \mu_{2,s}$, that is, $\varrho(s)\,\mathrm{d}\mu_{1,u}(\varrho) = \varrho(s)\,\mathrm{d}\mu_{2,u}(\varrho)$ where $u = \pi(s)$. Hence $\mu_{1,u}\big|G_s^+ = \mu_{2,u}\big|G_s^+$. This completes the proof since the measures involved are concentrated on G_s^+, by the lemma cited. $\square$

If μ is a measure and f is a function then we denote by $f\mu$ the measure with density f with respect to μ whenever this makes sense.

Theorem 5. *A $*$-semigroup is Stieltjes perfect if and only if it is perfect and Stieltjes flat.*

P r o o f. The 'if' part is Lemma 3. For the converse, since every Stieltjes perfect semigroup is Stieltjes flat (Corollary 1) it suffices to show that every Stieltjes perfect semigroup is perfect. We will show below that every Stieltjes perfect semigroup with zero is perfect. Taking this for granted, we can now quickly complete the proof. Suppose S is a Stieltjes perfect semigroup; we have to show that S is perfect. Since

S is Stieltjes perfect, so is $\tilde{S}$ (Theorem 1). Since the latter semigroup has a zero it is perfect, that is, S is quasi-perfect. Since S is Stieltjes flat it is flat (Theorem 3). Being quasi-perfect and flat, S is perfect.

It remains to be shown that if S is a Stieltjes perfect semigroup with zero then S is perfect. Suppose $\varphi \in \mathscr{P}(S)$; we have to show $\varphi \in \mathscr{H}_{\mathrm{det}}(S)$. By [8, Lemma 3.5], the $*$-subsemigroup $T = S^{\#}$ of S is Stieltjes perfect. For $s \in S$ and $n \in \{0,1,2,3\}$, define $\varphi_{s,n}\colon S \to \mathbb{C}$ by $\varphi_{s,n}(x) = \varphi(x) + \mathrm{i}^n\varphi(s+x) + \mathrm{i}^{-n}\varphi(s^*+x) + \varphi(s+s^*+x)$ for $x \in S$. Then $\varphi_{s,n} \in \mathscr{P}(S)$. To see this, suppose $r \in \mathbb{N}$, $x_1, \ldots, x_r \in S$, and $c_1, \ldots, c_r \in \mathbb{C}$; we have to show $\sum\limits_{p,q=1}^{r} c_p\overline{c_q}\varphi_{s,n}(x_p + x_q^*) \geqslant 0$. But the left-hand side is equal to $\sum\limits_{p,q=1}^{2r} c_p\overline{c_q}\varphi(x_p + x_q^*)$ where $x_{r+p} = s + x_p$ and $c_{r+p} = \mathrm{i}^n c_p$ for $p = 1, \ldots, r$; hence it is nonnegative since $\varphi \in \mathscr{P}(S)$. Thus $\varphi_{s,n} \in \mathscr{P}(S)$. By Lemma 5 it follows that $\varphi_{s,n}\big|T \in \mathscr{P}_c(T)$. Since T is Stieltjes perfect there is a unique measure $\mu_{s,n} \in F_+(T_+^*)$ such that $\varphi_{s,n}\big|T = \mathscr{L}\mu_{s,n}$. Since $\varphi(s+x) = \frac{1}{4}\sum\limits_{n=0}^{3} \mathrm{i}^{-n}\varphi_{s,n}(x)$ for $x \in T$, it follows that

$$\varphi(s+x) = \int_{T_+^*} \varrho(x)\,\mathrm{d}\mu_s(\varrho) \tag{8}$$

for all $x \in T$ where μ_s is the complex measure $\frac{1}{4}\sum\limits_{n=0}^{3} \mathrm{i}^{-n}\mu_{s,n} \in F(T_+^*)$. Now the family $(\mu_s)_{s\in S}$ is positive definite (for the proof, to save space we refer to [4, p. 205, l. 7–18).

