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EXTREMAL SOLUTIONS AND STRONG RELAXATION FOR SECOND ORDER MULTIVALUED BOUNDARY VALUE PROBLEMS

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Abstract. In this paper we study semilinear second order differential inclusions involving a multivalued maximal monotone operator. Using notions and techniques from the nonlinear operator theory and from multivalued analysis, we obtain "extremal" solutions and we prove a strong relaxation theorem.

Keywords: maximal monotone operator, pseudomonotone operator, Hartman condition, convex and nonconvex problems, extremal solutions, strong relaxation

MSC 2000: 34A60

1. Introduction

In this paper we study the following nonlinear multivalued boundary value problem:

$$\tag{1.1} \begin{cases} (\|x'(t)\|_{\mathbb{R}^N}^{p-2}x'(t))' \in A(x(t)) + \operatorname{ext} F(t, x(t)) \quad \text{a.e. on } [0,T], \\ x(0) = x(T) = 0, \end{cases}$$

where $p \geqslant 2$, $A\colon \mathbb{R}^N \supseteq D(A) \longrightarrow 2^{\mathbb{R}^N}$ is a maximal monotone map, $F\colon [0,T] \times \mathbb{R}^N \longrightarrow 2^{\mathbb{R}^N}$ is a multivalued vector field and $\xi\colon \mathbb{R}^N \times \mathbb{R}^N \longrightarrow 2^{\mathbb{R}^N \times \mathbb{R}^N}$ is a maximal monotone map describing the boundary conditions. By $\operatorname{ext} F(t,\zeta)$, we denote the set of extreme points of $F(t,\zeta)$. We remark that $\operatorname{ext} F(t,\zeta)$ need not be closed and the multifunction $\mathbb{R}^N \ni \zeta \longmapsto \operatorname{ext} F(t,\zeta) \in 2^{\mathbb{R}^N}$ needs not have any continuous properties, even if the multifunction $\mathbb{R}^N \ni \zeta \longmapsto F(t,\zeta) \in 2^{\mathbb{R}^N}$ is regular

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enough (see Hu-Papageorgiou [19]). The solutions of problem (1.1) are called "extremal solutions" and their set is denoted by $S_e \subseteq C^1([0,T];\mathbb{R}^N)$. The nonemptiness of S_e is the first major question that we address here. We show that under reasonable hypotheses on $F(t,\zeta)$, the solution set S_e of problem (1.1) is nonempty. It is an interesting open problem whether this existence result remains valid if we replace the Dirichlet boundary condition by the general nonlinear boundary conditions of problem (1.1) or even by the periodic or Neumann boundary conditions. The second major problem that is studied in this paper is whether the solutions of the original "convexified" problem can be approximated in the $C^1([0,T];\mathbb{R}^N)$-norm by extremal solutions. Such a result is also of interest in control theory, since in that context the extremal solutions are the states which are generated by extremal (bang-bang) controls which are physically realizable. Such an approximation (density) result is known as "strong relaxation". In this direction, we have only a partial result, namely we prove it only when $p = 2$, i.e. the differential operator is the Laplacian $x \longmapsto x''$. More precisely, let us consider the following problems

$$(1.2) \qquad \begin{cases} x''(t) \in A(x(t)) + \operatorname{ext} \overline{\operatorname{conv}}\, F(t,x(t)) \quad \text{a.e. on } [0,T], \\ x(0) = x(T) = 0 \end{cases}$$

and

$$(1.3) \qquad \begin{cases} x''(t) \in A(x(t)) + \overline{\operatorname{conv}}\, F(t,x(t)) \quad \text{a.e. on } [0,T], \\ x(0) = x(T) = 0. \end{cases}$$

We denote the solution set of problem (1.2) by $S_{ec} \subseteq C^1([0,T];\mathbb{R}^N)$ and the solution set of problem (1.3) by $S_c \subseteq C^1([0,T];\mathbb{R}^N)$. We show that $\overline{S}_{ec}^{C([0,T];\mathbb{R}^N)} = S_c$ (strong relaxation). It is a very interesting open problem whether this density result is still valid, if we have the more general differential operator $x \longmapsto (\|x'(\cdot)\|_{\mathbb{R}^N}^{p-2} x'(\cdot))'$, with $p \geqslant 2$. Also another open problem is whether in problem (1.1) the vector p-Laplacian can be replaced by a more general operator of the form $x \longmapsto (a(x(\cdot))\varphi(x'(\cdot)))'$.

We should mention that for first order multivalued Cauchy problems, the most important papers in this direction are those of De Blasi-Pianigiani [4], [5], [6], [7], who essentially initiated the subject and developed the so-called "Baire category method", which culminated in a continuous selection theorem (see Hu-Papageorgiou [18, p. 260]), which is a crucial tool in our study of extremal solutions.

Our basic hypothesis on F is the so called Hartman condition, which permits the derivation of *a priori* bounds for the solutions of (1.1). This condition was first employed by Hartman [15] (see also Hartman [16], p. 433]) for the vector Dirichlet problem

$$\begin{cases} x'' = f(t,x), \\ x(0) = x(T) = 0, \end{cases}$$

where the function $f\colon [0,T] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is continuous. Later, it was used by Knobloch [23] for the vector periodic problem for a vector field which is locally Lipschitz in $\zeta \in \mathbb{R}^N$. Variants and extensions can be found in the book of Gaines-Mawhin [11] and the references therein. Very recently the periodic problem was revisited by Mawhin [25], who used the vector p-Laplacian differential operator.

Recently there have been several papers involving the p-Laplacian differential operator. We mention the papers of Dang-Oppenheimer [2], De Coster [8], Guo [13], Kandilakis-Papageorgiou [22] (scalar equations) and Boccardo-Drábek-Giachetti-Kučera [1] and Mawhin [25] (vector problems). On the other hand for differential inclusions the previous papers deal with the semilinear problem (i.e. $p = 2$, the one-dimensional Laplacian differential operator) and assume that $A \equiv 0$. However, they go beyond the Dirichlet problem. We refer to the papers of Erbe-Krawcewicz [9], Frigon [10], Halidias-Papageorgiou [14] and Kandilakis-Papageorgiou [21]. It should be mentioned that only Halidias-Papageorgiou [14] address the existence of extremal solutions, always for the semilinear (i.e. $p = 2$) problem with $A \equiv 0$. So our work here appears to be the most general one in the direction of extremal solutions and strong relaxation for second order multivalued boundary value problems.

Our approach is based on notions and results from multivalued analysis and from the theory of nonlinear operators of monotone type. For the convenience of the reader, in the next section we recall some basic definitions and facts from these areas. Our main references are the books of Hu-Papageorgiou [19], [20] and Gasiński-Papageorgiou [12].

