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PROBLEMS CONCERNING n-WEAK AMENABILITY OF A BANACH ALGEBRA

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Abstract. In this paper we extend the notion of n-weak amenability of a Banach algebra $\mathscr{A}$ when $n \in \mathbb{N}$. Technical calculations show that when $\mathscr{A}$ is Arens regular or an ideal in $\mathscr{A}^{**}$, then $\mathscr{A}^*$ is an $\mathscr{A}^{(2n)}$-module and this idea leads to a number of interesting results on Banach algebras. We then extend the concept of n-weak amenability to $n \in \mathbb{Z}$.

Keywords: Banach algebra, weakly amenable, Arens regular, n-weakly amenable

MSC 2000: 46H20, 46H40

1. Introduction

Let $\mathscr{A}$ be a Banach algebra, X a Banach $\mathscr{A}$-bimodule. Then we denote by X^* the topological dual space of X; the value of $x^* \in X^*$ at $x \in X$ is denoted by $\langle x, x^* \rangle$. We recall that X^* is a Banach $\mathscr{A}$-bimodule under the actions

$$\langle x, ax^* \rangle = \langle xa, x^* \rangle, \quad \langle x, x^*a \rangle = \langle ax, x^* \rangle \quad (a \in \mathscr{A},\ x \in X,\ x^* \in X^*).$$

A derivation $D\colon \mathscr{A} \longrightarrow X$ is a (bounded) linear map such that

$$D(ab) = D(a)b + aD(b) \quad (a, b \in \mathscr{A}).$$

For each $x \in X$, $\delta_x(a) = ax - xa$ is a derivation, which is called inner. The first cohomology group $H^1(\mathscr{A}, X)$ is the quotient of the space of derivations by the inner derivations, and in many situations triviality of this space is of considerable importance. In particular, $\mathscr{A}$ is called contractible if $H^1(\mathscr{A}, X) = \{0\}$ for every Banach $\mathscr{A}$-bimodule X, $\mathscr{A}$ is called amenable if $H^1(\mathscr{A}, X^*) = \{0\}$ for every Banach $\mathscr{A}$-bimodule X, $\mathcal{A}$ is called n-weakly amenable if $H^1(\mathscr{A}, \mathscr{A}^{(n)}) = \{0\}$, and weakly amenable if $\mathscr{A}$ is 1-weakly amenable. For the theory of amenable and weakly amenable Banach algebras see [1], [2], [4], [6], [8] and [9] for example.

Let $\mathscr{A}$ be a Banach algebra. Given $a^* \in \mathscr{A}^*$ and $F \in \mathscr{A}^{**}$, then Fa^* and a^*F are defined in $\mathscr{A}^*$ by the formulae

$$\langle a, Fa^* \rangle = \langle a^*a, F \rangle, \quad \langle a, a^*F \rangle = \langle aa^*, F \rangle \quad (a \in \mathscr{A}).$$

Next, for $F, G \in \mathscr{A}^{**}$, $F \,\square\, G$ and $F \,\triangle\, G$ are defined in $\mathscr{A}^{**}$ by the formulae

$$\langle a^*, F \,\square\, G \rangle = \langle Ga^*, F \rangle, \quad \langle a^*, F \,\triangle\, G \rangle = \langle a^*F, G \rangle \quad (a^* \in \mathscr{A}^*).$$

Then $\mathscr{A}^{**}$ is a Banach algebra with respect to either of the products $\square$ and $\triangle$. These products are called the first and second Arens products on $\mathscr{A}^{**}$, respectively. The algebra $\mathscr{A}$ is called Arens regular if the two products $\square$ and $\triangle$ coincide. For the general theory of Arens products, see [5] and [10], for example.

Let $\mathscr{A}$ be a Banach algebra, $n \in \mathbb{N} \cup \{0\}$ and let $P_n \colon \mathscr{A}^{(n)} \longrightarrow \mathscr{A}^{(n+2)}$ be the natural embedding, i.e., $\langle \varphi_{n+1}, P_n\varphi_n \rangle = \langle \varphi_n, \varphi_{n+1} \rangle$ $(\varphi_n \in \mathscr{A}^{(n)},\ \varphi_{n+1} \in \mathscr{A}^{(n+1)})$, where $\mathscr{A}^{(0)} = \mathscr{A}$ and $\mathscr{A}^{(n)}$ is the nth dual of $\mathscr{A}$. We shall require the following standard properties of the Arens products. Suppose (a_α) and (b_β) are nets in $\mathscr{A}$ with $P_0 a_\alpha \longrightarrow F$ and $P_0 b_\beta \longrightarrow G$ in $(\mathscr{A}^{**}, \sigma)$, where $\sigma = \sigma(\mathscr{A}^{**}, \mathscr{A}^*)$ is the weak* topology on $\mathscr{A}^{**}$. Then $F \,\square\, G = \lim\limits_\alpha \lim\limits_\beta P_0(a_\alpha b_\beta)$ and $F \,\triangle\, G = \lim\limits_\beta \lim\limits_\alpha P_0(a_\alpha b_\beta)$ in $(\mathscr{A}^{**}, \sigma)$. Also, for $a \in \mathscr{A}$ and $F \in \mathscr{A}^{**}$, we have $P_0(a) \,\triangle\, F = P_0(a) \,\square\, F$ and $F \,\triangle\, P_0(a) = F \,\square\, P_0(a)$.

By easy calculations we can obtain the following properties of the P_n maps.

Lemma 1.1. *Let $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$. Then*

(i) $P_n^{**} P_n = P_{n+2} P_n$;

(ii) $P_n^* P_{n+1} = \mathrm{id}$;

(iii) $P_n^{(2m+1)} P_{n+2m+1} \ldots P_{n+3} P_{n+1} = P_{n+2m-1} \ldots P_{n+3} P_{n+1}$;

(iv) $P_n^{(2m)} P_{n+2m-2} = P_{n+2m} P_n^{(2m-2)}$.

Lemma 1.2. *Let $\mathscr{A}$ be a Banach algebra, $n \in \mathbb{N}$ and let $D \colon \mathscr{A} \longrightarrow \mathscr{A}^{(n)}$ be a derivation. Then $P_{n-1}^* P_{n+1}^* \ldots P_{n+2m-3}^* D^{(2m)} P_{2m-2} P_{2m-4} \ldots P_0 = D$ $(m \in \mathbb{N})$.*

