Czechoslovak Mathematical Journal

Yaşar Bolat; Ömer Akin

Oscillatory behaviour of a higher order nonlinear neutral delay type functional differential equation with oscillating coefficients

Czechoslovak Mathematical Journal, Vol. 55 (2005), No. 4, 893–900

Persistent URL: `http://dml.cz/dmlcz/128031`

Czechoslovak Mathematical Journal, 55 (130) (2005), 893–900

OSCILLATORY BEHAVIOUR OF A HIGHER ORDER NONLINEAR NEUTRAL DELAY TYPE FUNCTIONAL DIFFERENTIAL EQUATION WITH OSCILLATING COEFFICIENTS

Yaşar Bolat and Ömer Akin, Ankara

(Received January 2, 2003)

Abstract. In this paper we are concerned with the oscillation of solutions of a certain more general higher order nonlinear neutral type functional differential equation with oscillating coefficients. We obtain two sufficient criteria for oscillatory behaviour of its solutions.

Keywords: differential equation, higher order nonlinear neutral differential equation, oscillation, oscillating coefficients

MSC 2000: 34K11

1. Introduction

We consider the higher order nonlinear differential equation

$$(1)\qquad [y(t)+P(t)y(\tau(t))]^{(n)}+\sum_{i=1}^{m}Q_i(t)f_i(y(\sigma_i(t)))=0$$

where $n\geqslant 2$; $P(t),Q_i(t),\tau(t)\in C[t_0,+\infty)$ for $i=1,2,\ldots,m$; $P(t)$ is an oscillating function, $Q_i(t)$ are positive valued for $i=1,2,\ldots,m$; $\sigma_i(t)\in C'[t_0,+\infty)$, $\sigma_i'(t)>0$, $\sigma_i(t)\leqslant t$; $\sigma_i(t)\to+\infty$ as $t\to\infty$ for $i=1,2,\ldots,m$; $\tau(t)\to+\infty$ as $t\to\infty$; $f_i(u)\in C(\mathbb{R},\mathbb{R})$ are nondecreasing functions, $uf_i(u)>0$ for $u\neq 0$ and $i=1,2,\ldots,m$.

Recently, much research has been done on the oscillatory and asymptotic behaviour of solutions of higher order neutral type functional differential equations. Most of the known results concern the cases when $P(t)=c\in\mathbb{R}$ and $P(t)>0$ (or <0) and hold for special cases of the equation (1) and related equations; see, for example [1]–[10] and the references cited therein.

The purpose of this paper is to study oscillatory behaviour of solutions of equation (1). For the general theory of differential equations, one can refer to [1]–[5]. Many references to some applications of the differential equations can be found in [5].

As is customary, a solution of Eq. (1) is said to be oscillatory if it has arbitrarily large zeros. Otherwise the solution is called nonoscillatory.

For the sake of convenience, the function $z(t)$ is defined by

$$z(t) = y(t) + P(t)y(\tau(t)). \tag{2}$$

2. Some auxiliary lemmas

Lemma 2.1. *Let $y(t)$ be a function such that it and each of its derivatives up to order $(n-1)$ inclusive is absolutely continuous and of constant sign in an interval $[t_0, +\infty)$. If $y^{(n)}(t)$ is of constant sign and not identically zero on any interval of the form $[t_1, +\infty)$ for some $t_1 \geqslant t_0$, then there exist a $t_x \geqslant t_0$ and an integer l, $0 \leqslant l \leqslant n$ with $n+l$ even for $y^{(n)}(t) \geqslant 0$, or $n+l$ odd for $y^{(n)}(t) \leqslant 0$, and such that for every $t \geqslant t_x$, $l > 0$ implies $y^{(k)}(t) > 0$, $k = 0, 1, 2, \ldots, l-1$ and $l \leqslant n-1$ implies $(-1)^{l+k}y^{(k)}(t) > 0$, $k = l, l+1, \ldots, n-1$ [1].*