2. Preliminaries

Let (Ω, Σ) be a measurable space and let X be a separable Banach space. We introduce the following notation:

$$P_{f(c)}(X) \overset{\mathrm{df}}{=} \{A \subseteq X\colon\ A \text{ is nonempty, closed (and convex)}\},$$
$$P_{(w)k(c)}(X) \overset{\mathrm{df}}{=} \{A \subseteq X\colon\ A \text{ is nonempty, (weakly-)compact (and convex)}\}.$$

A multifunction $F\colon \Omega \longrightarrow P_f(X)$ is said to be measurable if for all $x \in X$, the function

$$\Omega \ni \omega \longmapsto d(x, F(\omega)) \overset{\mathrm{df}}{=} \inf\{\|x-y\|_X\colon\ y \in F(\omega)\} \in \mathbb{R}_+^N$$

is Σ-measurable. A multifunction $F\colon \Omega \longrightarrow 2^X \setminus \{\emptyset\}$ is said to be *graph measurable*, if

$$\operatorname{Gr} F \stackrel{\mathrm{df}}{=} \{(\omega, x) \in \Omega \times X\colon x \in F(\omega)\} \in \Sigma \times \mathcal{B}(X),$$

with $\mathcal{B}(X)$ being the Borel σ-field of X. For $P_f(X)$-valued multifunctions, measurability implies graph measurability, while the converse is true if Σ is complete (i.e. $\Sigma = \widehat{\Sigma} =$ the universal σ-field). Recall that, if μ is a measure on Σ and Σ is μ-complete, then $\Sigma = \widehat{\Sigma}$. Now, let (Ω, Σ, μ) be a finite measure space. For a given multifunction $F\colon \Omega \longrightarrow 2^X \setminus \{\emptyset\}$ and $1 \leqslant p \leqslant +\infty$, we introduce the set

$$S_F^p \stackrel{\mathrm{df}}{=} \{f \in L^p(\Omega; X)\colon f(\omega) \in F(\omega) \mu \text{ a.e. on } \Omega\}.$$

In general, this set may be empty. It is easy to check that if the map $\Omega \ni \omega \longmapsto \inf\{\|x\|_X\colon x \in F(\omega)\}$ is in $L^p(\Omega)$, then $S_F^p \neq \emptyset$.

Let Y, Z be Hausdorff topological spaces. A multifunction $G\colon Y \longrightarrow 2^Z \setminus \{\emptyset\}$ is said to be *lower semicontinuous* (respectively *upper semicontinuous*), if for every closed set $C \subseteq Z$, the set $G^+(C) \stackrel{\mathrm{df}}{=} \{y \in Y\colon G(y) \subseteq C\}$ (respectively $G^-(C) \stackrel{\mathrm{df}}{=} \{y \in Y\colon G(y) \cap C \neq \emptyset\}$) is closed in Y. An upper semicontinuous multifunction with closed values has a closed graph (i.e. $\operatorname{Gr} G \stackrel{\mathrm{df}}{=} \{(y, z) \in Y \times Z\colon z \in G(y)\}$ is closed), while the converse is true if G is locally compact (i.e. if for every $y \in Y$, there exists a neighbourhood U of y such that $\overline{G(U)}$ is compact in Z). Also, if Z is a metric space, then G is lower semicontinuous if and only if for every sequence $\{y_n\}_{n \geqslant 1} \subseteq Y$ such that $y_n \longrightarrow y$ in Y, we have that

$$G(y) \subseteq \liminf_{n \to +\infty} G(y_n),$$

where

$$\liminf_{n \to +\infty} G(y_n) \stackrel{\mathrm{df}}{=} \Big\{z \in Z\colon \lim_{n \to +\infty} d(z, G(y_n)) = 0\Big\}$$

or equivalently

$$\liminf_{n \to +\infty} G(y_n) \stackrel{\mathrm{df}}{=} \Big\{z \in Z\colon z = \lim_{n \to +\infty} z_n \text{ where } z_n \in G(y_n), \quad \text{for } n \geqslant 1\Big\}.$$

If Z is a metric space, then on $P_f(Z)$ we can define a generalized metric h, known in the literature as the *Hausdorff metric*, by setting

$$h(B, C) \stackrel{\mathrm{df}}{=} \max\Big\{\sup_{b \in B} d(b, C), \sup_{c \in C} d(c, B)\Big\} \quad \forall\, B, C \in P_f(Z).$$

If Z is complete, then $(P_f(Z), h)$ is complete too. A multifunction $F\colon Y \longrightarrow P_f(Z)$ is said to be *Hausdorff continuous* (*h-continuous* for short), if it is continuous from Y into $(P_f(Z), h)$.

Next, let X be a reflexive Banach space and X^* its topological dual. A map $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ is said to be *monotone* if for all $(x, x^*), (y, y^*) \in \operatorname{Gr} A$, we have $\langle x^* - y^*, x - y \rangle \geqslant 0$ (by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X, X^*)). If, additionally, the fact that $\langle x^* - y^*, x - y \rangle = 0$ implies that $x = y$, then we say that A is *strictly monotone*. The map A is said to be *maximal monotone*, if it is monotone and the fact that $\langle x^* - y^*, x - y \rangle \geqslant 0$ for all $(x, x^*) \in \operatorname{Gr} A$ implies that $(y, y^*) \in \operatorname{Gr} A$. So, according to this definition, the graph of a maximal monotone map is maximal monotone with respect to inclusion among the graphs of all monotone maps from X into 2^{X^*}. It is easy to see that a maximal monotone map A has a demiclosed graph, i.e. $\operatorname{Gr} A$ is sequentially closed in $X \times X_w^*$ and in $X_w \times X^*$ (here by X_w and X_w^* we denote the spaces X and X^*, respectively, furnished with their weak topologies). If $A\colon X \longrightarrow X^*$ is everywhere defined and single-valued, we say that A is *demicontinuous* if for every sequence $\{x_n\}_{n \geqslant 1} \subseteq X$ such that $x_n \longrightarrow x$ in X, we have $A(x_n) \longrightarrow A(x)$ weakly in X^*. If a map $A\colon X \longrightarrow X^*$ is monotone and demicontinuous, then it is also maximal monotone. A map $A\colon X \supseteq D(A) \longrightarrow 2^{X^*}$ is said to be *coercive* if $D(A) \subseteq X$ is bounded or if $D(A)$ is unbounded and we have

$$\frac{\inf\{\langle x^*, x \rangle\colon\ x^* \in A(x)\}}{\|x\|_X} \longrightarrow +\infty \quad \text{as } \|x\|_X \to +\infty, \text{ with } x \in D(A).$$

A maximal monotone and coercive map is surjective.