Proof. It is enough to show that $P_{n+(2m-3)}^* D^{(2m)} P_{2m-2} = D^{(2m-2)}$ for all $m \in \mathbb{N}$. For $\varphi \in \mathscr{A}^{(2m-2)}$ and $\psi \in \mathscr{A}^{(n+2m-3)}$ we have

$$\begin{aligned} \langle \psi, P_{n+2m-3}^* D^{(2m)} P_{2m-2}(\varphi) \rangle &= \langle D^{(2m-1)} P_{n+2m-3}(\psi), P_{2m-2}(\varphi) \rangle \\ \langle \varphi, D^{(2m-1)} P_{n+2m-1}(\psi) \rangle &= \langle \psi, D^{(2m-2)}(\varphi) \rangle, \end{aligned}$$

and so $P_{n+(2m-3)}^* D^{(2m)} P_{2m-2} = D^{(2m-2)}$. $\square$

2. When $\mathscr{A}^{(m)}$ is an $\mathscr{A}^{(2n)}$-module?

Let $\mathscr{A}$ be a Banach algebra. Clearly $\mathscr{A}^{(4)}$ is a Banach algebra with four Arens products. We denote these algebras by $(\mathscr{A}^4, \Box\Box) = ((\mathscr{A}^{**}, \Box)^{**}, \Box)$, $(\mathscr{A}^4, \triangle\Box) = ((\mathscr{A}^{**}, \triangle)^{**}, \Box)$, $(\mathscr{A}^4, \Box\triangle) = ((\mathscr{A}^{**}, \Box)^{**}, \triangle)$, $(\mathscr{A}^4, \triangle\triangle) = ((\mathscr{A}^{**}, \triangle)^{**}, \triangle)$. For $a \in \mathscr{A}$ and $\varphi \in \mathscr{A}^{(4)}$ it is easy to check that

$$P_2P_0(a) \Box\Box\, \varphi = P_2P_0(a) \Box\triangle\, \varphi = P_2P_0(a) \triangle\Box\, \varphi = P_2P_0(a) \triangle\triangle\, \varphi,$$
$$\varphi \Box\Box\, P_2P_0(a) = \varphi \Box\triangle\, P_2P_0(a) = \varphi \triangle\Box\, P_2P_0(a) = \varphi \triangle\triangle\, P_2P_0(a).$$

Let $\mathscr{A}$ be a Banach algebra and $n \in \mathbb{N}$. Consider the maps $(a^*, \varphi_{2n}) \mapsto a^* \cdot \varphi_{2n}$ and $(a^*, \varphi_{2n}) \mapsto \varphi_{2n} \cdot a^*$ from $\mathscr{A}^* \times \mathscr{A}^{(2n)}$ into $\mathscr{A}^*$ defined by

$$\langle a, a^* \cdot \varphi_{2n}\rangle = \langle P_{2n-3} \dots P_3P_1(aa^*), \varphi_{2n}\rangle,$$
$$\langle a, \varphi_{2n} \cdot a^*\rangle = \langle P_{2n-3} \dots P_3P_1(a^*a), \varphi_{2n}\rangle \quad (a \in \mathscr{A}).$$

Then $a^* \cdot \varphi_{2n} = a^*P_1^*P_3^* \dots P_{2n-3}^*(\varphi_{2n})$ and $\varphi_{2n} \cdot a^* = P_1^*P_3^* \dots P_{2n-3}^*(\varphi_{2n})a^*$. Clearly these maps are continuous and bilinear. Note that with respect to these actions $\mathscr{A}^*$ is not necessarily a Banach $\mathscr{A}^{(2n)}$-module. By dualizing these actions we obtain continuous bilinear maps from $\mathscr{A}^{(m)} \times \mathscr{A}^{(2n)}$ into $\mathscr{A}^{(m)}$ for every $m \in \mathbb{N}$. For example, for $F \in \mathscr{A}^{**}$ and $\varphi_{2n} \in \mathscr{A}^{(2n)}$ we have

$$\begin{aligned}\langle a^*, F \cdot \varphi_{2n}\rangle &= \langle \varphi_{2n} \cdot a^*, F\rangle \\ &= \langle P_1^*P_3^* \dots P_{2n-3}^*(\varphi_{2n})a^*, F\rangle \\ &= \langle a^*, F \,\Box\, P_1^*P_3^* \dots P_{2n-3}^*(\varphi_{2n})\rangle \quad (a^* \in \mathscr{A}^*),\end{aligned}$$

and so $F\cdot\varphi_{2n} = F\Box P_1^*P_3^* \dots P_{2n-3}^*(\varphi_{2n})$. Similarly, $\varphi_{2n}\cdot F = P_1^*P_3^* \dots P_{2n-3}^*(\varphi_{2n})\triangle F$. From now on we regard these actions as $\mathscr{A}^{(2n)}$-actions on $\mathscr{A}^{(m)}$ induced from $\mathscr{A}^*$.

Now consider the maps $(F, \varphi_{2n}) \mapsto F\cdot\varphi_{2n}$ and $(F, \varphi_{2n}) \mapsto \varphi_{2n}\cdot F$ from $\mathscr{A}^{**}\times\mathscr{A}^{(2n)}$ into $\mathscr{A}^{**}$ defined by

$$\langle a^*, F \cdot \varphi_{2n}\rangle = \langle P_{2n-3} \dots P_3P_1(a^*F), \varphi_{2n}\rangle,$$
$$\langle a^*, \varphi_{2n} \cdot F\rangle = \langle P_{2n-3} \dots P_3P_1(Fa^*), \varphi_{2n}\rangle \quad (a^* \in \mathscr{A}^*).$$

Clearly these are continuous bilinear maps, $F \cdot \varphi_{2n} = F \triangle P_1^*P_3^* \dots P_{2n-3}^*(\varphi_{2n})$ and similarly $\varphi_{2n} \cdot F = P_1^*P_3^* \dots P_{2n-3}^*(\varphi_{2n}) \Box F$. Note that these actions are different from the actions induced from $\mathscr{A}^*$. Again by dualizing these actions we have continuous bilinear maps from $\mathscr{A}^{(m)} \times \mathscr{A}^{(2n)}$ into $\mathscr{A}^{(m)}$ for every $m \geqslant 2$. So we have $\mathscr{A}^{(2n)}$-actions on $\mathscr{A}^{(m)}$ $(m \geqslant 2)$ induced from $\mathscr{A}^{**}$.

Let $\mathscr{A}$ be a Banach algebra and let $n, k \in \mathbb{N}$ be such that $n \geqslant 2k$. Set $\mathscr{B} = (\mathscr{A}^{(2k)}, \cdot)$, where $\cdot$ is one of the 2^k Arens products on $\mathscr{A}^{(2k)}$. Then $\mathscr{B}$ is a Banach algebra and $\mathscr{B}^*$ is a Banach $\mathscr{B}$-module. By a similar argument we have continuous bilinear maps from $\mathscr{B}^* \times \mathscr{A}^{(2n)}$ into $\mathscr{B}^*$ and from $\mathcal{B}^{**} \times \mathscr{A}^{(2n)}$ into $\mathscr{B}^{**}$. Therefore for every $m \geqslant 2k+1$ we have $\mathscr{A}^{(2n)}$-actions on $\mathscr{A}^{(m)}$ induced from $\mathscr{B}^*$ and for every $m \geqslant 2k+2$ we have $\mathscr{A}^{(2n)}$-actions on $\mathscr{A}^{(m)}$ induced from $\mathscr{B}^{**}$.