From the positive definiteness of the family (μ_s), by [8, Lemma 3.7], it follows that

$$\sum_{j,k=1}^{n} g_j\overline{g_k}\mu_{s_j+s_k^*} \geqslant 0 \tag{9}$$

whenever $n \in \mathbb{N}$, $s_1, \ldots, s_n \in S$, and $g_1, \ldots, g_n \in \bigcap\limits_{j,k=1}^{n} L^2(\mu_{s_j+s_k^*})$.

Define $U = \pi(S)$ for brevity. Let R and $\sim$ be as in Lemma 6, so U is canonically isomorphic to the quotient $*$-semigroup $S/{\sim}$. Since U is an abelian inverse semigroup, it is perfect.

For $a, s \in S$ we have

$$\mu_{a+a^*+s} = \varrho(a+a^*)\,\mathrm{d}\mu_s(\varrho). \tag{10}$$

Indeed, for $x \in T$ we have by (8), $\int_{T_+^*} \varrho(x)\,\mathrm{d}\mu_{a+a^*+s}(\varrho) = \varphi(a+a^*+s+x) = \int_{T_+^*} \varrho(a+a^*+x)\,\mathrm{d}\mu_s(\varrho) = \int_{T_+^*} \varrho(x)\varrho(a+a^*)\,\mathrm{d}\mu_s(\varrho)$. The above equality follows by the uniqueness of μ_s, cf. [4, 6.5.2].

Given $s, t \in S$ such that sRt, choose $a, b \in S$ such that $s = a+b$ and $t = 2a+a^*+b$. Since $t = a + a^* + s$, (10) implies $\mu_t = \varrho(a+a^*)\,\mathrm{d}\mu_s(\varrho)$. Since the mapping $\varrho \mapsto \varrho\big|T$ is a bijection of S_+^* onto T_+^* we see that T_+^* can be identified with S_+^*. With this identification,

$$\varrho(t)\,\mathrm{d}\mu_s(\varrho) = \varrho(s)\,\mathrm{d}\mu_t(\varrho). \tag{11}$$

Indeed, $\varrho(s)\,\mathrm{d}\mu_t(\varrho) = \varrho(a+b)\,\mathrm{d}\mu_t(\varrho) = \varrho(a+b)\varrho(a+a^*)\,\mathrm{d}\mu_s(\varrho) = \varrho(2a+a^*+b)\,\mathrm{d}\mu_s(\varrho) = \varrho(t)\,\mathrm{d}\mu_s(\varrho)$. Thus (11) holds if sRt. We shall see presently that it even holds under the weaker assumption $s \sim t$.

For s, t, a, and b as in the preceding paragraph, we have $G_s = G_a \cap G_b = G_t$. Since the binary relation Q in S defined by the condition that sQt if and only if $G_s = G_t$ is an equivalence relation containing R, and since $\sim$ is the least such equivalence relation, we infer that if $s, t \in S$ are such that $s \sim t$ then $G_s = G_t$. Hence, for $u \in U$ there is a unique subset H_u of S_+^* such that $H_u = G_s$ for all $s \in \pi^{-1}(u)$.

If $s, t \in S$ are such that sRt then by (11), $\varrho(s)^{-1}\,\mathrm{d}\mu_s(\varrho)\big|G_s = \varrho(t)^{-1}\,\mathrm{d}\mu_t(\varrho)\big|G_t$. (We have used the fact that $G_s = G_t$.) Since the binary relation P in S defined by the condition that sPt if and only if $\varrho(s)^{-1}\,\mathrm{d}\mu_s(\varrho)\big|G_s = \varrho(t)^{-1}\,\mathrm{d}\mu_t(\varrho)\big|G_t$ is an equivalence relation containing R, and since $\sim$ is the least such equivalence relation, we infer that

$$\varrho(s)^{-1}\,\mathrm{d}\mu_s(\varrho)\big|G_s = \varrho(t)^{-1}\,\mathrm{d}\mu_t(\varrho)\big|G_t \tag{12}$$

whenever $s, t \in S$ are such that $s \sim t$. It is now clear that (11) holds whenever $s, t \in S$ are such that $s \sim t$.