An operator $A\colon X \longrightarrow 2^{X^*}$ is said to be *pseudomonotone* if

(a) for all $x \in X$, we have $A(x) \in P_{wkc}(X^*)$;

(b) A is upper semicontinuous from every finite dimensional subspace Z of X into X_w^*;

(c) if $x_n \longrightarrow x$ weakly in X, $x_n^* \in A(x_n)$ and $\limsup\limits_{n\to+\infty} \langle x_n^*, x_n - x \rangle \leqslant 0$, then for every $y \in X$, there exists $x^*(y) \in A(x)$ such that

$$\langle x^*(y), x - y \rangle \leqslant \liminf_{n\to+\infty} \langle x_n^*, x_n - y \rangle.$$

If A is bounded (i.e. it maps bounded sets into bounded ones) and satisfies condition (c), then it satisfies condition (b) too. An operator $A\colon X \longrightarrow 2^{X^*}$ is said to be *generalized pseudomonotone* if for all $x_n^* \in A(x_n)$, with $n \geqslant 1$, such that $x_n \longrightarrow x$ weakly in X, $x_n^* \longrightarrow x^*$ weakly in X^* and $\limsup\limits_{n\to+\infty} \langle x_n^*, x_n - x \rangle \leqslant 0$, we have

$$x^* \in A(x) \quad \text{and} \quad \langle x_n^*, x_n \rangle \longrightarrow \langle x^*, x \rangle.$$

Every maximal monotone operator is generalized pseudomonotone. Also a pseudomonotone operator is generalized pseudomonotone. The converse is true if the operator is everywhere defined and bounded. A pseudomonotone operator which is also coercive is surjective.

Let Y, Z be Banach spaces and let $K\colon Y \longrightarrow Z$ be a map. We say that K is *completely continuous* if the fact that $y_n \longrightarrow y$ weakly in Y implies that $K(y_n) \longrightarrow K(y)$ in Z. We say that K is *compact* if it is continuous and maps bounded sets into relatively compact sets. In general, these two notions are distinct. However, if Y is reflexive, then complete continuity implies compactness. Moreover, if Y is reflexive and K is linear, then the two notions are equivalent. Also a multifunction $F\colon Y \longrightarrow 2^Z \setminus \{\emptyset\}$ is said to be *compact* if it is upper semicontinuous and maps bounded sets in Y into relatively compact sets in Z.

3. Extremal solutions

We start with an existence theorem for problem (1.1). We shall need the following hypotheses on the multifunctions A and F:

H(A) $A\colon \mathbb{R}^N \longrightarrow 2^{\mathbb{R}^N}$ is a maximal monotone map, such that $\operatorname{dom} A = \mathbb{R}^N$ and $0 \in A(0)$.

H(F)$_1$ $F\colon [0,T] \times \mathbb{R}^N \longrightarrow P_{kc}(\mathbb{R}^N)$ is a multifunction such that

(i) the multifunction $[0,T] \ni t \longmapsto F(t,\zeta) \in 2^{\mathbb{R}^N}$ is measurable, for all $\zeta \in \mathbb{R}^N$;

(ii) for almost all $t \in [0,T]$, the multifunction $\mathbb{R}^N \ni \zeta \longmapsto F(t,\zeta) \in 2^{\mathbb{R}^N}$ is h-continuous;

(iii) for all $k > 0$, there exists $a_k \in L^{p'}([0,T])_+$ such that for almost all $t \in [0,T]$, all $\zeta \in \mathbb{R}^N$ with $\|\zeta\|_{\mathbb{R}^N} \leqslant k$ and all $u \in F(t,\zeta)$, we have $\|u\|_{\mathbb{R}^N} \leqslant a_k(t)$;

(iv) there exists $M > 0$ such that for almost all $t \in [0,T]$, all $\zeta \in \mathbb{R}^N$ with $\|\zeta\|_{\mathbb{R}^N} = M$ and all $u \in F(t,\zeta)$, we have $(u,\zeta)_{\mathbb{R}^N} \geqslant 0$ (Hartman condition).

Theorem 3.1. *If hypoheses* H(A) *and* H(F)$_1$ *hold, then problem* (1.1) *has a solution* $\overline{x} \in C^1([0,T];\mathbb{R}^N)$.

P r o o f. Let $\varphi\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be the homeomorphism defined by $\varphi(\zeta) \overset{\text{df}}{=} \|\zeta\|_{\mathbb{R}^N}^{p-2}\zeta$ and let $p_M\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be the M-radial retraction defined by

$$p_M(\zeta) \overset{\text{df}}{=} \begin{cases} \zeta & \text{if } \|\zeta\|_{\mathbb{R}^N} \leqslant M, \\ \dfrac{M\zeta}{\|\zeta\|_{\mathbb{R}^N}} & \text{if } \|\zeta\|_{\mathbb{R}^N} > M, \end{cases}$$

with M being as in hypothesis $\mathrm{H(F)}_1$ (iv). We consider the following modification of the multifunction $F(t,\zeta)$:

$$F_1(t,\zeta) \stackrel{\mathrm{df}}{=} F(t,p_M(\zeta)) + \varphi(\zeta) - \varphi(p_M(\zeta)),$$

which clearly satisfies the same conditions as F.

In what follows, by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $(W_0^{1,p}([0,T];\mathbb{R}^N), W^{-1,p'}([0,T];\mathbb{R}^N))$ (where $1/p+1/p'=1$). Consider the operator

$$U\colon\ W_0^{1,p}([0,T];\mathbb{R}^N) \longrightarrow W^{-1,p'}([0,T];\mathbb{R}^N)$$

defined by

$$\langle U(x),y\rangle = \int_0^T (\|x'(t)\|_{\mathbb{R}^N}^{p-2}x'(t),y'(t))_{\mathbb{R}^N}\,\mathrm{d}t \quad \forall\, x,y\in W_0^{1,p}([0,T];\mathbb{R}^N).$$

Clearly U is bounded, demicontinuous, and monotone, thus maximal monotone. Also it is strictly monotone. Because U is everywhere defined, we infer that it is pseudomonotone (see Section 2). Also let

$$\hat{A}\colon\ W_0^{1,p}([0,T];\mathbb{R}^N) \longrightarrow 2^{L^{p'}([0,T];\mathbb{R}^N)}$$

be defined by

$$\hat{A}(x) \stackrel{\mathrm{df}}{=} S^{p'}_{A(x(\cdot))} \quad \forall\, x\in W_0^{1,p}([0,T];\mathbb{R}^N).$$