Proposition 2.1. *Let $\mathscr{A}$ be an Arens regular Banach algebra and $n \in \mathbb{N}$. Then $\mathscr{A}^*$ is a Banach $\mathscr{A}^{(2n)}$-bimodule with actions induced from $\mathscr{A}^*$ and any of Arens products on $\mathscr{A}^{(2n)}$. In particular, $\mathscr{A}^{(m)}$ is a Banach $\mathscr{A}^{(2n)}$-bimodule by actions induced from $\mathscr{A}^*$.*

Proof. When $n = 1$, one can immediately see that $\mathscr{A}^*$ is a left Banach $(\mathscr{A}^{**}, \Box)$-module and a right Banach $(\mathscr{A}^{**}, \triangle)$-module. Since $\mathscr{A}$ is Arens regular, $\mathscr{A}^*$ is a left and right Banach $\mathscr{A}^{**}$-module. For $a \in \mathscr{A}$, $a^* \in \mathscr{A}$ and $F, G \in \mathscr{A}^{**}$ we have

$$\begin{aligned}\langle a, (Fa^*)G\rangle &= \langle (aF)a^*, G\rangle = \langle a^*, G \,\Box\, (aF)\rangle \\ &= \langle a^*G, P_0(a) \,\Box\, F\rangle = \langle F(a^*G), P_0(a)\rangle \\ &= \langle a, F(a^*G)\rangle,\end{aligned}$$

and so $(Fa^*)G = F(a^*G)$. Hence $\mathscr{A}^*$ is a Banach $\mathscr{A}^{**}$-bimodule. Now suppose the result has been proved for n. We may assume that $\mathscr{A}^{(2n+2)} = ((\mathscr{A}^{(2n)})^{**}, \Box)$. Let $a \in \mathscr{A}$, $a^* \in \mathscr{A}^*$, $\varphi, \psi \in \mathscr{A}^{(2n+2)}$ and let (φ_α) , (ψ_β) be nets in $\mathscr{A}^{(2n)}$ such that $P_{2n}(\varphi_\alpha) \longrightarrow \varphi$ and $P_{2n}(\psi_\beta) \longrightarrow \psi$ in the weak* topology. Then

$$\begin{aligned}\langle a, a^* \cdot (\varphi \,\Box\, \psi)\rangle &= \lim_\alpha \lim_\beta \langle P_{2n-3} \dots P_3 P_1(aa^*), \varphi_\alpha \psi_\beta\rangle \\ &= \lim_\alpha \lim_\beta \langle a, a^* \cdot (\varphi_\alpha \psi_\beta)\rangle \\ &= \lim_\alpha \lim_\beta \langle a, (a^* \cdot \varphi_\alpha) \cdot \psi_\beta\rangle \\ &= \lim_\alpha \lim_\beta \langle a, a^* P_1^* P_3^* \dots P_{2n-3}^*(\varphi_\alpha) P_1^* P_3^* \dots P_{2n-3}^*(\psi_\beta)\rangle \\ &= \lim_\alpha \lim_\beta \langle P_{2n-3} \dots P_3 P_1(aa^* P_1^* P_3^* \dots P_{2n-3}^*(\varphi_\alpha)), \psi_\beta\rangle \\ &= \lim_\alpha \langle aa^* P_1^* P_3^* \dots P_{2n-3}^*(\varphi_\alpha), P_1^* P_3^* \dots P_{2n-1}^*(\psi)\rangle \\ &= \lim_\alpha \langle P_1^* \dots P_{2n-1}^*(\psi) aa^*, P_1^* P_3^* \dots P_{2n-3}^*(\varphi_\alpha)\rangle \\ &= \langle a, a^* P_1^* \dots P_{2n-1}^*(\varphi) P_1^* \dots P_{2n-1}^*(\psi)\rangle \\ &= \langle a, (a^* \cdot \varphi) \cdot \psi\rangle,\end{aligned}$$

and so $a^* \cdot (\varphi \,\square\, \psi) = (a^* \cdot \varphi) \cdot \psi$. Similarly $(\varphi \,\square\, \psi) \cdot a^* = \varphi \cdot (\psi \cdot a^*)$. On the other hand,

$$\begin{aligned}(\varphi \cdot a^*) \cdot \psi &= (P_1^* \dots P_{2n-1}^*(\varphi) a^*) P_1^* \dots P_{2n-1}^*(\psi) \\ &= P_1^* \dots P_{2n-1}^*(\varphi)(a^* P_1^* \dots P_{2n-1}^*(\psi)) \\ &= \varphi \cdot (a^* \cdot \psi).\end{aligned}$$

Hence $\mathscr{A}^*$ is a Banach $\mathscr{A}^{(2n+2)}$-bimodule. So we are done by induction. □

Proposition 2.2. *Let $\mathscr{A}$ be an Arens regular Banach algebra and $n \in \mathbb{N}$. Then with any of the Arens products on $\mathscr{A}^{(2n)}$, the $\mathscr{A}^{(2n)}$-actions on $\mathscr{A}^{**}$ induced from $\mathscr{A}^*$ and $\mathscr{A}^{**}$ coincide. In particular, $\mathscr{A}^{**}$ is a Banach $\mathscr{A}^{(2n)}$-bimodule with any of these actions.*

Proof is straightforward. □

Let $\mathscr{A}$ be a Banach algebra and $m, n \in \mathbb{N}$. Then $\mathscr{A}^{(2m)}$ is a Banach algebra with one of the 2^m Arens products. We recall that every closed subalgebra of an Arens regular Banach algebra is Arens regular. In particular, when $\mathscr{A}^{(2m)}$ is Arens regular for a Banach algebra $\mathscr{A}$ and $m \in \mathbb{N}$, then $\mathscr{A}^{**}, \mathscr{A}^{(4)}, \dots, \mathscr{A}^{(2m-2)}$ are Arens regular, and these algebras have only one Arens product. The following proposition is a generalization of Proposition 2.1 and Proposition 2.2.

Proposition 2.3. *Let $\mathscr{A}$ be a Banach algebra and let $n, m \in \mathbb{N}$ be such that $n \geqslant 2m$. If $\mathscr{A}^{(2m)}$ is Arens regular, then $\mathscr{A}^{(2m+1)}$ and $\mathscr{A}^{(2m+2)}$ are Banach $\mathscr{A}^{(2n)}$-bimodules with actions induced from $\mathscr{A}^{(2m+1)}$. Moreover, the $\mathscr{A}^{(2n)}$-actions on $\mathscr{A}^{(2m+2)}$ induced from $\mathscr{A}^{(2m+1)}$ and $\mathscr{A}^{(2m+2)}$ coincide.*

Definition 2.4. Let $\mathscr{A}$ be a Banach algebra. $\mathscr{A}$ is called completely Arens regular, if for every $n \in \mathbb{N}$, $\mathscr{A}^{(2n)}$ is Arens regular.

It is well known that every C^*-algebra is Arens regular and the second dual of a C^*-algebra is a C^*-algebra. Therefore, every C^*-algebra is completely Arens regular.

