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A CLASS OF STATISTICAL AND σ-CONSERVATIVE MATRICES

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Abstract. In [5] and [10], statistical-conservative and σ-conservative matrices were characterized. In this note we have determined a class of statistical and σ-conservative matrices studying some inequalities which are analogous to Knopp's Core Theorem.

Keywords: statistical convergence, invariant means, core theorems, matrix transformations

MSC 2000: 40C05, 40J05, 46A45

1. Introduction

Let K be a subset of $\mathbb{N}$, the set of all positive integers. The natural density δ of K is defined by

$$\delta(K) = \lim_n \frac{1}{n} |\{k \leqslant n\colon\ k \in K\}|$$

where $|\{k \leqslant n\colon\ k \in K\}|$ denotes the number of elements of K not exceeding n. A sequence x is said to be statistically convergent to a number l, if $\delta(\{k\colon\ |x_k - l| \geqslant \varepsilon\}) = 0$ for every ε. In this case we write $\text{st-lim}\, x = l$, [3]. By S and S_0 we denote the space of all statistically convergent sequences and the space of sequences which statistically convergent to zero, respectively. Note that a convergent sequence is also statistically convergent and a statistically convergent sequence need not be bounded.

Let ℓ_∞ and c be the Banach spaces of bounded and convergent sequences $x = (x_k)$ with the usual supremum norm. Let σ be a one-to-one mapping of $\mathbb{N}$ into itself and $T\colon\ \ell_\infty \longrightarrow \ell_\infty$ a linear operator defined by $Tx = (Tx_k) = (x_{\sigma(k)})$. An element $\varphi \in \ell'_\infty$, the conjugate space of ℓ_∞, is called an invariant mean or a σ-mean if and only if i) $\varphi(x) \geqslant 0$ when the sequence $x = (x_k)$ has $x_k \geqslant 0$ for all k, ii) $\varphi(e) = 1$ where $e = (1, 1, 1, \ldots)$ and iii) $\varphi(Tx) = \varphi(x)$ for all $x \in \ell_\infty$. Let M be the set of all σ-means on ℓ_∞. A sublinear functional P on ℓ_∞ is said to generate σ-means if

$\varphi \in \ell'_\infty$ and $\varphi \leqslant P \Rightarrow \varphi$ is a σ-mean, to dominate σ-means if $\varphi \leqslant P$ for all $\varphi \in M$ where $\varphi \leqslant P$ means that $\varphi(x) \leqslant P(x)$ for all $x \in \ell_\infty$.

It is shown [7] that the sublinear functional

$$V(x) = \sup_n \lim_p \sup t_{pn}(x)$$

both generates and dominates σ-means where

$$t_{pn}(x) = \frac{1}{p+1}(x_n + x_{\sigma(n)} + \ldots + x_{\sigma^p(n)}), \quad t_{-1,n}(x) = 0.$$

A bounded sequence x is called σ-convergent to s if $V(x) = -V(-x) = s$. In this case we write $\sigma\text{-}\lim x = s$. Let V_σ denote the set of all σ-convergent sequences. We assume throughout this paper that $\sigma^p(n) \neq n$ for all $n \geqslant 0$ and $p \geqslant 1$, where $\sigma^p(n)$ is the pth iterate of σ at n. Thus, a σ-mean extends the limit functional onto c in the sense that $\varphi(x) = \lim x$ for all $x \in c$, [8]. Consequently, $c \subset V_\sigma$.

By (iii), it is clear that $(Tx - x) \in Z$ for $x \in \ell_\infty$, where Z is the set of all σ-convergent sequences with σ-limit zero.

For $x \in \ell_\infty$, we write

$$l(x) = \liminf x, \quad L(x) = \limsup x, \quad W(x) = \inf_{z \in Z} L(x + z).$$

It is known that $V(x) = W(x)$ on ℓ_∞, [7].

Let $A = (a_{nk})$ be an infinite matrix of real numbers and $x = (x_k)$ a real sequence such that $Ax = (A_n(x)) = \left(\sum_k a_{nk} x_k\right)$ exists for each n. Then the sequence $Ax = (A_n(x))$ is called an A-transform of x. For two sequence spaces E and F we say that the matrix A map E into F if Ax exits and belongs to F for each $x \in E$. By (E, F) we denote the set of all matrices which map E into F. If E and F are equipped with the limits E-lim and F-lim, respectively, $A \in (E, F)$ and $F\text{-}\lim_n A_n(x) = E\text{-}\lim_k x_k$ for all $x \in E$, then we say that A regularly maps E into F and write $A \in (E, F)_{\text{reg}}$.