Note that since $W_0^{1,p}([0,T];\mathbb{R}^N) \subseteq C([0,T];\mathbb{R}^N)$ and $\operatorname{dom} A = \mathbb{R}^N$, for every $x\in W_0^{1,p}([0,T];\mathbb{R}^N)$ we see that $A(\overline{B}_{\|x\|_{C([0,T];\mathbb{R}^N)}}(0))$ is compact and so $\hat{A}$ has nonempty, weakly compact and convex values in $L^{p'}([0,T];\mathbb{R}^N)$. Moreover, via Theorem III.1.33, p. 309 of Hu-Papageorgiou [19], we see that $\hat{A}$ is maximal monotone as a map from $W_0^{1,p}([0,T];\mathbb{R}^N)$ into $W^{-1,p'}([0,T];\mathbb{R}^N)$. Then by virtue of Theorem III.3.3, p. 334 of Hu-Papageorgiou [19], the map

$$W_0^{1,p}([0,T];\mathbb{R}^N) \ni x \longmapsto U(x)+\hat{A}(x) \in W^{-1,p'}([0,T];\mathbb{R}^N)$$

is maximal monotone and it is everywhere defined. Let $h\in\hat{A}(x)$. Since $0\in\hat{A}(0)$, we have

$$\begin{aligned}\langle U(x)+h,x\rangle &= \langle U(x),x\rangle + \langle h,x\rangle\\ &\geqslant \langle U(x),x\rangle = \int_0^T \|x'(t)\|_{\mathbb{R}^N}^p\,\mathrm{d}t = \|x'\|_p^p,\end{aligned}$$

so the function $x \longmapsto U(x) + \hat{A}(x)$ is coercive (by Poincare's inequality; see Hu-Papageorgiou [20, Theorem A.1.24, p. 866]). Let

$$E \stackrel{\mathrm{df}}{=} \{g \in L^{p'}([0,T];\mathbb{R}^N) \colon \ \|g\|_{p'} \leqslant \|a_M\|_{p'}\}.$$

For every $g \in E$, we see that the operator inclusion

$$g \in U(x) + \hat{A}(x) \tag{3.1}$$

has a solution $x \in W_0^{1,p}([0,T];\mathbb{R}^N)$. Using the fact that the map $\mathbb{R}^N \ni \zeta \longmapsto \varphi(\zeta) = \|\zeta\|_{\mathbb{R}^N}^{p-2}\zeta \in \mathbb{R}^N$ is a homeomorphism, we have $x \in C^1([0,T];\mathbb{R}^N)$. Let K be the set of all solutions of problem (3.1) as g varies in E. Because of the coercivity of the map $x \longmapsto U(x) + \hat{A}(x)$, the set $K \subseteq W_0^{1,p}([0,T];\mathbb{R}^N)$ is bounded.

Claim 1. *The set K is weakly closed (hence weakly compact) in $W_0^{1,p}([0,T];\mathbb{R}^N)$.*

P r o o f. Since the weak $W_0^{1,p}([0,T];\mathbb{R}^N)$-topology on K is metrizable, we may work with sequences. So, let $\{x_n\}_{n\geqslant 1} \subseteq K$ be a sequence and assume that

$$x_n \longrightarrow x \quad \text{weakly in } W_0^{1,p}([0,T];\mathbb{R}^N).$$

We have

$$\langle U(x_n), x_n - x\rangle + \langle h_n, x_n - x\rangle = \langle g_n, x_n - x\rangle,$$

for some $h_n \in \hat{A}(x_n)$ and $g_n \in E$. Because of the compactness of the embedding $W_0^{1,p}([0,T];\mathbb{R}^N) \subseteq C([0,T];\mathbb{R}^N)$, we have $x_n \longrightarrow x$ in $C([0,T];\mathbb{R}^N)$. Hence

$$c_1 \stackrel{\mathrm{df}}{=} \sup_{n\geqslant 1} \|x_n\|_{C([0,T];\mathbb{R}^N)} < +\infty.$$

So, it follows that the set $A\left(\overline{B}_{c_1}(0)\right) \subseteq \mathbb{R}^N$ is compact (since $\operatorname{dom} A = \mathbb{R}^N$) and from this we infer that the sequence $\{h_n\}_{n\leqslant 1} \subseteq L^{p'}([0,T];\mathbb{R}^N)$ is bounded. As

$$\langle h_n, x_n - x\rangle = \langle h_n, x_n - x\rangle_{pp'} \longrightarrow 0,$$
$$\langle g_n, x_n - x\rangle = \langle g_n, x_n - x\rangle_{pp'} \longrightarrow 0,$$

so also

$$\lim_{n\to+\infty} \langle U(x_n), x_n - x\rangle = 0.$$

Because U is pseudomonotone and bounded, we conclude that $U(x_n) \longrightarrow U(x)$ weakly in $W^{-1,p'}([0,T];\mathbb{R}^N)$. Also, passing to a subsequence if necessary, we may assume that $h_n \longrightarrow h$ and $g_n \longrightarrow g$ weakly in $L^{p'}([0,T];\mathbb{R}^N)$. Clearly $g \in E$ and because $h_n(t) \in A(x_n(t))$ almost everywhere on $[0,T]$, using Proposition VII.3.9, p. 694 of Hu-Papageorgiou [19], we see that

$$h(t) \in \overline{\operatorname{conv}} \limsup_{n\to+\infty} A(x_n(t)) \subseteq A(x(t)) \quad \text{a.e. on } [0,T]$$

(the last inclusion follows from the upper semicontinuity of A and the fact that A is $P_{kc}(\mathbb{R}^N)$-valued). Hence $h \in \hat{A}_1(x)$. In the limit as $n \to +\infty$, we obtain

$$g = U(x) + h, \quad \text{with } h \in \hat{A}_1(x),$$

hence K is weakly closed in $W_0^{1,p}([0,T];\mathbb{R}^N)$ and Claim 1 is proved. □

Let $i\colon W_0^{1,p}([0,T];\mathbb{R}^N) \longrightarrow C([0,T];\mathbb{R}^N)$ be the embedding operator. We know that i is completely continuous and so $i(K) \subseteq C([0,T];\mathbb{R}^N)$ is compact. We have

$$\operatorname{conv} i(K) = i(\operatorname{conv} K) \subseteq i(\overline{\operatorname{conv}}\, K) \subseteq \overline{i(\operatorname{conv} K)} = \overline{\operatorname{conv}}\, i(K),$$

so

$$i(\overline{\operatorname{conv}}\, K) = \overline{\operatorname{conv}}\, i(K) = K_1 \subseteq W_0^{1,p}([0,T];\mathbb{R}^N) \subseteq C([0,T];\mathbb{R}^N)$$

and K_1 is weakly compact in $W_0^{1,p}([0,T];\mathbb{R}^N)$ and compact in $C([0,T];\mathbb{R}^N)$.