Lemma 2.2. *If the function $y(t)$ is as in Lemma 2.1 and*

$$y^{(n-1)}(t)y^{(n)}(t) \leqslant 0 \quad \text{for all } t \geqslant t_x,$$

then for every λ, $0 < \lambda < 1$, there exists a constant $M > 0$ such that

$$|y(\lambda t)| \geqslant M t^{n-1} |y^{(n-1)}(t)| \quad \text{for all large } t \text{ [1].}$$

3. Main results

Theorem 3.1. *Assume that n is odd and*

(C_1) $\lim\limits_{t\to\infty} P(t) = 0$,

(C_2) $\int_{t_0}^{+\infty} s^{n-1} \sum\limits_{i=1}^{m} Q_i(s)\,\mathrm{d}s = +\infty$.

Then every bounded solution of Eq. (1) is either oscillatory or tends to zero as $t \to +\infty$.

P r o o f. Assume that Eq. (1) has a bounded nonoscillatory solution $y(t)$. Without loss of generality, assume that $y(t)$ is eventually positive (the proof is similar when $y(t)$ is eventually negative). That is, $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\sigma_i(t)) > 0$ for $t \geqslant t_1 \geqslant t_0$ and $i = 1, 2, \ldots, m$. Assume further that $y(t)$ does not tend to zero as $t \to \infty$. By (1), (2) we have for $t \geqslant t_1$

$$z^{(n)}(t) = -\sum_{i=1}^{m} Q_i(t) f_i(y(\sigma_i(t))) < 0. \tag{3}$$

That is, $z^{(n)}(t) < 0$. It follows that $z^{(j)}(t)$ $(j = 0, 1, 2, \ldots, n-1)$ is strictly monotone and eventually of constant sign. Since $P(t)$ is oscillatory function, there exists a $t_2 \geqslant t_1$ such that if $t \geqslant t_2$ then $z(t) > 0$. Since $y(t)$ is bounded, by virtue of (C_1) and (2), there is a $t_3 \geqslant t_2$ such that $z(t)$ is also bounded for $t \geqslant t_3$. Because n is odd and $z(t)$ is bounded, by Lemma 2.1, when $l = 0$ (otherwise $z(t)$ is not bounded) there exists $t_4 \geqslant t_3$ such that for $t \geqslant t_4$ we have $(-1)^k z^{(k)}(t) > 0$ $(k = 0, 1, 2, \ldots, n-1)$. In particular, since $z'(t) < 0$ for $t \geqslant t_4$, $z(t)$ is decreasing. Since $z(t)$ is bounded, we may write $\lim\limits_{t\to\infty} z(t) = L$ $(-\infty < L < +\infty)$. Assume that $0 \leqslant L < +\infty$. Let $L > 0$. Then there exists a constant $c > 0$ and a $t_5 \geqslant t_4$ such that $z(t) > c > 0$ for $t \geqslant t_5$. Since $y(t)$ is bounded, $\lim\limits_{t\to\infty} P(t) y(\tau(t)) = 0$ by (C_1). Therefore, there exists a constant $c_1 > 0$ and a $t_6 \geqslant t_5$ such that $y(t) = z(t) - P(t) y(\tau(t)) > c_1 > 0$ for $t \geqslant t_6$. So, we may find a t_7 with $t_7 \geqslant t_6$ such that $y(\sigma_i(t)) > c_1 > 0$ for $t \geqslant t_7$. From (3) we have

$$z^{(n)}(t) = -\sum_{i=1}^{m} Q_i(t) f_i(c_1) < 0 \quad (t \geqslant t_7). \tag{4}$$