Proposition 2.5. *Let $\mathscr{A}$ be a completely Arens regular Banach algebra. Then $\mathscr{A}^{(m)}$ is a Banach $\mathscr{A}^{(2n)}$-module with actions induced $\mathscr{A}^{(m)}$.*

Proof. A direct consequence of Proposition 2.3. □

Lemma 2.6. *Let $\mathscr{A}$ be a Banach algebra and $P_0(\mathscr{A})$ a left (right) ideal in $\mathscr{A}^{**}$. Then $\mathscr{A}^*$ is a Banach $(\mathscr{A}^{**},\Box)$-module($(\mathscr{A}^{**},\triangle)$)-module).*

Proof. For $a^*\in\mathscr{A}^*$, $a\in\mathscr{A}$ and $F,G\in\mathscr{A}^{**}$ we have

$$\begin{aligned}\langle a, a^*(F\,\Box\,G)\rangle &= \langle G(aa^*),F\rangle = \langle G\,\triangle\,P_0(a)a^*,F\rangle\\ &= \langle a^*, F\,\triangle\,G\,\triangle\,P_0(a)\rangle = \langle (a^*F)G, P_0(a)\rangle\\ &= \langle a,(a^*F)G\rangle\end{aligned}$$

and

$$\begin{aligned}\langle a, F(a^*G)\rangle &= \langle a^*G\,\Box\,P_0(a),F\rangle = \langle a^*, G\,\Box\,P_0(a)\,\Box\,F\rangle\\ &= \langle F(a^*G),P_0(a)\rangle = \langle a,F(a^*G)\rangle.\end{aligned}$$

Therefore $\mathscr{A}^*$ is a Banach $(\mathscr{A}^{**},\Box)$-module. $\square$

Lemma 2.7. *Let $\mathscr{A}$ be a Banach algebra. Then $P_0(\mathscr{A})$ is an ideal in $\mathscr{A}^{**}$ with any of the Arens products if and only if $P_2P_0(\mathscr{A})$ is an ideal in $\mathscr{A}^{(4)}$ with any of the Arens products.*

Proof. Let $P_0(\mathscr{A})$ be an ideal in $\mathscr{A}^{**}$. For $a\in\mathscr{A}$ and $\varphi\in\mathscr{A}^{(4)}$, one can immediately see that

$$P_2P_0(a)\,\Box\,\Box\varphi = P_2(P_0(a)\,\Box\,P_1^*(\varphi))\quad\text{and}\quad \varphi\,\Box\,\Box P_2P_0(a) = P_2(P_1^*(\varphi)\,\Box\,P_0(a)).$$

Therefore $P_2P_0(\mathscr{A})$ is an ideal in $\mathscr{A}^{(4)}$ with any of the Arens products on $\mathscr{A}^{(4)}$. Conversely, let $P_2P_0(\mathscr{A})$ be an ideal in $\mathscr{A}^{(4)}$. Take $a\in\mathscr{A}$ and $F\in\mathscr{A}^{**}$. It is easy to see that $P_2(P_0(a)\Box F) = P_2P_0(a)\Box\Box P_2(F)\in P_2P_0(\mathscr{A})$. Hence $P_0(a)\Box F\in P_0(\mathscr{A})$ and similarly $F\,\Box\,P_0(a)\in P_0(\mathscr{A})$. Therefore $P_0(\mathscr{A})$ is an ideal in $\mathscr{A}^{**}$. $\square$

Proposition 2.8. *Let $\mathscr{A}$ be a Banach algebra and $P_0(\mathscr{A})$ an ideal in $\mathscr{A}^{**}$. Then $\mathscr{A}^*$ is a Banach $\mathscr{A}^{(2n)}$-module with any of the Arens products on $\mathscr{A}^{(2n)}$ $(n\in\mathbb{N})$.*

Proof. When $n=1$ the result is true by Lemma 2.6. Now suppose, inductively, the result has been proved for $n-1$. We may assume that $\mathscr{A}^{(2n+2)} = ((\mathscr{A}^{(2n)})^{**},\Box)$. Let $a\in\mathscr{A}$, $a^*\in\mathscr{A}^*$ and $\varphi,\psi\in\mathscr{A}^{(2n+2)}$, and let (φ_α), (ψ_β) be nets in $\mathscr{A}^{(2n)}$ such that $P_{2n}(\varphi_\alpha)\longrightarrow\varphi$ and $P_{2n}(\psi_\beta)\longrightarrow\psi$ in the weak* topology. Now we have

$$\begin{aligned}\langle a, a^*\cdot(\varphi\,\Box\,\psi)\rangle &= \langle P_{2n-1}\dots P_3P_1(aa^*),\varphi\,\Box\,\psi\rangle\\ &= \lim_\alpha\lim_\beta\langle a^*, P_1^*P_3^*\dots P_{2n-3}^*(\varphi_\alpha)\,\triangle\,P_1^*P_3^*\dots P_{2n-3}^*(\psi_\beta)\,\triangle\,P_0(a)\rangle\\ &= \lim_\alpha\langle aa^*P_1^*P_3^*\dots P_{2n-3}^*(\varphi_\alpha), P_1^*P_3^*\dots P_{2n-1}^*(\psi)\rangle\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\alpha} \langle a^*, P_1^* P_3^* \dots P_{2n-3}^*(\varphi_\alpha) \,\Box\, P_1^* P_3^* \dots P_{2n-1}^*(\psi) \,\Box\, P_0(a) \rangle \\
&= \langle aa^*, P_1^* P_3^* \dots P_{2n-1}^*(\varphi) \,\Box\, P_1^* P_3^* \dots P_{2n-1}^*(\psi) \rangle \\
&= \langle a, (a^* \cdot \varphi) \cdot \psi \rangle,
\end{aligned}
$$

and so $a^* \cdot (\varphi \,\Box\, \psi) = (a^* \cdot \varphi) \cdot \psi$. Similarly $(\varphi \,\Box\, \psi) \cdot a^* = \varphi \cdot (\psi \cdot a^*)$. Since $\mathscr{A}^*$ is a Banach $\mathscr{A}^{**}$-module,

$$
\begin{aligned}
(\varphi \cdot a^*) \cdot \psi &= (P_1^* P_3^* \dots P_{2n-1}^*(\varphi) a^*) P_1^* P_3^* \dots P_{2n-1}^*(\psi) \\
&= P_1^* P_3^* \dots P_{2n-1}^*(\varphi)(a^* P_1^* P_3^* \dots P_{2n-1}^*(\psi)) \\
&= \varphi \cdot (a^* \cdot \psi).
\end{aligned}
$$

Consequently, $\mathscr{A}^*$ is a Banach $\mathscr{A}^{(2n)}$-module. $\square$

Proposition 2.9. *Let $\mathscr{A}$ be a Banach algebra. Then $P_2((\mathscr{A}^{**}, \Box))$ is a left (right, two-sided) ideal in $(\mathscr{A}^{(4)}, \Box\Box)$ if and only if P_2^* is an $\mathscr{A}^{(4)}$-module homomorphism between left (right, two-sided) Banach $\mathscr{A}^{(4)}$-modules.*

P r o o f. Let $P_2((\mathscr{A}^{**}, \Box))$ be a left ideal in $(\mathscr{A}^{(4)}, \Box\Box)$. For $F \in \mathscr{A}^{**}$, $\varphi_4 \in \mathscr{A}^{(4)}$ and $\varphi_5 \in \mathscr{A}^{(5)}$ we have

$$
\begin{aligned}
\langle F, P_2^*(\varphi_4 \varphi_5) \rangle &= \langle P_2(F), \varphi_4 \varphi_5 \rangle = \langle P_2(F) \,\Box\Box\, \varphi_4, \varphi_5 \rangle \\
&= \langle P_2^*(\varphi_5), P_2(F) \,\Box\Box\, \varphi_4 \rangle = \langle F, \varphi_4 P_2^*(\varphi_5) \rangle.
\end{aligned}
$$