Next, let $\hat{F}_1\colon K_1 \longrightarrow P_{wkc}(L^{p'}([0,T];\mathbb{R}^N))$ be the multifunction defined by

$$\hat{F}_1(x) \overset{\text{df}}{=} S^{p'}_{F_1(\cdot,x(\cdot))} \quad \forall\, x \in K_1.$$

Using Theorem II.8.31, p. 260 of Hu-Papageorgiou [19], we obtain a continuous map

$$\tilde{u}\colon K_1 \longrightarrow L^1_w([0,T];\mathbb{R}^N),$$

such that

$$\tilde{u}(x) \in \operatorname{ext} \hat{F}_1(x) = \operatorname{ext} S^{p'}_{F_1(\cdot,x(\cdot))} \quad \forall\, x \in K_1.$$

We denote by $L^1_w([0,T];\mathbb{R}^N)$, the space $L^1([0,T];\mathbb{R}^N)$ with the weak norm, defined for all $f \in L^1([0,T];\mathbb{R}^N)$ by

$$\|f\|_w \overset{\text{df}}{=} \sup\left\{ \left\| \int_s^t f(\tau)\,\mathrm{d}\tau \right\|_{\mathbb{R}^N} : \; 0 \leqslant s \leqslant t \leqslant T \right\}.$$

In particular $\tilde{u}$ is continuous from K_1 into $L^1([0,T];\mathbb{R}^N)$ furnished with the weak topology (see Hu-Papageorgiou [19, p. 194–195]) and because of the definition of F_1 and hypothesis $\mathrm{H(F)}_1$ (iii), we see that $\tilde{u}$ is in fact continuous from K_1 into

$L^{p'}([0,T];\mathbb{R}^N)_w$. Applying Dugundji's extention theorem (see Hu-Papageorgiou [19, Theorem I.2.88, p. 70]), we obtain a continuous map

$$\hat{u}\colon\ C([0,T];\mathbb{R}^N) \longrightarrow L^{p'}([0,T];\mathbb{R}^N)_w,$$

such that $\hat{u}|_{K_1} = \tilde{u}$. Exploiting the compactness of the embedding $W_0^{1,p}([0,T];\mathbb{R}^N) \subseteq C([0,T];\mathbb{R}^N)$, we see that the function

$$\overline{u} = \hat{u}|_{W_0^{1,p}([0,T];\mathbb{R}^N)}\colon\ W_0^{1,p}([0,T];\mathbb{R}^N)_w \longrightarrow L^{p'}([0,T];\mathbb{R}^N)_w$$

is continuous.

Consider the Dirichlet problem

$$\begin{cases} (\|x'(t)\|_{\mathbb{R}^N}^{p-2}x'(t))' \in A(x(t)) + \overline{u}(x)(t) \quad \text{a.e. on } [0,T], \\ x(0) = x(T) = 0. \end{cases} \tag{3.2}$$

Since $\overline{u}\colon\ W_0^{1,p}([0,T];\mathbb{R}^N) \longrightarrow W^{-1,p'}([0,T];\mathbb{R}^N)$ is completely continuous (from the compactness of the embedding $L^{p'}([0,T];\mathbb{R}^N) \subseteq W^{-1,p'}([0,T];\mathbb{R}^N)$), we see that the function

$$W_0^{1,p}([0,T];\mathbb{R}^N) \ni x \longmapsto U(x) + \hat{A}(x) + \overline{u}(x) \in 2^{W^{-1,p'}([0,T];\mathbb{R}^N)}$$

is pseudomonotone and of course coercive (from Dugundji's theorem and the definitions of F_1 and $\overline{u}$, we have for all $x \in W_0^{1,p}([0,T];\mathbb{R}^N)$, $\|\overline{u}(x)\|_{p'} \leqslant \|a_M\|_{p'}$). Then, we can find $\overline{x} \in W_0^{1,p}([0,T];\mathbb{R}^N)$, such that

$$0 \in U(\overline{x}) + \hat{A}(\overline{x}) + \overline{u}(\overline{x}).$$

Evidently $\overline{x} \in K_1$ and so $\overline{u}(\overline{x}) = \tilde{u}(\overline{x})$ and from Theorem II.4.6, p. 192 of Hu-Papageorgiou [19], we conclude that

$$\tilde{u}(\overline{x}) \in \operatorname{ext} S^{p'}_{F_1(\cdot,\overline{x}(\cdot))} = S^{p'}_{\operatorname{ext} F_1(\cdot,\overline{x}(\cdot))}.$$

Claim 2. *For all $t \in [0,T]$, we have $\|\overline{x}(t)\|_{\mathbb{R}^N} \leqslant M$.*

P r o o f. Suppose that the claim is not true. Then, we can find $t_1, t_2 \in [0,T]$, with $t_1 < t_2$, such that

$$\begin{aligned} &\|\overline{x}(t_1)\|_{\mathbb{R}^N} = M, \\ &\|\overline{x}(t_2)\|_{\mathbb{R}^N} = \max_{t\in[0,T]} \|\overline{x}(t)\|_{\mathbb{R}^N} > M, \text{ and} \\ &\|\overline{x}(t)\|_{\mathbb{R}^N} > M \quad \forall\, t \in (t_1, t_2]. \end{aligned}$$

Since $0 \in U(\overline{x}) + \hat{A}(\overline{x}) + \tilde{u}(\overline{x})$, we obtain that

$$(\|\overline{x}'(t)\|_{\mathbb{R}^N}^{p-2}\overline{x}'(t))' = \overline{y}(t) + f(t) + \varphi(\overline{x}(t)) - \varphi(p_M(\overline{x}(t)))$$

almost everywhere on $[0,T]$ with $f \in S^{p'}_{F(\cdot,p_M(\overline{x}(\cdot)))}$ and $\overline{y} \in \hat{A}(\overline{x})$. Then, for almost all $t \in (t_1,t_2]$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\overline{x}'(t)\|_{\mathbb{R}^N}^{p-2}\overline{x}'(t),\overline{x}(t))_{\mathbb{R}^N}\\
&\qquad = ((\|\overline{x}'(t)\|_{\mathbb{R}^N}^{p-2}\overline{x}'(t))',\overline{x}(t))_{\mathbb{R}^N} + (\|\overline{x}'(t)\|_{\mathbb{R}^N}^{p-2}\overline{x}'(t),\overline{x}'(t))_{\mathbb{R}^N}\\
&\qquad = (\overline{y}(t) + f(t) + \varphi(\overline{x}(t)) - \varphi(p_M(\overline{x}(t))),\overline{x}(t))_{\mathbb{R}^N} + \|\overline{x}'(t)\|_{\mathbb{R}^N}^{p}\\
&\qquad \geqslant \frac{\|\overline{x}(t)\|_{\mathbb{R}^N}}{M}(f(t),p_M(\overline{x}(t)))_{\mathbb{R}^N} + \|\overline{x}(t)\|_{\mathbb{R}^N}^{p} - M^{p-1}\|\overline{x}(t)\|_{\mathbb{R}^N}.
\end{aligned}$$

By virtue of hypothesis $\mathrm{H(F)}_1$ (iv), we have

$$\frac{\|\overline{x}(t)\|_{\mathbb{R}^N}}{M}(f(t),p_M(\overline{x}(t)))_{\mathbb{R}^N} \geqslant 0 \quad \text{a.e. on } (t_1,t_2],$$

and so we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\overline{x}'(t)\|_{\mathbb{R}^N}^{p-2}\overline{x}'(t),\overline{x}(t))_{\mathbb{R}^N} \geqslant \|\overline{x}(t)\|_{\mathbb{R}^N}(\|\overline{x}(t)\|_{\mathbb{R}^N}^{p-1} - M^{p-1}) > 0 \tag{3.3}$$

almost everywhere on $(t_1,t_2]$.