We will call the matrices (c, c), (c, V_σ) and $(c, S \cap \ell_\infty)$ conservative, σ-conservative and statistical (st-) conservative matrices. It is known [6] that A is conservative if and only if $\|A\| = \sup_n \sum_k |a_{nk}| < \infty$, $a_k = \lim_n a_{nk}$ for each k, and $a = \lim_n \sum_k a_{nk}$. If A is conservative, the number $\chi = \chi(A) = a - \sum_k a_k$ called the characteristic of A is of importance in summability.

Schaefer [10] has proved that A is σ-conservative if and only if $\|A\| < \infty$, $\alpha_k = \sigma\text{-}\lim_n a_{nk}$ for each k, and $\alpha = \sigma\text{-}\lim \sum_k a_{nk}$.

Kolk [5] has shown that a matrix A is st-conservative if and only if $\|A\| < \infty$, $t_k = \text{st-}\lim_n a_{nk}$ for each k, and $t = \text{st-}\lim \sum_k a_{nk}$.

In the case A is σ-conservative or st-conservative, similarly, we can define numbers $\chi_\sigma = \chi_\sigma(A) = \alpha - \sum \alpha_k$ or $\chi_{\mathrm{st}} = \chi_{\mathrm{st}}(A) = t - \sum t_k$. If $\chi_\sigma \neq 0$, A is σ-coregular; otherwise, it is σ-conull. The matrix A is called st-coregular if $\chi_{\mathrm{st}} \neq 0$; otherwise, we call it st-conull.

For any real λ we write $\lambda^+ = \max\{0, \lambda\}$, $\lambda^- = \max\{-\lambda, 0\}$. Then $\lambda = \lambda^+ + \lambda^-$ and $|\lambda| = \lambda^+ - \lambda^-$.

Fridy and Orhan [3] have introduced the notions of the statistical boundedness, statistical-limit superior (st-lim sup) and inferior (st-lim inf), and also determined necessary and sufficient conditions for a matrix A to yield $L(Ax) \leqslant \beta(x)$ for all $x \in \ell_\infty$, where $\beta(x) = \text{st-}\limsup x$. Recently, Lie and Fridy [4] have characterized the class of matrices A such that $\beta(Ax) \leqslant \beta(x)$ for all $x \in \ell_\infty$.

Das [2] has characterized a class of conservative matrices in terms of inequalities involving sublinear functionals on ℓ_∞. In this paper, we shall determine a class of conservative, σ-conservative and st-conservative matrices using the same technique.

Now, we list some known results:

Lemma 1.1 [2, Theorem 1 (c)]. *Let $\mathscr{A} = (a_{nk}(i))$ be conservative. Then, for some constant $\lambda \geqslant |\chi|$ and for all $x \in \ell_\infty$,*

$$\limsup_n \sup_i \sum_k (a_{nk}(i) - a_k)x_k \leqslant \frac{\lambda + \chi}{2} L(x) - \frac{\lambda - \chi}{2} l(x)$$

if and only if

$$\limsup_n \sup_i \sum_k |a_{nk}(i) - a_k| \leqslant \lambda. \tag{1.1}$$

Lemma 1.2 [2, Lemma 1]. *Let $\mathscr{A} = (a_{nk}(i))$ be conservative and $\lambda \geqslant 0$. Then* (1.1) *holds if and only if*

$$\limsup_n \sup_i \sum_k (a_{nk}(i) - a_k)^+ \leqslant \frac{\lambda + \chi}{2}$$

and

$$\limsup_n \sup_i \sum_k (a_{nk}(i) - a_k)^- \leqslant \frac{\lambda - \chi}{2}.$$

2. Main results

Theorem 2.1. *Let A be conservative. Then, for some constant $\lambda \geqslant |\chi|$ and for all $x \in \ell_\infty$,*

$$\limsup_n \sum_k (a_{nk} - a_k)x_k \leqslant \frac{\lambda+\chi}{2}\beta(x) + \frac{\lambda-\chi}{2}\alpha(-x) \tag{2.1}$$

if and only if

$$\limsup_n \sum_k |a_{nk} - a_k| \leqslant \lambda, \tag{2.2}$$

$$\lim_n \sum_{k\in E} |a_{nk} - a_k| = 0 \tag{2.3}$$

for every $E \subseteq \mathbb{N}$ with $\delta(E) = 0$, where $\beta(x) = \text{st-}\limsup x$ and $\alpha(x) = \text{st-}\liminf x$.