By (10) we have, in particular, $\mu_{s+s^*} = \varrho(s+s^*)\,\mathrm{d}\mu_0(\varrho)$ for each $s \in S$. Since the family $(\mu_s)_{s \in S}$ is positive definite we have $0 \leqslant \mu_0 + c\mu_s + \overline{c}\mu_{s^*} + |c|^2\mu_{s+s^*} = \mu_0 + c\mu_s + \overline{c}\mu_{s^*} + |c|^2\varrho(s)^2\,\mathrm{d}\mu_0(\varrho)$ for $s \in S$ and $c \in \mathbb{C}$. Hence, if A is a measurable set disjoint with G_s then $0 \leqslant \mu_0(A) + 2\mathrm{Re}(c\mu_s(A))$ for all $c \in \mathbb{C}$, so $\mu_s(A) = 0$. Thus μ_s is concentrated on G_s. By (12) we infer that for each $u \in U$ there is a unique complex measure μ_u on $\mathscr{A}_0(S_+^*)$ such that $\mu_s = \varrho(s)\,\mathrm{d}\mu_u(\varrho)$ for all $s \in \pi^{-1}(u)$. Moreover, μ_u is concentrated on H_u.

By (9), for $s \in S$ and $n \in \{0, 1, 2, 3\}$ we have $0 \leqslant \varrho(s)^2\,\mathrm{d}\mu_0(\varrho) + \mathrm{i}^n\varrho(s)\,\mathrm{d}\mu_s(\varrho) + \mathrm{i}^{-n}\varrho(s)\,\mathrm{d}\mu_{s^*}(\varrho) + \mu_{s+s^*} = 2\varrho(s)^2\,\mathrm{d}\beta_n(\varrho)$ where $\beta_n = \mu_0 + \varrho(s)^{-1}\mathrm{Re}(\mathrm{i}^n\mu_s)$. If A is a measurable subset of G_s then $\beta_n(A) \geqslant 0$ by the above since the function $\varrho \mapsto \varrho(s)^2$ is positive on G_s. On the other hand, if A is a measurable set disjoint with G_s then $\beta_n(A) = \mu_0(A) \geqslant 0$. (The measure μ_0 is positive since the family $(\mu_t)_{t \in S}$ is positive definite.) Collecting, we see that $\beta_n \geqslant 0$. Since $\sum\limits_{n=0}^{3} \beta_n = 4\mu_0 \in F_+(S_+^*)$, we see that $\beta_n \in F_+(S_+^*)$ for all $n \in \{0,1,2,3\}$. Hence $\varrho(s)^{-1}\,\mathrm{d}\mu_s = \frac{1}{4}\sum_{n=0}^{3} \mathrm{i}^{-n}\beta_n \in F(S_+^*)$. Comparing with the definition of μ_u for $u \in U$, we see that $\mu_u \in F(S_+^*)$ for all $u \in U$.

Let us show that the family $(\mu_u)_{u\in U}$ is positive definite. Suppose $n \in \mathbb{N}$, $u_1,\dots,u_n \in U$, and $c_1,\dots,c_n \in \mathbb{C}$; we have to show $\sum_{j,k=1}^{n} c_j\overline{c_k}\mu_{u_j+u_k^*} \geqslant 0$. For each $j \in \{1,\dots,n\}$ we can choose $s_j \in \pi^{-1}(u_j)$, and then the left-hand side is equal to $\sum_{j,k=1}^{n} f_j\overline{f_k}\mu_{s_j+s_k^*}$ where $f_j(\varrho) = c_j\varrho(s_j)^{-1}$. For $N \in \mathbb{N}$ and for each j, define

$$f_{j,N}(\varrho) = \begin{cases} f_j(\varrho) & if |f_j(\varrho)| \leqslant N, \\ 0 & \text{otherwise.} \end{cases}$$

Since the measures $|\mu_{s_j+s_k^*}|$ are bounded and the functions $f_{j,N}$ are bounded, the latter are in the space $\bigcap_{j,k=1}^{n} L^2(\mu_{s_j+s_k^*})$, so by (9), $\sum_{j,k=1}^{n} f_{j,N}\overline{f_{k,N}}\mu_{s_j+s_k^*} \geqslant 0$. The desired inequality follows by letting $N \to \infty$, by bounded convergence. (We are using the fact that $\varrho(s_j)^{-1}\varrho(s_k)^{-1}\,\mathrm{d}\mu_{s_j+s_k^*} = \mu_{\pi(s_j+s_k^*)}$, which is a finite measure.)