Of course $0 < t_2 < T$. Then for $r(t) \overset{\mathrm{df}}{=} \|x(t)\|_{\mathbb{R}^N}^2$, we have $r'(t_2) = 0$ and so $(x'(t_2),x(t_2))_{\mathbb{R}^N} = 0$. From (3.3), we see that the function

$$(t_1,t_2] \ni t \longmapsto (\|\overline{x}'(t)\|_{\mathbb{R}^N}^{p-2}\overline{x}'(t),\overline{x}(t))_{\mathbb{R}^N} \in \mathbb{R}$$

is strictly increasing. This means that

$$\|\overline{x}'(t)\|_{\mathbb{R}^N}^{p-2}(\overline{x}'(t),\overline{x}(t))_{\mathbb{R}^N} < \|\overline{x}'(t_2)\|_{\mathbb{R}^N}^{p-2}(\overline{x}'(t_2),\overline{x}(t_2))_{\mathbb{R}^N} \quad \forall\, t \in [t_1,t_2),$$

so

$$(\overline{x}'(t),\overline{x}(t))_{\mathbb{R}^N} = \frac{1}{2}r'(t) < 0 \quad \forall\, t \in [t_1,t_2).$$

Thus

$$M^2 < \|\overline{x}(t_2)\|_{\mathbb{R}^N}^2 < \|\overline{x}(t_1)\|_{\mathbb{R}^N}^2,$$

which is a contradition to the fact that $\|\overline{x}(t_2)\|_{\mathbb{R}^N} = \max\limits_{t\in[0,T]} \|\overline{x}(t)\|_{\mathbb{R}^N} \geqslant \|\overline{x}(t_1)\|_{\mathbb{R}^N}$. So Claim 2 is proved. $\square$

Because of Claim 2, we have $p_M(\overline{x}(t)) = \overline{x}(t)$ for all $t \in [0,T]$ and therefore

$$\begin{cases} (\|\overline{x}'(t)\|_{\mathbb{R}^N}^{p-2}\overline{x}'(t))' \in A(\overline{x}(t)) + \operatorname{ext} F(t,\overline{x}(t)) \quad \text{a.e. on } [0,T], \\ \overline{x}(0) = \overline{x}(T) = 0, \end{cases}$$

and so $\overline{x} \in C^1([0,T];\mathbb{R}^N)$ is an extremal solution. $\square$

4. Strong relaxation

For the strong relaxation theorem, we shall need the following stronger conditions on the multifunction $F(t,\zeta)$:

$H(F)_2$ $F\colon [0,T]\times\mathbb{R}^N \longrightarrow P_k(\mathbb{R}^N)$ is a multifunction such that

(i) the function $[0,T] \ni t \longmapsto F(t,\zeta) \in 2^{\mathbb{R}^N}$ is measurable, for all $\zeta \in \mathbb{R}^N$;

(ii) for almost all $t \in [0,T]$ and all $\zeta,\zeta' \in \mathbb{R}^N$, we have $h(\overline{\operatorname{conv}}\, F(t,\zeta), \overline{\operatorname{conv}}\, F(t,\zeta')) \leqslant k(t)\|\zeta-\zeta'\|_{\mathbb{R}^N}$ with $k \in L^1([0,T])_+$ such that $\|k\|_1 < 1$;

(iii) for all $k > 0$, there exists $a_k \in L^1([0,T])_+$ such that for almost all $t \in [0,T]$, all $\zeta \in \mathbb{R}^N$ with $\|\zeta\|_{\mathbb{R}^N} \leqslant k$, and all $u \in F(t,\zeta)$, we have $\|u\|_{\mathbb{R}^N} \leqslant a_k(t)$;

(iv) there exists $M > 0$ such that for almost all $t \in [0,T]$, all $\zeta \in \mathbb{R}^N$ with $\|\zeta\|_{\mathbb{R}^N} = M$, and all $u \in F(t,\zeta)$, we have $(u,\zeta)_{\mathbb{R}^N} \geqslant 0$.

Remark 4.1. It is well known even for first order multivalued Cauchy problems that simple h-continuity of $F(t,\cdot)$ does not suffice to have a relaxation result (see Hu-Papageorgiou [20, p. 217]). This is the reason why we use the stronger hypothesis $H(F)_2$ (ii).

Recall that by $S_{ec} \subseteq C^1([0,T];\mathbb{R}^N)$ we denote the solution set of problem (1.2), and by $S_c \subseteq C^1([0,T];\mathbb{R}^N)$, the solution set of problem (1.3).

Theorem 4.2. *If hypotheses* H(A) *and* $H(F)_2$ *hold, then* $\overline{S}_{ec}^{C^1([0,T];\mathbb{R}^N)} = S_c$.

Proof. Let φ and p_M be as in the proof of Theorem 3.1. Let $K_1 \subseteq W_0^{1,2}([0,T];\mathbb{R}^N) \subseteq C([0,T];\mathbb{R}^N)$ be as in the proof of Theorem 3.1. We know that K_1 is compact in $C([0,T];\mathbb{R}^N)$ and weakly compact in $W_0^{1,2}([0,T];\mathbb{R}^N)$. Let $x \in S_c$. Then, by definition, we have

$$(4.1) \qquad \begin{cases} x''(t) \in A(x(t)) + f(t) \quad \text{a.e. on } [0,T], \\ x(0) = x(T) = 0, \end{cases}$$

with $f \in S^2_{\overline{\operatorname{conv}}\, F(\cdot,x(\cdot))}$. Also, for a given $y \in K_1$ and $\varepsilon > 0$, let

$$\Gamma_{y,\varepsilon}\colon [0,T] \longrightarrow 2^{\mathbb{R}^N}\setminus\{\emptyset\}$$

be defined by

$$\Gamma_{y,\varepsilon}(t) \stackrel{\mathrm{df}}{=} \Big\{\zeta \in \overline{\operatorname{conv}}\, F_1(t,y(t)) \colon \\ \|f(t)-\zeta\|_{\mathbb{R}^N} < \frac{\varepsilon}{2M_1T} + d(f(t), \overline{\operatorname{conv}}\, F_1(t,y(t)))\Big\},$$

with

$$F_1(t,\zeta) \stackrel{\mathrm{df}}{=} F(t,p_M(\zeta)) - \varphi(\zeta) + \varphi(p_M(\zeta))$$