Hence P_2^* is an $\mathscr{A}^{(4)}$-module homomorphism between left Banach $\mathscr{A}^{(4)}$-modules. Conversely, for $F \in \mathscr{A}^{**}$, $\varphi_4 \in \mathscr{A}^{(4)}$ it is easy to see that

$$
P_2(F) \,\Box\Box\, \varphi_4 = P_2(P_1^*(P_2(F) \,\Box\Box\, \varphi_4)),
$$

so $P_2((\mathscr{A}^{**}, \Box))$ is a left ideal in $(\mathscr{A}^{(4)}, \Box\Box)$. $\square$

3. N-weak amenability for $N \in Z$

Lemma 3.1. *Let $\mathscr{A}$ be a Banach algebra and $D\colon \mathscr{A} \longrightarrow \mathscr{A}^*$ a derivation. Then*
(i) *$D^{**}\colon (\mathscr{A}^{**}, \Box) \longrightarrow (\mathcal{A}^{**})^*$ is satisfied in*

$$
D^{**}(F \,\Box\, G) = D^{**}(F)G + P_0^{**}(F) D^{**}(G) \quad (F, G \in \mathscr{A}^{**}),
$$

(ii) *$D^{**}\colon (\mathscr{A}^{**}, \triangle) \longrightarrow (\mathscr{A}^{**})^*$ is satisfied in*

$$
D^{**}(F \,\triangle\, G) = D^{**}(F) P_0^{**}(G) + F D^{**}(G) \quad (F, G \in \mathscr{A}^{**}).
$$

P r o o f. (i) Let $F, G \in \mathscr{A}^{**}$ and let (a_α), (b_β) be nets in $\mathscr{A}$ such that $P_0(a_\alpha) \longrightarrow F$ and $P_0(b_\beta) \longrightarrow G$ in the weak* topology. We have

$$\begin{aligned}\langle H, D^{**}(F \,\square\, G)\rangle &= \langle D^*(H), F \,\square\, G\rangle \\ &= \lim_\alpha \lim_\beta \langle D(a_\alpha)b_\beta + a_\alpha D(b_\beta), H\rangle \\ &= \lim_\alpha \lim_\beta \langle b_\beta, HD(a_\alpha) + D^*(Ha_\alpha)\rangle \\ &= \lim_\alpha \langle a_\alpha, D^*(G \,\square\, H) + P_0^*(D^{**}(G)H)\rangle \\ &= \langle H, D^{**}(F)G + P_0^{**}(F)D^{**}(G)\rangle\end{aligned}$$

and so $D^{**}(F \,\square\, G) = D^{**}(F)G + P_0^{**}(F)D^{**}(G)$.

(ii) The proof is similar to (i). $\square$

Corollary 3.2. *Let $\mathscr{A}$ be a Banach algebra and $D\colon \mathscr{A} \longrightarrow \mathscr{A}^*$ a derivation. Then*

(i) *$D^{**}\colon (\mathscr{A}^{**}, \square) \longrightarrow (\mathscr{A}^{**})^*$ is a derivation if and only if $P_0^{**}(F)D^{**}(G) = FD^{**}(G)$ for $F, G \in \mathscr{A}^{**}$;*

(ii) *$D^{**}\colon (\mathscr{A}^{**}, \triangle) \longrightarrow (\mathscr{A}^{**})^*$ is a derivation if and only if $D^{**}(F)P_0^{**}(G) = D^{**}(F)G$ for $F, G \in \mathscr{A}^{**}$.*

Definition 3.3. Let $\mathscr{A}$ be a Banach algebra $m, n \in \mathbb{N}$, and $1 \leqslant m < 2n$. The Banach algebra $\mathscr{A}^{(2n)}$ is called $(-m)$-weakly amenable, if $\mathscr{A}^{(2n-m)}$ is a Banach $\mathscr{A}^{(2n)}$-bimodule with actions induced from $\mathscr{A}^{(2n-m)}$ and $H^1(\mathscr{A}^{(2n)}, \mathscr{A}^{(2n-m)}) = \{0\}$.

Theorem 3.4. *Let $\mathscr{A}$ be a Banach algebra and $P_0(\mathscr{A})$ a left (right) ideal in $\mathscr{A}^{**}$. If $(\mathscr{A}^{**}, \square)((\mathscr{A}^{**}, \triangle))$ is (-1)-weakly amenable, then $\mathscr{A}$ is weakly amenable.*

P r o o f. By Lemma 2.6, $\mathscr{A}^*$ is a Banach $(\mathscr{A}^{**}, \square)$-module. Let $D\colon \mathscr{A} \longrightarrow \mathscr{A}^*$ be a derivation. Put $d = P_0^* D^{**}\colon (\mathscr{A}^{**}, \square) \longrightarrow \mathscr{A}^*$. For $F, G \in \mathscr{A}^{**}$, $a \in \mathscr{A}$ we have

$$\langle a, P_0^*(D^{**}(F)G)\rangle = \langle G \,\square\, P_0(a), D^{**}(F)\rangle = \langle d(F), G \triangle P_0(a)\rangle = \langle a, d(F)G\rangle$$

and

$$\langle a, P_0^*(P_0^{**}(F)D^{**}(G))\rangle = \langle P_0^*(D^{**}(G)P_0(a)), F\rangle = \langle d(G)P_0(a), F\rangle = \langle a, Fd(G)\rangle.$$

Therefore, by Lemma 3.1, d is a derivation. Since $H^1((\mathscr{A}^{**}, \square), \mathscr{A}^*) = \{0\}$, there exists $a^* \in \mathscr{A}^*$ such that $d = \delta_{a^*}$. Using Lemma 1.2 we obtain

$$\begin{aligned}aa^* - a^*a &= P_0(a)a^* - a^*P_0(a) = dP_0(a) \\ &= P_0^* D^{**} P_0(a) = D(a) \quad (a \in \mathscr{A}).\end{aligned}$$

Hence $D = \delta_{a^*}$ is an inner derivation. $\square$

Theorem 3.5. *Let $\mathscr{A}$ be a Banach algebra. If $P_0(\mathscr{A})$ is an ideal in $\mathscr{A}^{**}$ and the Banach algebra $\mathscr{A}^{(2n)}$ ($n \in \mathbb{N}$) with one of 2^n Arens products is $(-2n+1)$-weakly amenable, then $\mathscr{A}$ is weakly amenable.*