Thus the family $(\mu_u)_{u\in U}$ is positive definite. That is, for each $B \in \mathscr{A}(S_+^*)$ the function $u \mapsto \mu_u(B)\colon U \to \mathbb{C}$ is positive definite. Since U is perfect, there is a unique measure $\mu^B \in F_+(U^*)$ such that $\mu_u(B) = \mathscr{L}\mu^B(u)$ for all $u \in U$.

As in the proof of [4, 6.5.4], the function $(A,B) \mapsto \mu^B(A)\colon \mathscr{A}(U^*)\times\mathscr{A}(S_+^*) \to \mathbb{R}_+$ is a bimeasure. As in [15, Lemma 1], it follows that there is a unique measure μ on $\mathscr{A}(U^*)\otimes\mathscr{A}(S_+^*)$ such that $\mu^B(A) = \mu(A\times B)$ for all $A \in \mathscr{A}(U^*)$ and $B \in \mathscr{A}(S_+^*)$. (Since we have S_+^* instead of S^* it is necessary to adapt the proof of [15, Lemma 1], slightly.)

Defining g and h as in Lemma 7, set $\nu = \mu^h$. In order to be able to apply Lemma 7, we need to verify that $\mu = \nu^g$. By Lemma 7 it suffices to show $\mu((G_{\pi(t)} \times G_s^+) \setminus (G_{\pi(s+t)} \times G_{s+t}^+)) = 0$ for $s,t \in S$. 'Divide and conquer'. Since $(G_{\pi(t)} \times G_s^+) \setminus (G_{\pi(s+t)} \times G_{s+t}^+) = ((G_{\pi(t)} \setminus G_{\pi(s+t)}) \times G_s^+) \cup (G_{\pi(t)} \times (G_s^+ \setminus G_{s+t}^+))$ we precisely have to show $\mu((G_{\pi(t)} \setminus G_{\pi(s+t)}) \times G_s^+) = 0$ and $\mu(G_{\pi(t)} \times (G_s^+ \setminus G_{s+t}^+)) = 0$. For the first identity, since $G_s^+ = \bigcup_{n=1}^{\infty} G_{s,n}^+$, it suffices to show $\mu((G_{\pi(t)} \setminus G_{\pi(s+t)}) \times G_{s,n}^+) = 0$ for $n \in \mathbb{N}$. Now $G_{s,n}^+ \in \mathscr{A}_0(S_+^*)$ and $\omega(\pi(t+t^*)) = |\omega(\pi(t))|^2 = 1$ for $\omega \in G_{\pi(t)}$, so $\mu(G_{\pi(t)} \times G_{s,n}^+) = \mu^{G_{s,n}^+}(G_{\pi(t)}) = \mathscr{L}\mu^{G_{s,n}^+}(\pi(t+t^*)) = \mu_{\pi(t+t^*)}(G_{s,n}^+)$ and similarly for $s+t$ instead of t, so we have to show $\mu_{\pi(t+t^*)}(G_{s,n}^+) = \mu_{\pi(s+s^*+t+t^*)}(G_{s,n}^+)$. We can even show that the measures $\mu_{\pi(t+t^*)}$ and $\mu_{\pi(s+s^*+t+t^*)}$ coincide on the set G_s^+. Indeed, on this set, $\mu_{\pi(t+t^*)} = \varrho(t)^{-2}\,\mathrm{d}\mu_{t+t^*}(\varrho) = \varrho(s+t)^{-2}\varrho(s)^2\,\mathrm{d}\mu_{t+t^*}(\varrho) = \varrho(s+t)^{-2}\,\mathrm{d}\mu_{s+t^*+t+t^*}(\varrho) = \mu_{\pi(s+s^*+t+t^*)}$ where we have used (10) and the defining property of μ_u for $u \in \pi(S)$, noting that $\varrho(s+s^*) = |\varrho(s)|^2 = \varrho(s)^2$. For the second identity, by the same token it suffices to show $\mu(G_{\pi(t)} \times (G^+_{s,n} \setminus G^+_{s+t})) = 0$ for $n \in \mathbb{N}$. Now $G^+_{s,n} \setminus G^+_{s+t} \in \mathscr{A}_0(S^*_+)$ and $\mu(G_{\pi(t)} \times (G^+_{s,n} \setminus G^+_{s+t})) = \mu^{G^+_{s,n} \setminus G^+_{s+t}}(G_{\pi(t)}) = \mathscr{L}\mu^{G^+_{s,n} \setminus G^+_{s+t}}(\pi(t + t^*)) = \mu_{\pi(t+t^*)}(G^+_{s,n} \setminus G^+_{s+t}) = 0$ since $\mu_{\pi(t+t^*)}$ is concentrated on the set G^+_t and $(G^+_{s,n} \setminus G^+_{s+t}) \cap G^+_t \subset (G^+_s \setminus G^+_{s+t}) \cap G_t = (G^+_s \cap G^+_t) \setminus G^+_{s+t} = G^+_{s+t} \setminus G^+_{s+t} = \emptyset$.