If we multiply (4) by t^{n-1} and integrate it from t_7 to t, we obtain

$$F(t) - F(t_7) \leqslant -f(c_1) \int_{t_7}^{t} \sum_{i=1}^{m} Q_i(s) s^{n-1} \,\mathrm{d}s \tag{5}$$

where

$$\begin{aligned} F(t) = {} & t^{n-1} z^{(n-1)}(t) - (n-1) t^{n-2} z^{(n-2)}(t) + (n-1)(n-2) t^{n-3} z^{(n-3)}(t) \\ & - \ldots - (n-1)(n-2)(n-3) \ldots 3 \cdot 2 t z'(t) \\ & + (n-1)(n-2)(n-3) \ldots 3 \cdot 2 \cdot 1 z(t). \end{aligned}$$

Since $(-1)^k z^{(k)}(t) > 0$ for $k = 0, 1, 2, \ldots, n-1$ and $t \geqslant t_4$, we have $F(t) > 0$ for $t \geqslant t_7$. From (5) we have

$$-F(t_7) \leqslant -f(c_1) \int_{t_7}^{t} \sum_{i=1}^{m} Q_i(s) s^{n-1} \,\mathrm{d}s.$$

From (C_2) we obtain

$$-F(t_7) \leqslant -f(c_1) \int_{t_7}^{t} \sum_{i=1}^{m} Q_i(s) s^{n-1}\,\mathrm{d}s = -\infty$$

as $t \to \infty$. This is a contradiction. So, $L > 0$ is impossible. Therefore, $L = 0$ is the only possible case. That is, $\lim_{t\to\infty} z(t) = 0$. Since $y(t)$ is bounded, by (C_1) we obtain

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) - \lim_{t\to\infty} P(t)y(t) = 0$$

from (2).

Now let us consider the case of $y(t) < 0$ for $t \geqslant t_1$. By (1) and (2),

$$z^{(n)}(t) = -\sum_{i=1}^{m} Q_i(t) f_i(y(\sigma_i(t))) > 0 \quad (t \geqslant t_1).$$

That is, $z^{(n)}(t) > 0$. It follows that $z^{(j)}(t)$ $(j = 0, 1, 2, \ldots, n-1)$ is strictly monotone and eventually of constant sign. Since $P(t)$ is oscillatory function, there exists a $t_2 \geqslant t_1$ such that if $t \geqslant t_2$ then $z(t) < 0$. Since $y(t)$ is bounded, by (C_1) and (2) there is a $t_3 \geqslant t_2$ such that $z(t)$ is also bounded for $t \geqslant t_3$. Assume that $x(t) = -z(t)$. Then $x^{(n)}(t) = -z^{(n)}(t)$. Therefore, $x(t) > 0$ and $x^{(n)}(t) < 0$ for $t \geqslant t_3$. From this we observe that $x(t)$ is bounded. Since n is odd, by Lemma 2.1 there is a $t_4 \geqslant t_3$ and $l = 0$ (otherwise, $x(t)$ is not bounded) such that $(-1)^k x^{(k)}(t) > 0$ for $k = 0, 1, 2, \ldots, n-1$ and $t \geqslant t_4$. That is, $(-1)^k z^{(k)}(t) < 0$ for $k = 0, 1, 2, \ldots, n-1$ and $t \geqslant t_4$. In particular, for $t \geqslant t_4$ we have $z'(t) > 0$. Therefore, $z(t)$ is increasing. So, we can assume that $\lim_{t\to\infty} z(t) = L$ $(-\infty < L \leqslant 0)$. As in the proof of $y(t) > 0$, we may prove that $L = 0$. As for the rest, it is similar to the case of $y(t) > 0$. That is, $\lim_{t\to\infty} y(t) = 0$. This contradicts to our assumption. Hence, the proof is completed. □

Theorem 3.2. *Assume that n is even and (C_1) holds. If the following condition is satisfied:*