Proof. Necessity: Since $\beta(x) \leqslant L(x)$ and $\alpha(-x) \leqslant -l(x)$ for all $x \in \ell_\infty$, we have

$$\limsup_n \sum_k (a_{nk} - a_k)x_k \leqslant \frac{\lambda+\chi}{2}L(x) - \frac{\lambda-\chi}{2}l(x).$$

Hence, the necessity of (2.2) follows from the special case of Lemma 1.1.

To show (2.3), define $b_{nk} = a_{nk} - a_k$ for $k \in E$; otherwise, let it be zero for all n, where E is any subset of $\mathbb{N}$ with $\delta(E) = 0$. Since A is conservative, the matrix $B = (b_{nk})$ satisfies the conditions of Corollary 12 of [11]. So, there exits a $y \in \ell_\infty$ such that $\|y\| \leqslant 1$ and

$$\limsup_n \sum_k |b_{nk}| = \limsup_n \sum_k b_{nk}y_k. \tag{2.4}$$

Now, for the same E we can choose a sequence (y_k) as

$$y_k = \begin{cases} 1, & k \in E, \\ 0, & k \notin E. \end{cases}$$

Thus, since $\text{st-}\lim y = \beta(y) = \alpha(y) = 0$, combining the supposition and (2.4) we have

$$\limsup_n \sum_{k\in E} |a_{nk} - a_k| \leqslant \frac{\lambda+\chi}{2}\beta(x) + \frac{\lambda-\chi}{2}\alpha(-x) = 0,$$

which implies (2.3).

Sufficiency: Let $x \in \ell_\infty$. If we write $E_1 = \{k\colon\ x_k > \beta(x) + \varepsilon\}$ and $E_2 = \{k\colon\ x_k < \alpha(x) - \varepsilon\}$ then $\delta(E_1) = \delta(E_2) = 0$. Hence the set $E = E_1 \cap E_2$ has also zero density. It can be written that

$$\sum_k (a_{nk} - a_k)x_k = \sum_{k\in E} (a_{nk} - a_k)x_k + \sum_{k\notin E} (a_{nk} - a_k)^+ x_k - \sum_{k\notin E} (a_{nk} - a_k)^- x_k.$$

Thus, since (2.3) implies that the first sum on the right-hand side is zero, from the special case of Lemma 1.2 we get

$$\limsup_n \sum_k (a_{nk} - a_k)x_k \leqslant \frac{\lambda + \chi}{2}\beta(x) + \frac{\lambda - \chi}{2}\alpha(-x),$$

which completes the proof. □

In the case $\chi > 0$ and $\lambda = \chi$ we conclude from Theorem 2.1 that for all $x \in \ell_\infty$

$$\limsup_n \sum_k (a_{nk} - a_k)x_k \leqslant \chi\beta(x) \tag{2.5}$$

if and only if (2.3) holds and

$$\lim_n \sum_k |a_{nk} - a_k| = \chi.$$

Moreover, if $A \in (c, c)_{\mathrm{reg}}$ and $\lambda = \chi$, then since $\chi = 1$ and $a_k = 0$ for each k, Theorem 2.1 is reduced to the Lemma of Fridy and Orhan [3].

If A is σ-conservative in Theorem 2.1, we have the following result which can be proved with the same argument as Theorem 2.1:

Theorem 2.2. *Let A be σ-conservative. Then, for some constant $\lambda \geqslant |\chi_\sigma|$ and for all $x \in \ell_\infty$,*

$$\limsup_p \sup_n \sum_k (a(p,n,k) - \alpha_k)x_k \leqslant \frac{\lambda + \chi_\sigma}{2}\beta(x) + \frac{\lambda - \chi_\sigma}{2}\alpha(-x) \tag{2.6}$$

if and only if

$$\limsup_p \sup_n \sum_k |a(p,n,k) - \alpha_k| \leqslant \lambda, \tag{2.7}$$

$$\lim_p \sup_n \sum_{k\in E} |a(p,n,k) - \alpha_k| = 0 \tag{2.8}$$

for $E \subseteq \mathbb{N}$ with $\delta(E) = 0$, where $a(p,n,k) = (p+1)^{-1} \sum\limits_{i=0}^{p} a_{\sigma^i(n),k}$.

When $A \in (c, V_\sigma)_{\mathrm{reg}}$ and $\lambda = \chi_\sigma$, Theorem 2.2 gives Theorem 2.3 of [1].