Proof. Let $D\colon \mathscr{A} \longrightarrow \mathscr{A}^*$ be a derivation. We claim that

$$d_n = P_0^* P_0^{***} \dots P_0^{(2n-1)} D^{(2n)}\colon \mathcal{A}^{(2n)} \longrightarrow \mathscr{A}^*$$

is a derivation. By Proposition 3.4, the result is true for $n = 1$. Now suppose, inductively, that the result has been proved for n. We may suppose that $\mathscr{A}^{(2n+2)} = ((\mathscr{A}^{(2n)})^{**}, \Box)$. For $a \in \mathscr{A}$, $a^* \in \mathscr{A}^*$, $\varphi, \psi \in \mathscr{A}^{(2n+2)}$, let (φ_α) and (ψ_β) be nets in $\mathscr{A}^{(2n)}$ such that $P_{2n}(\varphi_\alpha) \longrightarrow \varphi$ and $P_{2n}(\psi_\beta) \longrightarrow \psi$ in the weak* topology. Then we have

$$\begin{aligned}
\langle a, d_{n+1}(\varphi \Box \psi)\rangle &= \lim_\alpha \lim_\beta \langle a, d_n(\varphi_\alpha)\psi_\beta + \varphi_\alpha d_n(\psi_\beta)\rangle \\
&= \lim_\alpha \lim_\beta \langle P_{2n-3} \dots P_3 P_1(a d_n(\varphi_\alpha)), \psi_\beta\rangle \\
&\quad + \lim_\alpha \lim_\beta \langle \psi_\beta, D^{(2n+1)} P_0^{(2n)} \dots P_0^{**}(a P_1^* \dots P_{2n-3}^*(\varphi_\alpha))\rangle \\
&= \lim_\alpha \langle a d_n(\varphi_\alpha), P_1^* \dots P_{2n-1}^*(\psi)\rangle \\
&\quad + \lim_\alpha \langle d_{n+1}(\psi) a, P_1^* \dots P_{2n-3}^*(\varphi_\alpha)\rangle \\
&= \langle a, d_{n+1}(\varphi) \cdot \psi + \varphi \cdot d_{n+1}(\psi)\rangle,
\end{aligned}$$

so d_{n+1} is a derivation. Since $H^1(\mathscr{A}^{(2n)}, \mathcal{A}^*) = \{0\}$, there exists $a^* \in \mathscr{A}^*$ such that $d_n = \delta_{a^*}$. Using Lemma 1.1 (iv) and Lemma 1.2, we conclude that

$$\begin{aligned}
aa^* - a^*a &= P_{2n-2} \dots P_2 P_0(a) \cdot a^* - a^* \cdot P_{2n-2} \dots P_2 P_0(a) \\
&= d_n P_{2n-2} \dots P_2 P_0(a) = D(a) \quad (a \in \mathscr{A}).
\end{aligned}$$

Hence $D = \delta_{a^*}$ is inner. □

Lemma 3.6. *Let $\mathscr{A}$ be a Banach algebra, $n \in \mathbb{N}$ and let $D\colon \mathscr{A} \longrightarrow \mathscr{A}^{(2n)}$ be a derivation. Then for every $F, G \in \mathscr{A}^{**}$*

(i) $D^{**}\colon (\mathscr{A}^{**}, \Box) \longrightarrow ((\mathscr{A}^{(2n)})^{**}, \Box)$ *holds in*

$$D^{**}(F \Box G) = D^{**}(F) \Box P_{2n-2}^{**} \dots P_2^{**} P_0^{**}(G) + P_{2n-2}^{**} \dots P_2^{**} P_0^{**}(F) \Box D^{**}(G).$$

(ii) $D^{**}\colon (\mathscr{A}^{**}, \triangle) \longrightarrow ((\mathscr{A}^{(2n)})^{**}, \triangle)$ *holds in*

$$D^{**}(F \triangle G) = D^{**}(F) \triangle P_{2n-2}^{**} \dots P_2^{**} P_0^{**}(G) + P_{2n-2}^{**} \dots P_2^{**} P_0^{**}(F) \triangle D^{**}(G).$$

Proof is straightforward. □

Proposition 3.7. *Let $\mathscr{A}$ be a Banach algebra, $n \in \mathbb{N}$ and let $D\colon \mathscr{A} \longrightarrow \mathscr{A}^{(2n)}$ be a derivation. If $\mathscr{A}^{(2n)}$ is Arens regular and*

$$D^{**}(\mathscr{A}^{**})\mathscr{A}^{(2n+1)} \cup \mathscr{A}^{(2n+1)}D^{**}(\mathscr{A}^{**}) \subseteq P_{2n-1}\dots P_3P_1(\mathscr{A}^*),$$

then $D^{**}\colon \mathscr{A}^{**} \longrightarrow (\mathscr{A}^{(2n)})^{**}$ is a derivation.

Proof. Since $\mathscr{A}^{(2n)}$ is Arens regular, $\mathscr{A}$ is Arens regular. For $\varphi_{2n+1} \in \mathscr{A}^{(2n+1)}$, $F, G \in \mathscr{A}^{**}$, there exists $a^* \in \mathscr{A}^*$ such that

$$\varphi_{2n+1} \cdot D^{**}(F) = P_{2n-1}\dots P_1(a^*).$$

By Lemma 1.1 we have

$$\begin{aligned}\langle \varphi_{2n+1}, D^{**}(F) \,\Box\, P_{2n-2}^{**}\dots P_0^{**}(G)\rangle &= \langle P_{2n-1}\dots P_1(a^*), P_{2n-2}^{**}\dots P_0^{**}(G)\rangle \\ &= \langle P_{2n-2}\dots P_2(G), \varphi_{2n+1}D^{**}(F)\rangle \\ &= \langle \varphi_{2n+1}, D^{**}(F)G\rangle.\end{aligned}$$

Similarly, $P_{2n-2}^{**}\dots P_0^{**}(F) \,\Box\, D^{**}(G) = FD^{**}(G)$. Hence D^{**} is a derivation by Lemma 3.6. □

Lemma 3.8. *Let $\mathscr{A}$ be a Banach algebra and $D\colon \mathscr{A} \longrightarrow \mathscr{A}^{(2n+1)}$ $(n \in \mathbb{N})$ a derivation. Then $D^{**}\colon (\mathscr{A}^{**}, \Box) \longrightarrow ((\mathscr{A}^{(2n)})^{**}, \Box)^*$ is valid in*

$$D^{**}(F \,\Box\, G) = D^{**}(F)P_{2n-2}^{**}\dots P_0^{**}(G) + P_{2n}^{**}\dots P_0^{**}(F)D^{**}(G) \quad (F, G \in \mathscr{A}^{**}).$$

Proof is straightforward. □

Proposition 3.9. *Let $\mathscr{A}$ be a Banach algebra and let $D\colon \mathscr{A} \longrightarrow \mathscr{A}^{(2n+1)}$ $(n \in \mathbb{N})$ be a derivation. If*

$$D^{**}(\mathscr{A}^{**}) \cdot \mathscr{A}^{(2n+2)} \cup \mathscr{A}^{(2n+2)} \cdot D^{**}(\mathscr{A}^{**}) \subseteq P_{2n+1}\dots P_1(\mathscr{A}^*),$$

then $D^{**}\colon (\mathscr{A}^{**}, \Box) \longrightarrow (\mathscr{A}^{**})^{(2n+1)}$ is a derivation.

Proof. By Lemma 3.8, it is clear. □

Lemma 3.10. *Let $\mathscr{A}$ be a Banach algebra and $D\colon \mathscr{A} \longrightarrow \mathscr{A}^*$ a derivation. Then for every φ and ψ in $\mathscr{A}^{(4)}$*

(i) $D^{(4)}\colon (\mathscr{A}^{(4)}, \Box\Box) \longrightarrow (\mathscr{A}^{(4)}, \Box\Box)^*$ *holds in* $D^{(4)}(\varphi\Box\Box\psi) = D^{(4)}(\varphi)\psi + P_0^{(4)}(\varphi) D^{(4)}(\psi)$;

(ii) $D^{(4)}\colon (\mathscr{A}^{(4)}, \triangle\triangle) \longrightarrow (\mathscr{A}^{(4)}, \triangle\triangle)^*$ *holds in* $D^{(4)}(\varphi\triangle\triangle\psi) = D^{(4)}(\varphi)P_0^{(4)}(\psi) + \varphi D^{(4)}(\psi)$.