Thus $\mu = \nu^g$. Arguing as in the proof of Lemma 7, we see that $\nu \in F_+(S^*)$ and $\mathscr{L}\nu(s) = \mathscr{L}\nu(s + 0) = \mathscr{L}\mu_s(0) = \varphi(s + 0) = \varphi(s)$ for $s \in S$. Thus φ is a moment function. It is determinate since the $*$-semigroup S, being Stieltjes perfect, is Stieltjes determinate, which is equivalent to its being determinate (Lemma 10). $\square$

By the proof of Theorem 5 and [8], Lemmas 3.2 and 3.5, we have the following corollary.

Corollary 2. *For an abelian $*$-semigroup S with zero the following four conditions are equivalent:*
- (i) *S is perfect;*
- (ii) *S is Stieltjes perfect;*
- (iii) *$S^\#$ is perfect;*
- (iv) *$S^\#$ is Stieltjes perfect.*

Remark. Stieltjes semiperfect finitely generated abelian semigroups with the identical involution were characterized in [13]. As remarked above, even for finitely generated abelian semigroups with zero and the identical involution, semiperfectness and Stieltjes semiperfectness are not equivalent.

For every $*$-semigroup S we denote by χ (or χ_S, if S has to be specified) the quotient mapping of S onto its greatest $\mathbb{C}$-separative $*$-homomorphic image. If S has a zero then every positive definite function on S factors via χ [9]. It easily follows that in this case, S is perfect (or semiperfect) if and only if $\chi(S)$ is such.

Remark. For a $*$-semigroup S without zero, positive definite functions on S do not in general factor via χ. For a sufficient condition, see [10].

For every $*$-semigroup S we denote by ϱ (or ϱ_S, if S has to be specified) the quotient mapping of S onto its greatest $\mathbb{R}_+$-separative $*$-homomorphic image. It is trivial to verify that every $\mathbb{R}$-separative $*$-semigroup carries the identical involution. An abelian semigroup S carrying the identical involution is $\mathbb{R}_+$-separative if and only if it is *torsion-free* in the sense that the conditions $a, b \in S$, $k \in \mathbb{N}$, and $ka = kb$ imply $a = b$. Indeed, for an arbitrary abelian semigroup S carrying the identical involution and for $a, b \in S$ we have $\varrho(a) = \varrho(b)$ if and only if there is some $k \in \mathbb{N}$ such that $ka = kb$ [6]. If S happens to be an abelian group then the term 'torsion-free' has its usual sense. For every abelian semigroup S we write $2S = \{2s\colon\ s \in S\}$.