To the proof of the next theorem we need two lemmas:

Lemma 2.3. *Let A be* st*-conservative and $\lambda > 0$. Then*

$$\text{st-}\limsup_n \sum_k |a_{nk} - t_k| \leqslant \lambda$$

if and only if

$$\text{st-}\limsup_n \sum_k (a_{nk} - t_k)^+ \leqslant \frac{\lambda + \chi_{\text{st}}}{2}$$

and

$$\text{st-}\limsup_n \sum_k (a_{nk} - t_k)^- \leqslant \frac{\lambda - \chi_{\text{st}}}{2}.$$

Proof. By the st-conservativeness of A we get

$$\text{st-}\limsup_n \sum_k (a_{nk} - t_k) = \chi_{\text{st}}.$$

Therefore, the result follows from the relations

$$\sum_k (a_{nk} - t_k) = \sum_k (a_{nk} - t_k)^+ - \sum_k (a_{nk} - t_k)^-$$

and

$$\sum_k |a_{nk} - t_k| = \sum_k (a_{nk} - t_k)^+ + \sum_k (a_{nk} - t_k)^-.$$

$\square$

Lemma 2.4. *Let $\|A\| < \infty$ and* st*-*$\lim_n |a_{nk}| = 0$*. Then there exists a $y \in \ell_\infty$ such that $\|y\| \leqslant 1$ and*

$$\text{st-}\limsup \sum_k a_{nk} y_k = \text{st-}\limsup \sum_k |a_{nk}|.$$

Proof. If st-$\lim_n |a_{nk}| = 0$, then $\delta(E) = \delta(\{n\colon |a_{nk}| > \varepsilon\}) = 0$ and so $|a_{nk}| \leqslant \varepsilon$ for $n \notin E$. Since $\|A\| < \infty$, $\Bigl(\sum_k |a_{nk}|\Bigr)_n$ is a bounded sequence so that st-$\limsup_n \sum_k |a_{nk}| < \infty$.

Let $\gamma =$ st-$\limsup_n \sum_k |a_{nk}|$ and let for a given $\varepsilon > 0$,

$$N(\varepsilon) = \Bigl\{n\colon \sum_k |a_{nk}| > \gamma - \varepsilon\Bigr\}.$$

Hence there exists an increasing sequence (n_r) in $N(\varepsilon)-E$ and a sequence (k_r) such that

$$\sum_{k\leqslant k_{r-1}} |a_{n_r,k}| < \frac{1}{r}, \qquad \sum_{k>k_{r-1}} |a_{n_r,k}| < \frac{1}{r}.$$

Now define a $y\in \ell_\infty$ such that for $k_{r-1}\leqslant k<k_r$

$$y_k = \begin{cases} 1, & a_{n_r,k}\geqslant 0,\\ -1, & a_{n_r,k}<0.\end{cases}$$

Then by the same argument as in Lemma 2 of [2] we can see that

$$\sum_k a_{n_r,k}y_k \geqslant \sum_k |a_{n_r,k}| - \frac{4}{r}$$

and applying the operator $\operatorname{st-lim\,sup}_r$ we have

$$\operatorname*{st-lim\,sup}_r \sum_k a_{n_r,k}y_k \geqslant \gamma-\varepsilon.$$

Since (n_r) and ε are arbitrary, we get

$$\operatorname*{st-lim\,sup}_n \sum_k a_{nk}y_k \geqslant \gamma,$$

which completes the proof, because for such a y it is always true that

$$\operatorname*{st-lim\,sup}_r \sum_k a_{n_r,k}y_k \leqslant \gamma.$$

$\square$

Theorem 2.5. *Let A be* st*-conservative. Then, for some constant $\lambda\geqslant|\chi_{\mathrm{st}}|$ and for all $x\in\ell_\infty$,*

$$\operatorname*{st-lim\,sup}_n \sum_k (a_{nk}-t_k)x_k \leqslant \frac{\lambda+\chi_{\mathrm{st}}}{2}L(x) - \frac{\lambda-\chi_{\mathrm{st}}}{2}l(x) \tag{2.9}$$

if and only if

$$\operatorname*{st-lim\,sup}_n \sum_k |a_{nk}-t_k| \leqslant \lambda. \tag{2.10}$$

P r o o f. Necessity: Define $B=(b_{nk})$ by $b_{nk}=(a_{nk}-t_k)$ for all n, k. Then, since A is st-conservative, the matrix B satisfies the hypothesis of Lemma 2.4. Hence we have

$$\begin{aligned}\text{st-}\limsup_n\sum_k|b_{nk}| &= \text{st-}\limsup_n\sum_k b_{nk}y_k\\ &\leqslant \frac{\lambda+\chi_{\rm st}}{2}L(y)-\frac{\lambda-\chi_{\rm st}}{2}l(y)\\ &\leqslant \Big(\frac{\lambda+\chi_{\rm st}}{2}+\frac{\lambda-\chi_{\rm st}}{2}\Big)\|y\|=\lambda,\end{aligned}$$

which is (2.10).