P r o o f. (i) Let $\xi, \varphi, \psi \in \mathscr{A}^{(4)}$ and let (F_α), (G_β) be nets in $\mathscr{A}^{**}$ such that $P_2(F_\alpha) \longrightarrow \varphi$ and $P_2(G_\beta) \longrightarrow \psi$ in the weak* topology. By Lemma 3.1 we have

$$\begin{aligned}
\langle \xi, D^{(4)}(\varphi \Box \Box \psi)\rangle &= \lim_\alpha \lim_\beta \langle D^{**}(F_\alpha \Box G_\beta), \xi\rangle \\
&= \lim_\alpha \lim_\beta \langle D^{**}(F_\alpha)G_\beta + P_0^{**}(F_\alpha)D^{**}(G_\beta), \xi\rangle \\
&= \lim_\alpha \langle \xi D^{**}(F_\alpha) + D^{(3)}(\xi \Box\Box P_0^{**}(F_\alpha)), \psi\rangle \\
&= \langle D^{(3)}(\psi \Box\Box \xi) + P_0^{(3)}(D^{(4)}(\psi)\xi), \varphi\rangle \\
&= \langle \xi, D^{(4)}(\varphi)\psi + P_0^{(4)}(\varphi)D^{(4)}(\psi)\rangle.
\end{aligned}$$

(ii) The proof is similar to (i). □

Proposition 3.11. *Let $\mathscr{A}$ be a Banach algebra and $D\colon \mathscr{A} \longrightarrow \mathscr{A}^*$ a derivation.*

(i) *If $D^{(4)}(\mathscr{A}^{(4)}) \cdot \mathscr{A}^{(4)} \subseteq P_3P_1(\mathscr{A}^*)$, then $D^{(4)}\colon (\mathscr{A}^{(4)}, \Box\Box) \longrightarrow (\mathscr{A}^{(4)})^*$ is a derivation.*

(ii) *If $\mathscr{A}^{(4)} \cdot D^{(4)}(\mathscr{A}^{(4)}) \subseteq P_3P_1(\mathscr{A}^*)$, then $D^{(4)}\colon (\mathscr{A}^{(4)}, \triangle\triangle) \longrightarrow (\mathscr{A}^{(4)})^*$ is a derivation.*

P r o o f. (i) Let $\xi, \varphi, \psi \in \mathscr{A}^{(4)}$, there exists $a^* \in \mathscr{A}^*$ such that $D^{(4)}(\varphi) \cdot \xi = P_3P_1(a^*)$. By Lemma 1.1 (iii) we have

$$\langle \xi, P_0^{(4)}(\varphi)D^{(4)}(\psi)\rangle = \langle P_0^{(3)}P_3P_1(a^*), \varphi\rangle = \langle \varphi, D^{(4)}(\psi)\xi\rangle = \langle \xi, \varphi D^{(4)}(\psi)\rangle.$$

Therefore $D^{(4)}$ is a derivation by Lemma 3.10.

(ii) The proof is similar to (i). □

We recall that an operator $T\colon X \longrightarrow Y$ between Banach spaces is weakly compact if and only if $T^{**}X^{**} \subset Y$ (considered as a subspace of Y^{**}) if and only if T^* is weakly compact.

Lemma 3.12. *Let $\mathscr{A}$ be a Banach algebra and $D\colon \mathscr{A} \longrightarrow \mathscr{A}^*$ a weakly compact operator. Then $D^{(2n)}(\mathscr{A}^{(2n)}) \subseteq P_{2n-1} \dots P_3 P_1(\mathscr{A}^*)$ $(n \in \mathbb{N})$.*

P r o o f. When $n = 1$, clearly the result is true. Now suppose, inductively, that the result has been proved for n. Let $\varphi, \xi \in \mathscr{A}^{(2n+2)}$ and let (φ_α) be a net in $\mathscr{A}^{(2n)}$ such that $P_{2n}(\varphi_\alpha) \longrightarrow \varphi$ in the weak* topology. Then

$$\begin{aligned}\langle \xi, D^{(2n+2)}(\varphi)\rangle &= \lim_\alpha \langle D^{(2n)}(\varphi_\alpha), \xi\rangle = \lim_\alpha \langle P^*_{2n-1}(\xi), D^{(2n)}(\varphi_\alpha)\rangle \\ &= \langle D^{(2n-1)} P^*_{2n-1}(\xi), P^*_{2n-1}(\varphi)\rangle = \langle P^*_{2n-1}(\xi), D^{(2n)} P^*_{2n-1}(\varphi)\rangle \\ &= \langle D^{(2n)} P^*_{2n-1}(\varphi), \xi\rangle = \langle \xi, P_{2n+1} D^{(2n)} P^*_{2n-1}(\varphi)\rangle.\end{aligned}$$

Consequently, $D^{(2n+2)}(\varphi) = P_{2n+1} D^{(2n)} P^*_{2n-1}(\varphi) \subseteq P_{2n+1} \dots P_3 P_1(\mathscr{A}^*)$. □

Dales, Rodrigues-Palacios and Velasco in [3] proved the following theorem.

Theorem 3.13. *Let $\mathscr{A}$ be an Arens regular Banach algebra and $D\colon \mathscr{A} \longrightarrow \mathscr{A}^*$ a weakly compact derivation. Then $D^{(**)}\colon \mathscr{A}^{(**)} \longrightarrow (\mathscr{A}^{**})^*$ is a derivation.*

Now we have the same result for $\mathscr{A}^{(4)}$.

Theorem 3.14. *Let $\mathscr{A}$ be an Arens regular Banach algebra and $D\colon \mathscr{A} \longrightarrow \mathscr{A}^*$ a weakly compact derivation. Then $D^{(4)}\colon (\mathscr{A}^{(4)}, \Box\Box) \longrightarrow (\mathscr{A}^{(4)})^*$ and $D^{(4)}\colon (\mathscr{A}^{(4)}, \triangle\triangle) \longrightarrow (\mathscr{A}^{(4)})^*$ are derivations.*

P r o o f. Let $\xi, \varphi, \psi \in \mathscr{A}^{(4)}$ and (F_α), (G_β), (H_γ) be nets in $\mathscr{A}^{**}$ such that $P_2(F_\alpha) \longrightarrow \varphi$, $P_2(G_\beta) \longrightarrow \psi$ and $P_2(H_\gamma) \longrightarrow \xi$ in the weak* topology, let $a^* \in \mathscr{A}^*$ and let a^*_α be a net in $\mathscr{A}^*$ such that $P_1(a^*_\alpha) = D^{**}(F_\alpha)$ and $P_1(a^*) = D^{**} P^*_1(\varphi)$. We have

$$\begin{aligned}\langle \xi, D^{(4)}(\varphi)\psi\rangle &= \langle \psi \,\Box\Box\, \xi, D^{(4)}(\varphi)\rangle = \lim_\alpha \lim_\beta \lim_\gamma \langle a^*_\alpha, G_\beta \,\Box\, H_\gamma\rangle \\ &= \lim_\alpha \lim_\beta \langle a^*_\alpha G_\beta, P^*_1(\xi)\rangle = \lim_\alpha \langle P^*_1(\psi) \,\Box\, P^*_1(\xi), D^{**} P^{**}_1(\varphi)\rangle \\ &= \langle \xi, P_3(P_1(a^*) P^*_1(\psi))\rangle = \langle \xi, P_3 P_1(a^* P^*_1(\psi))\rangle,\end{aligned}$$

and so $D^{(4)}(\mathscr{A}^{(4)})\mathscr{A}^{(4)} \subseteq P_3 P_1(\mathscr{A}^*)$ and by Proposition 3.11, $D^{(4)}\colon (\mathscr{A}^{(4)}, \Box\Box) \longrightarrow (\mathscr{A}^{(4)})^*$ is a derivation. The other part is similar. □