and $M_1 \stackrel{\mathrm{df}}{=} \sup\{\|z\|_{C([0,T];\mathbb{R}^N)};\ z \in K_1\}$. Clearly $\operatorname{Gr}\Gamma_{y,\varepsilon} \in \mathcal{L}([0,T]) \times \mathcal{B}(\mathbb{R}^N)$ (with $\mathcal{L}([0,T])$ being the Lebesgue σ-field of $[0,T]$) and we can use the Yankov-von Neumann-Aumann selection theorem (see Hu-Papageorgiou [19, Theorem II.2.14, p. 158]) and obtain $z \in S^2_{\overline{\operatorname{conv}}\, F_1(\cdot,y(\cdot))}$ such that $z(t) \in \Gamma_\varepsilon(t)$ almost everywhere on $[0,T]$. So, we can define the multifunction $L_\varepsilon\colon\ K_1 \longrightarrow 2^{L^2([0,T];\mathbb{R}^N)}$ as follows

$$L_\varepsilon(y) \stackrel{\mathrm{df}}{=} \Big\{z \in S^2_{\overline{\operatorname{conv}}\, F_1(\cdot,y(\cdot))} \colon \\ \|f(t)-z(t)\|_{\mathbb{R}^N} < \frac{\varepsilon}{2M_1T} + d(f(t), \overline{\operatorname{conv}}\, F_1(t,y(t))) \quad \text{a.e. on } [0,T]\Big\}.$$

We have just seen that L_ε has nonempty values, which are clearly decomposable. Moreover, the function $y \longmapsto L_\varepsilon(y)$ is lower semicontinuous (see Hu-Papageorgiou [19, p. 239]) and then so is $y \longmapsto \overline{L_\varepsilon(y)}$. Therefore we can find a continuous map

$$u_\varepsilon\colon\ K_1 \longmapsto L^2([0,T];\mathbb{R}^N),$$

such that

$$u_\varepsilon(y) \in \overline{L_\varepsilon(y)} \quad \forall\, y \in K_1$$

(see Hu-Papageorgiou [19, Theorem II.8.7, p. 245]). From Hu-Papageorgiou [19, Theorem II.8.31, p. 260], we can find a continuous map

$$v_\varepsilon\colon\ K_1 \longrightarrow L^1_w([0,T];\mathbb{R}^N),$$

such that

$$v_\varepsilon(y) \in \operatorname{ext} S^2_{\overline{\operatorname{conv}}\, F_1(\cdot,y(\cdot))} = S^2_{\operatorname{ext}\overline{\operatorname{conv}}\, F_1(\cdot,y(\cdot))} \quad \forall\, y \in K_1$$

and

$$\|v_\varepsilon(y) - u_\varepsilon(y)\|_w < \varepsilon \quad \forall\, y \in K_1$$

(recall that $\|\cdot\|_w$ is the weak norm on $L^1([0,T];\mathbb{R}^N)$ introduced in the proof of Theorem 3.1).

Now, let $\varepsilon_n \searrow 0$ and set $u_n = u_{\varepsilon_n}$, $v_n = v_{\varepsilon_n}$. Consider the following boundary value problem:

$$\begin{cases} x_n''(t) \in A(x_n(t)) + v_n(x_n)(t) \quad \text{a.e. on } [0,T], \\ x_n(0) = x_n(T) = 0. \end{cases} \tag{4.2}$$

As before, we show that problem (4.2) has a solution $x_n \in C^1([0,T];\mathbb{R}^N)$, such that

$$\|x_n\|_{C([0,T];\mathbb{R}^N)} \leqslant M, \quad \forall\, n \geqslant 1$$

(see problem (3.2) in the proof of Theorem 3.1). We have $\{x_n\}_{n\geqslant 1} \subseteq K_1$ and so, passing to a subsequence if necessary, we may assume that

$$\begin{aligned} x_n &\longrightarrow y \quad \text{in } C([0,T];\mathbb{R}^N), \\ x_n &\longrightarrow y \quad \text{weakly in } W_0^{1,2}([0,T];\mathbb{R}^N). \end{aligned}$$

From (4.1) and (4.2), exploiting the monotonicity of A, we obtain

$$\langle x_n'' - x'', x - x_n \rangle \leqslant \langle v_n(x_n) - f, x - x_n \rangle,$$

and so, by Green's identity, we have

$$\begin{aligned} \|x_n' - x'\|_2^2 \leqslant & \int_0^T (v_n(x_n)(t) - u_n(x_n)(t), x(t) - x_n(t))_{\mathbb{R}^N}\, \mathrm{d}t \\ & + \int_0^T (u_n(x_n)(t) - f(t), x(t) - x_n(t))_{\mathbb{R}^N}\, \mathrm{d}t. \end{aligned}$$

Because, by construction, we have $\|v_n(x_n) - u_n(x_n)\|_w \longrightarrow 0$, so also

$$\left| \int_0^T (v_n(x_n)(t) - u_n(x_n)(t), x(t) - x_n(t))_{\mathbb{R}^N}\, \mathrm{d}t \right| \leqslant \varepsilon_n'$$

for some $\varepsilon_n' \searrow 0$.

Recall that $u_n(x_n)(t) \in \overline{\operatorname{conv}}\, F_1(t, x_n(t))$ almost everywhere on $[0,T]$ and since $\|x_n\|_{C([0,T];\mathbb{R}^N)} \leqslant M$, we have $F_1(t, x_n(t)) = F(t, x_n(t))$. Hence

$$\begin{aligned} &\int_0^T (u_n(x_n)(t) - f(t), x(t) - x_n(t))_{\mathbb{R}^N}\, \mathrm{d}t \\ &\qquad \leqslant \int_0^T \|u_n(x_n)(t) - f(t)\|_{\mathbb{R}^N} \|x(t) - x_n(t)\|_{\mathbb{R}^N}\, \mathrm{d}t \end{aligned}$$

$$\leqslant \int_0^T \Big[\frac{\varepsilon_n}{2M_1T} + h(\overline{\operatorname{conv}}\, F(t,x_n(t)), \overline{\operatorname{conv}}\, F(t,x(t)))\Big] \|x(t)-x_n(t)\|_{\mathbb{R}^N}\,\mathrm{d}t$$
$$\leqslant \varepsilon_n + \int_0^T k(t)\|x(t)-x_n(t)\|^2_{\mathbb{R}^N}\,\mathrm{d}t$$
$$\leqslant \varepsilon_n + \int_0^T k(t)\int_0^T \|x'(s)-x_s'(s)\|^2_{\mathbb{R}^N}\,\mathrm{d}s\,\mathrm{d}t.$$

Therefore, finally, we can write

$$\|x_n'-x'\|_2^2 \leqslant \varepsilon_n'' + \|k\|_1\|x_n'-x'\|_2^2,$$

with $\varepsilon_n'' = \varepsilon_n + \varepsilon_n'$, and so

$$\limsup_{n\to+\infty} \|x_n'-x'\|_2^2 \leqslant \|k\|_1 \limsup_{n\to+\infty} \|x_n'-x'\|_2^2.$$

Because by hypothesis $\mathrm{H}(\mathrm{F})_2$ (ii) we have $\|k\|_1 < 1$, it follows that

$$\limsup_{n\to+\infty} \|x_n'-x'\|_2 = 0,$$

so $\|y'-x'\|_2 = 0$ and $x_n' \longrightarrow x'$ in $L^2([0,T];\mathbb{R}^N)$ and thus $y' = x'$.