Sufficiency: Let (2.10) hold and $x\in\ell_\infty$. Then for any $\varepsilon>0$ there exits a $k_0\in\mathbb{N}$ such that $l(x)-\varepsilon<x_k<L(x)+\varepsilon$ whenever $k>k_0$. Now, we can write

$$\sum_k(a_{nk}-t_k)x_k=\sum_{k\leqslant k_0}(a_{nk}-t_k)x_k+\sum_{k>k_0}(a_{nk}-t_k)^+x_k-\sum_{k>k_0}(a_{nk}-t_k)^-x_k.$$

By the st-conservativeness of A and Lemma 2.3 we obtain

$$\begin{aligned}\text{st-}\limsup_n\sum_k(a_{nk}-t_k)x_k &\leqslant (L(x)+\varepsilon)\Big(\frac{\lambda+\chi_{\rm st}}{2}\Big)-(l(x)-\varepsilon)\Big(\frac{\lambda-\chi_{\rm st}}{2}\Big)\\ &=\frac{\lambda+\chi_{\rm st}}{2}L(x)-\frac{\lambda-\chi_{\rm st}}{2}l(x)+\lambda\varepsilon,\end{aligned}$$

which yields (2.9), since ε is arbitrary. $\square$

Theorem 2.6. *Let A be* st-*conservative. Then, for some constant $\lambda\geqslant|\chi_{\rm st}|$ and for all $x\in\ell_\infty$,*

$$\text{st-}\limsup_n\sum_k(a_{nk}-t_k)x_k\leqslant\frac{\lambda+\chi_{\rm st}}{2}\beta(x)+\frac{\lambda-\chi_{\rm st}}{2}\alpha(-x)\tag{2.11}$$

if and only if (2.10) *holds and*

$$\text{st-}\lim_n\sum_{k\in E}|a_{nk}-t_k|=0\tag{2.12}$$

for every $E\subseteq\mathbb{N}$ with $\delta(E)=0$.

P r o o f. Necessity: If (2.11) holds, since $\beta(x)\leqslant L(x)$ and $\alpha(-x)\leqslant -l(x)$, (2.10) follows from Theorem 2.5. To show the necessity of (2.12), for any $E\subseteq\mathbb{N}$ with $\delta(E)=0$ let us define a matrix $B=(b_{nk})$ by $b_{nk}=a_{nk}-t_k$, $k\in E$; otherwise it equals zero for all n. Then, clearly, B satisfies the conditions of Lemma 2.4 and therefore there exists a $y \in \ell_\infty$ such that $\|y\| \leqslant 1$ and

$$\text{st-}\limsup_n \sum_k b_{nk} y_k = \text{st-}\limsup_n \sum_k |b_{nk}|.$$

Now, for the same E we choose the sequence y as

$$y_k = \begin{cases} 1, & k \in E, \\ 0, & k \notin E. \end{cases}$$

Hence, since $\text{st-}\lim y = \beta(y) = \alpha(y) = 0$, (2.11) implies that

$$\text{st-}\limsup_n \sum_{k \in E} |a_{nk} - t_k| \leqslant \frac{\lambda + \chi_{\text{st}}}{2} \beta(y) + \frac{\lambda - \chi_{\text{st}}}{2} \alpha(-y) = 0,$$

which is (2.12).

Sufficiency: Let the conditions of the theorem hold and let $x \in \ell_\infty$. Put the set E as in Theorem 2.1. Now, we can write

$$\sum_k (a_{nk} - t_k) x_k = \sum_{k \in E} (a_{nk} - t_k) x_k + \sum_{k \notin E} (a_{nk} - t_k)^+ x_k - \sum_{k \notin E} (a_{nk} - t_k)^- x_k.$$

Thus, by (2.12) and Lemma 2.3, (2.11) is obtained since

$$\text{st-}\limsup_n \sum_k (a_{nk} - t_k) x_k \leqslant \frac{\lambda + \chi_{\text{st}}}{2} \beta(x) + \frac{\lambda - \chi_{\text{st}}}{2} \alpha(-x) + \lambda\varepsilon$$

and ε is arbitrary. $\square$

We also should state that Theorem 2.6 is the dual of Theorem 3 in [4] when $A \in (c, S \cap \ell_\infty)_{\text{reg}}$ and $\lambda = \chi_{\text{st}}$.