Corollary 3. *A $*$-semigroup is perfect if and only if it is flat and its greatest $\mathbb{R}_+$-separative $*$-homomorphic image is quasi-perfect (or equivalently, perfect).*

P r o o f. Suppose S is a $*$-semigroup; we have to show that S is perfect if and only if S is flat and $\varrho(S)$ is quasi-perfect (or equivalently, perfect). If S is perfect then it is of course flat, and its $*$-homomorphic image $\varrho(S)$ is perfect [15]. Conversely, suppose S is flat and $\varrho(S)$ is quasi-perfect; we have to show that S is perfect. Since S is flat, it suffices to show that S is quasi-perfect, that is, $S \cup \{0\}$ is perfect. Since this semigroup has a zero it suffices (as shown below) to show that $\varrho(S \cup \{0\})$ is perfect. But the latter semigroup can be identified with $\varrho(S) \cup \{0\}$, which is perfect since $\varrho(S)$ is quasi-perfect.

It remains to be shown that if S is a $*$-semigroup with zero such that $\varrho(S)$ is perfect then so is S. Since S has a zero, it suffices to show that $\chi(S)$ is perfect. If $s, t \in S$ are such that $\chi(s) = \chi(t)$ then trivially, $\varrho(s) = \varrho(t)$. Hence, there is a unique mapping $\varrho'\colon \chi(S) \to \varrho(S)$ such that $\varrho = \varrho' \circ \chi$. We leave it as an exercise to verify that ϱ' can be identified with the quotient mapping of $\chi(S)$ onto its greatest $\mathbb{R}_+$-separative $*$-homomorphic image.

In other words, we may assume that S is $\mathbb{C}$-separative. By Corollary 2 it suffices to show that $S^{\#}$ is perfect. By [14, Theorem 4], $S^{\#}$ is isomorphic to $\varrho(S)$, hence perfect. $\square$

3. The perfectness of G-conelike semigroups for dense subgroups G of $\mathbb{Q}$

If G is a subgroup of $(\mathbb{Q}, +)$ and if S is a subsemigroup of a $\mathbb{Q}$-vector space then S is said to be G-*conelike* if for each $s \in S$ there is some $a \in \mathbb{Q}$ such that $\alpha s \in S$ for all $\alpha \in G$ such that $\alpha \geqslant a$. A $\mathbb{Q}$-conelike semigroup is called simply 'conelike'.

The perfectness of conelike $*$-subsemigroups of finite-dimensional rational vector spaces with arbitrary involution (containing the zero of the space) was shown by Nishio and the second-mentioned author [21]. Since every $*$-semigroup which is generated by the union of its perfect $*$-subsemigroups is perfect [15], the assumption on the dimension is superfluous.

The following result generalizes this to semigroups that are G-conelike for some dense subgroup G of $\mathbb{Q}$. Since a $*$-semigroup S with zero is perfect if and only if $\varrho(S)$ is perfect it suffices to consider $\mathbb{R}_+$-separative semigroups. A G-conelike semigroup without zero is of course quasi-perfect (hence perfect iff it is flat).

Theorem 6. *If G is a dense subgroup of $\mathbb{Q}$ then every G-conelike semigroup with zero is perfect.*

P r o o f. Suppose S is a G-conelike subsemigroup of some rational vector space carrying the identical involution and such that $0 \in S$; we have to show that S is perfect. Since every $*$-semigroup which is generated by the union of its perfect $*$-subsemigroups is perfect it suffices to show that each $s \in S$ belongs to some perfect subsemigroup of S. Such a semigroup is the set $\{ns\colon n \in \mathbb{N}\} \cup (S \cap \{\alpha s\colon \alpha \in G\})$. Indeed, by an isomorphism it suffices to show that the semigroup $G_s = \mathbb{N} \cup \{\alpha \in G\colon \alpha s \in S\}$ is perfect. Since S is conelike, G_s contains the semigroup $G_a = \{0\} \cup (G \cap \,]a, \infty[)$ for some $a > 0$. Since every subsemigroup of $\mathbb{Q}_+$ containing a perfect subsemigroup of $\mathbb{Q}_+$ properly containing $\{0\}$ is perfect [8] it suffices to show that G_a is perfect, or equivalently, the semigroup $G \cap \,]a, \infty[$ is quasi-perfect. Since every ideal of a perfect semigroup is quasi-perfect it suffices to note that by [15, Proposition 2], the semigroup $G \cap \mathbb{Q}_+$ is Stieltjes perfect, hence perfect. $\square$