Corollary 3.15. *Let $\mathscr{A}$ be an Arens regular Banach algebra such that $(\mathscr{A}^{(4)}, \Box\Box)$ or $(\mathscr{A}^{(4)}, \triangle\triangle)$ is weakly amenable and each derivation from $\mathcal{A}$ to $\mathscr{A}^*$ is weakly compact. Then $\mathscr{A}$ is weakly amenable.*

P r o o f. Let $D\colon \mathscr{A} \longrightarrow \mathscr{A}^*$ be a derivation. We may suppose that $(\mathscr{A}^{(4)}, \Box\Box)$ is weakly amenable. By Theorem 3.14, $D^{(4)}\colon (\mathscr{A}^{(4)}, \Box\Box) \longrightarrow (\mathscr{A}^{(4)})^*$ is a derivation. So there exists $\varphi_5 \in (\mathscr{A}^{(4)})^*$ such that $D^{(4)} = \delta_{\varphi_5}$. Set $a^* = P_0^* P_2^*(\varphi_5)$. Then by Lemma 1.2 we have

$$\begin{aligned} aa^* - a^*a &= P_0^* P_2^*(P_2 P_0(a)\varphi_5 - \varphi_5 P_2 P_0(a)) \\ &= P_0^* P_2^* D^{(4)} P_2 P_0(a) = D(a) \quad (a \in \mathscr{A}). \end{aligned}$$

Therefore $D = \delta_{a^*}$ is inner. Hence $\mathscr{A}$ is weakly amenable. $\Box$

Proposition 3.16. *Let $\mathscr{A}$ be a Banach algebra, $D\colon \mathscr{A} \longrightarrow \mathscr{A}^*$ a derivation and $\mathscr{A}^{(2n)} = ((\ldots((\mathscr{A}^{**}, \Box)^{**}, \Box)\ldots)^{**}, \Box)$ $(n \in \mathbb{N})$. Then*

(i) $D^{(2n)}\colon \mathscr{A}^{(2n)} \longrightarrow (\mathscr{A}^{(2n)})^*$ *holds in*

$$D^{(2n)}(\varphi \,\Box\, \psi) = D^{(2n)}(\varphi)\psi + P_0^{(2n)}(\varphi) D^{(2n)}(\psi) \quad (\varphi, \psi \in \mathscr{A}^{(2n)}).$$

(ii) *If $D^{(2n)}(\mathscr{A}^{(2n)}) \cdot \mathscr{A}^{(2n)} \subseteq P_{2n-1} \ldots P_3 P_1(\mathscr{A}^*)$, then $D^{(2n)}$ is a derivation.*

(iii) *If $\mathscr{A}^{(2n-2)}$ is Arens regular and D is weakly compact, then $D^{(2n)}$ is a derivation.*

Corollary 3.17. *Let $\mathscr{A}$ be a completely regular Banach algebra such that $\mathscr{A}^{(2n)}$ is weakly amenable for some $n \in \mathbb{N}$, and each derivation from $\mathscr{A}$ to $\mathscr{A}^*$ is weakly compact. Then $\mathscr{A}$ is weakly amenable.*

Lemma 3.18. *Let $\mathscr{A}$ be an Arens regular Banach algebra such that $(\mathscr{A}^{(4)}, \Box\Box)$ or $(\mathscr{A}^{(4)}, \triangle\triangle)$ is (-2)-weakly amenable. Then $\mathscr{A}$ is 2-weakly amenable.*

P r o o f. Let $D\colon \mathscr{A} \longrightarrow \mathscr{A}^{**}$ be a derivation, and let $(\mathscr{A}^{(4)}, \Box\Box)$ be (-2)-weakly amenable. Set $d = P_1^* D^{**} P_1^*\colon (\mathscr{A}^{(4)}, \Box\Box) \longrightarrow \mathscr{A}^{**}$. For $a^* \in \mathscr{A}^*$, $\varphi, \psi \in \mathscr{A}^{(4)}$ let $(F_\alpha), (G_\beta)$ be nets in $\mathscr{A}^{**}$ such that $P_2(F_\alpha) \longrightarrow \varphi$ and $P_2(G_\beta) \longrightarrow \psi$ in the weak* topology. Then

$$\begin{aligned} \langle a^*, d(\varphi \,\Box\Box\, \psi)\rangle &= \langle P_1 D^* P_1(a^*), \varphi \,\Box\Box\, \psi\rangle \\ &= \lim_\alpha \lim_\beta \langle P_1(a^*), D^{**}(F_\alpha \,\Box\, G_\beta)\rangle \\ &= \lim_\alpha \lim_\beta \langle a^*, P_1^* D^{**}(F_\alpha) G_\beta + F_\alpha P_1^* D^{**}(G_\beta)\rangle \\ &= \lim_\alpha \langle a^* P_1^* D^{**}(F_\alpha), P_1^*(\psi)\rangle + \langle D^* P_1(a^* F_\alpha), P_1^*(\psi)\rangle \\ &= \langle P_1^*(\psi) a^*, d(\varphi)\rangle + \langle d(\psi) a^*, P_1^*(\varphi)\rangle \\ &= \langle a^*, d(\varphi) \cdot \psi + \varphi \cdot d(\psi)\rangle. \end{aligned}$$

Therefore d is a derivation. Since $H^1(\mathscr{A}^{(4)}, \mathscr{A}^{**}) = \{0\}$, there exists $F \in \mathscr{A}^{**}$ such that $d = \delta_F$. It is easy to see that $D = \delta_F$. So $\mathscr{A}$ is 2-weakly amenable. $\square$

Proposition 3.19. *Let $\mathscr{A}$ be an Arens regular Banach algebra such that $\mathscr{A}^{(2n+2)}$ ($n \in \mathbb{N}$) with one of Arens products is $(-2n)$-weakly amenable. Then $\mathscr{A}$ is 2-weakly amenable.*

Proof. Let $D\colon \mathscr{A} \longrightarrow \mathscr{A}^{**}$ be a derivation. By Lemma 3.18 and by induction, $d = P_1^* D^{**} P_1^* P_3^* \dots P_{2n-1}^*\colon \mathscr{A}^{(2n+2)} \longrightarrow \mathscr{A}^{**}$ is a derivation. Since $H^1(\mathscr{A}^{(2n+2)}, \mathscr{A}^{**}) = \{0\}$, there exists $F \in \mathscr{A}^{**}$ such that $d = \delta_F$. It is easy to see that $D = \delta_F$ is inner. $\square$

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