Theorem 2.7. *Let A be* st*-conservative. Then, for some constant $\lambda \geqslant |\chi_{\text{st}}|$ and for all $x \in \ell_\infty$,*

$$\text{st-}\limsup_n \sum_k (a_{nk} - t_k) x_k \leqslant \frac{\lambda + \chi_{\text{st}}}{2} V(x) + \frac{\lambda - \chi_{\text{st}}}{2} V(-x) \tag{2.13}$$

if and only if (2.10) *holds and*

$$\text{st-}\lim_n \sum_k |a_{nk} - a_{n,\sigma(k)} - (t_k - t_{\sigma(k)})| = 0. \tag{2.14}$$

P r o o f. Necessity: Since $q_\sigma(x) \leqslant L(x)$ and $q_\sigma(-x) \leqslant -l(x)$ for all $x \in \ell_\infty$, the necessity of (2.10) follows from Theorem 2.5. Define $C = (c_{nk})$ by $c_{nk} = b_{nk} - b_{n,\sigma(k)}$ for all n, k where b_{nk} is as in Theorem 2.5. Then we have from Lemma 2.4 a $y \in \ell_\infty$ such that $\|y\| \leqslant 1$ and

$$\operatorname{st-lim\,sup}_n \sum_k |c_{nk}| = \operatorname{st-lim\,sup}_n \sum_k c_{nk} y_k.$$

Let us choose y such that $y_k = 0$, $k \notin \sigma(\mathbb{N})$. Hence, since $(y_k - y_{\sigma(k)}) \in Z$, (2.13) implies that

$$\begin{aligned}
\operatorname{st-lim\,sup}_n \sum_k |c_{nk}| &= \operatorname{st-lim\,sup}_n \sum_k c_{nk} y_{\sigma(k)} \\
&= \operatorname{st-lim\,sup}_n \sum_k b_{nk}(y_k - y_{\sigma(k)}) \\
&\leqslant \frac{\lambda + \chi_{\mathrm{st}}}{2} V(y_k - y_{\sigma(k)}) + \frac{\lambda - \chi_{\mathrm{st}}}{2} V(y_{\sigma(k)} - y_k) = 0,
\end{aligned}$$

which is (2.14).

Sufficiency: Let the conditions (2.10) and (2.14) hold. By the same argument as in Theorem 23 of [9], one can easily see that for any $x \in \ell_\infty$

$$\sum_k b_{nk}(x_k - x_{\sigma(k)}) = \sum_k c_{nk} x_{\sigma(k)}$$

where the matrices B and C are as above.

Hence, since $(x_k - x_{\sigma(k)}) \in Z$, (2.14) implies that $B \in (Z, S_0 \cap \ell_\infty)$. We also see from the assumption that (2.9) holds. Thus, taking infimum over $z \in Z$ in (2.9) we get that

$$\begin{aligned}
\inf_{z \in Z} \Bigl(\operatorname{st-lim\,sup}_n \sum_k b_{nk}(x_k + z_k) \Bigr) &\leqslant \frac{\lambda + \chi_{\mathrm{st}}}{2} L(x+z) - \frac{\lambda - \chi_{\mathrm{st}}}{2} l(x+z) \\
&= \frac{\lambda + \chi_{\mathrm{st}}}{2} W(x) + \frac{\lambda - \chi_{\mathrm{st}}}{2} W(-x).
\end{aligned}$$

On the other hand, since $\operatorname{st-lim} Bz = 0$ for $z \in Z$,

$$\begin{aligned}
&\inf_{z \in Z} \Bigl(\operatorname{st-lim\,sup}_n \sum_k b_{nk}(x_k + z_k) \Bigr) \\
&\geqslant \operatorname{st-lim\,sup}_n \sum_k b_{nk} x_k + \inf_{z \in Z} \Bigl(\operatorname{st-lim\,sup}_n \sum_k b_{nk} z_k \Bigr) \\
&= \operatorname{st-lim\,sup}_n \sum_k b_{nk} x_k.
\end{aligned}$$

Since $q_\sigma(x) = W(x)$ for all $x \in \ell_\infty$, we conclude that (2.13) holds and the proof is completed. $\square$

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