4. The perfectness of semi-$*$-divisible $*$-semigroups

Theorem 7. *Every semi-$*$-divisible semigroup is perfect.*

P r o o f. Suppose S is a semi-$*$-divisible semigroup; we have to show that S is perfect, that is, flat and quasi-perfect. To see that S is flat, suppose φ is a singular positive definite function on S; we have to show $\varphi = 0$. Since S is semi-$*$-divisible we have $s + s^* \in S + S + S$, so $\varphi(s + s^*) = 0$ for $s \in S$. For arbitrary $s, t \in S$ we now have $|\varphi(s + t)|^2 \leqslant \varphi(s + s^*)\varphi(t + t^*)$ by the Cauchy-Schwarz inequality, so $\varphi(s + t) = 0$. Thus S is flat. It remains to be shown that S is quasi-perfect, that is, $\tilde{S}$ is perfect. Now $\tilde{S}$ is obviously semi-$*$-divisible. In other words, we may assume that S has a zero. Since S has a zero it suffices to show that $\varrho(S)$ is perfect. Now $\varrho(S)$, being a $*$-homomorphic image of the semi-$*$-divisible semigroup S, is again semi-$*$-divisible. Thus we may assume that S is $\mathbb{R}_+$-separative. It follows that each archimedean component of S is embeddable in a torsion-free abelian group [9].

Since every $*$-semigroup which is generated by the union of its perfect $*$-subsemigroups is perfect it suffices to show that each $s \in S$ belongs to some perfect subsemigroup of S. Let H be the archimedean component of S containing s. Let G be the group $H - H$, which is torsion-free by the above. Let X be the least face of S containing H. Recall that $X + H \subset H$. Define a homomorphism $g\colon X \to G$ by $g(x) = (x + s) - s$ (difference in the group G) for $x \in X$. It easily follows that $x + h = g(x) + h$ for all $h \in H$. Define a sequence $(s_n)_{n=0}^{\infty}$ in X by induction as follows. Firstly, $s_0 = s \in H \subset X$. Secondly, suppose $n \geqslant 1$ and that s_{n-1} has been defined and belongs to X. Since S is semi-$*$-divisible we can choose $s_n \in S$ and $k_n \in \mathbb{N}$ such that $k_n \geqslant 2$ and $2s_{n-1} = s_{n-1} + k_n s_n$. Since the left-hand side is in X, it follows by the definition of a face that $s_n \in X$. This completes the induction. Let K be the subgroup of $\mathbb{Q}$ generated by the set $\{(k_1 \dots k_n)^{-1}\colon\ n \in \mathbb{N}_0\}$. (If $n = 0$ then $k_1 \dots k_n = 1$ by definition.) Clearly, K is dense in $\mathbb{Q}$. For $\alpha \in K$ such that $\alpha \geqslant 1$, the element αs is well-defined (in the enveloping rational vector space of G) and belongs to S. Indeed, with $\beta = \alpha - 1$ we have $\beta \in K$ and $\beta \geqslant 0$, hence $\beta = p/k_1 \dots k_n$ for some $n, p \in \mathbb{N}_0$. Now $\alpha s = (\beta + 1)s = \beta s + s$. To see that this is in S, since $x + s = g(x) + s$ for all $x \in X$ it suffices to show $\beta s \in g(X)$. For $m \in \mathbb{N}$, since $2s_{m-1} = s_{m-1} + k_m s_m$ we have $2g(s_{m-1}) = g(2s_{m-1}) = g(s_{m-1} + k_m s_m) = g(s_{m-1}) + k_m g(s_m)$. Since G is a group it follows that $g(s_{m-1}) = k_m g(s_m)$. Hence by induction, $s = g(s) = g(s_0) = k_1 \dots k_n g(s_n)$, so $\beta s = pg(s_n) = g(ps_n) \in g(X)$. We see that the set $\{0\} \cup \{\alpha s\colon\ \alpha \in K, \alpha \geqslant 1\}$ is a well-defined subsemigroup of S containing s. Since it is K-conelike and has a zero, it is perfect. This completes the proof. $\square$

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