From this and the Dirichlet boundary conditions, we infer that $x = y$. Therefore

$$x_n \longrightarrow x \quad \text{in } W_0^{1,2}([0,T];\mathbb{R}^N).$$

From (4.2), we see that $\{x_n''\}_{n\geqslant 1} \subseteq L^2([0,T];\mathbb{R}^N)$ is bounded and so, passing to a subsequence if necessary, we may assume that

$$x_n'' \longrightarrow z \quad \text{weakly in } L^2([0,T];\mathbb{R}^N),$$

and since $\{x_n\}_{n\geqslant 1} \subseteq C^1([0,T];\mathbb{R}^N)$ and $x \in C^1([0,T];\mathbb{R}^N)$, we can easily see that $z = x''$. Also from the above construction, we have

$$v_n(x_n) \longrightarrow f \quad \text{weakly in } L^2([0,T];\mathbb{R}^N).$$

Since $x_n'' - v_n(x_n) \in \hat{A}(x_n)$ and $\hat{A}$ is maximal monotone, we have

$$x'' - f \in \hat{A}(x).$$

Therefore

$$x_n \longrightarrow x \quad \text{weakly in } W^{2,2}([0,T];\mathbb{R}^N),$$

and from the compactness of the embedding $W^{2,2}([0,T];\mathbb{R}^N) \subseteq C^1([0,T];\mathbb{R}^N)$, we conclude that

$$x_n \longrightarrow x \quad \text{in } C^1([0,T];\mathbb{R}^N).$$

Finally, note that $x_n \in S_{ec}$ and that $S_c \subseteq C^1([0,T];\mathbb{R}^N)$ is closed. □

5. Examples

In this section, we present two examples of interest which fit into our general framework and illustrate the general character of our work here.

Example 1. Let $\varphi\colon \mathbb{R}^N \longrightarrow \mathbb{R}$ be a continuous convex function and $F\colon [0,T] \times \mathbb{R}^N \longrightarrow P_{fc}(\mathbb{R}^N)$ a multifunction satisfying hypotheses $\mathrm{H(F)}_2$. Consider the following "gradient" system with in general nonsmooth potential φ:

$$\tag{5.1} \begin{cases} (\|x'(t)\|_{\mathbb{R}^N}^{p-2} x'(t))' \in \partial\varphi(x(t)) + F(t,x(t)) \quad \text{a.e. on } [0,T], \\ x(0) = x(T) = 0. \end{cases}$$

If $A = \partial\varphi$, then A is maximal monotone with $\operatorname{dom} A = \mathbb{R}^N$. So the results of this paper apply to the system and its solutions can be obtained as the uniform limit of extremal solutions (i.e. solutions corresponding to the orientor field $\operatorname{ext} F(t,\zeta)$). The results of this paper apply to problem (5.1).

Example 2. Suppose that $p = 2$, $A\colon \mathbb{R}^N \longrightarrow 2^{\mathbb{R}^N}$ satisfies hypothesis H(A) and let $f\colon [0,T] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a function such that

(i) the function $t \longmapsto f(t,\zeta)$ is measurable;
(ii) $\|f(t,\zeta) - f(t,\zeta')\|_{\mathbb{R}^N} \leqslant k_1(t)\|\zeta - \zeta'\|_{\mathbb{R}^N}$ almost everywhere on $[0,T]$, for some $k_1 \in L^1([0,T])$;
(iii) $f(\cdot,0) \in L^1([0,T];\mathbb{R}^N)$;
(iv) for almost all $t \in [0,T]$ and all $\zeta \in \mathbb{R}^N$ with $\|\zeta\|_{\mathbb{R}^N} = M$, we have

$$(f(t,\zeta),\zeta)_{\mathbb{R}^N} \geqslant 0.$$

Also let $U\colon [0,T] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a multifunction such that

(i) the function $t \longmapsto U(t,\zeta)$ is measurable for all $\zeta \in \mathbb{R}^N$;
(ii) $h(U(t,\zeta), U(t,\zeta')) \leqslant k_2(t)\|\zeta - \zeta'\|_{\mathbb{R}^N}$ almost everywhere on $[0,T]$, for some $k_2 \in L^1([0,T])$;
(iii) for all $r > 0$, almost all $t \in [0,T]$, all $\zeta \in \mathbb{R}^N$ with $\|\zeta\|_{\mathbb{R}^N} \leqslant r$ and all $u \in U(t,\zeta)$, we have $\|u\|_{\mathbb{R}^N} \leqslant a_r(t)$, where $a_r \in L^1([0,T])_+$.

Finally let $B \in L^\infty([0,T];\mathbb{R}^{N\times N})$ be such that

(i) for almost all $t \in T$, all $\zeta \in \mathbb{R}^N$ with $\|\zeta\|_{\mathbb{R}^N} = M$ and all $u \in U(t,\zeta)$, we have $(B(t)u,\zeta)_{\mathbb{R}^N} \geqslant 0$.

Suppose that $\|k_1\|_1 + \|B\|_\infty \|k_2\|_1 \leqslant 1$ and that $F(t,\zeta) = f(t,\zeta) + B(t)U(t,\zeta)$. We can easily verify that the last multifunction satisfies hypotheses $\mathrm{H(F)}_2$. So if we consider the feedback control problem:

$$\begin{cases} x''(t) \in A(x(t)) + f(t,x(t)) + B(t)u(t) \quad \text{a.e. on } [0,T], \\ x(0) = x(T), \quad u(t) \in U(t,x(t)) \ \text{a.e. on } [0,T], \end{cases}$$

then every state of it can be approximated with respect to the L^∞-norm (i.e. uniformly in $t \in [0,T]$) by a state generated by a bang-bang control, i.e. a control $u(t)$ such that $u(t) \in \operatorname{ext} U(t,x(t))$ for almost all $t \in [0,T]$. In practice such a controls are easier to realize. Moreover, if we have some continuous objective functional $J(x)$ to minimize, then we can always find an ε-optimal state procedure by a bang-bang control.

There are three open problems left by the work initiated in this paper. The first is to extend the results of this paper to the periodic and Neumann problems. In this respect the corresponding papers on first order inclusions by De Blasi-Górniewicz-Pianigiani [3], Hu-Kandilakis-Papageorgiou [17] and Hu-Papageorgiou [18] may be helpful. The second is to prove a stronger relaxation theorem for problems driven by the p-Laplacian. The third is to allow $\operatorname{dom} A \neq \mathbb{R}^N$ (cf. hypothesis H(A)). In this way we could fit in our framework variational inequalities.

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