(C_3) *There is a function $\varphi(t)$ such that $\varphi(t) \in C'[t_0, +\infty)$. Moreover,*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \varphi(s) \sum_{i=1}^{m} Q_i(s)\,\mathrm{d}s = +\infty$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\frac{[\varphi'(s)]^2}{\varphi(s)\sigma_i'(s)\sigma_i^{n-2}(s)}\right] \mathrm{d}s < +\infty$$

for $\varphi(t)$ and $i = 1, 2, \ldots, m$. Then every bounded solution of Eq. (1.1) *is oscillatory.*

Proof. Assume that Eq. (1) has a bounded nonoscillatory solution $y(t)$. Without loss of generality, assume that $y(t)$ is eventually positive (the proof is similar when $y(t)$ is eventually negative). That is, $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\sigma_i(t)) > 0$ for $t \geqslant t_1 \geqslant t_0$. By (1), (2) we have for $t \geqslant t_1$

$$z^{(n)}(t) = -\sum_{i=1}^{m} Q_i(t) f_i(y(\sigma_i(t))) < 0. \tag{6}$$

That is, $z^{(n)}(t) < 0$. It follows that $z^{(j)}(t)$ $(j = 0, 1, 2, \ldots, n-1)$ is strictly monotone and eventually of constant sign. Since $P(t)$ is oscillatory function, there exists a $t_2 \geqslant t_1$ such that for $t \geqslant t_2$ we have $z(t) > 0$. Since $y(t)$ is bounded, by (C_1) and (2) there is a $t_3 \geqslant t_2$, such that $z(t)$ is also bounded for $t \geqslant t_3$. Because n is even, by Lemma 2.1 when $l = 1$ (otherwise, $z(t)$ is not bounded) there exists $t_4 \geqslant t_3$ such that for $t \geqslant t_4$ we have

$$(-1)^{k+1} z^{(k)}(t) > 0 \quad (k = 0, 1, 2, \ldots, n-1). \tag{7}$$

In particular, since $z'(t) > 0$ for $t \geqslant t_4$, $z(t)$ is increasing. Since $y(t)$ is bounded, $\lim\limits_{t\to\infty} P(t)y(\tau(t)) = 0$ by (C_1). Then there exists a $t_5 \geqslant t_4$ and a positive integer δ such that by (2)

$$y(t) = z(t) - P(t)y(\tau(t)) > \frac{1}{\delta} z(t) > 0$$

for $t \geqslant t_5$. We may find a $t_6 \geqslant t_5$ such that for $t \geqslant t_6$ and $i = 1, 2, \ldots, m$

$$y(\sigma_i(t)) > \frac{1}{\delta} z(\sigma_i(t)) > 0. \tag{8}$$

From (6), (8) and the properties of f we have

$$\begin{aligned} z^{(n)}(t) &\leqslant -\sum_{i=1}^{m} Q_i(t) f_i\Big(\frac{1}{\delta} z(\sigma_i(t))\Big) \\ &= -\sum_{i=1}^{m} Q_i(t) \frac{f_i\big(\delta^{-1} z(\sigma_i(t))\big)}{z(\sigma_i(t))} z(\sigma_i(t)) \end{aligned} \tag{9}$$

for $t \geqslant t_6$. Since $z(t) > 0$ is bounded and increasing, $\lim\limits_{t\to\infty} z(t) = L$ $(0 < L < +\infty)$. By the continuity of f, we have

$$\lim_{t\to\infty} \frac{f_i\big(\delta^{-1} z(\sigma_i(t))\big)}{z(\sigma_i(t))} = \frac{f_i\big(L/\delta\big)}{L} > 0.$$

Then there is a $t_7 \geqslant t_6$ such that for $t \geqslant t_7$, $i = 1, 2, \ldots, m$ we have

$$\lim_{t\to\infty} \frac{f_i\big(\delta^{-1} z(\sigma_i(t))\big)}{z(\sigma_i(t))} \geqslant \frac{f_i\big(L/\delta\big)}{2L} = \alpha > 0. \tag{10}$$

By (9) and (10),

$$z^{(n)}(t) \leqslant -\alpha \sum_{i=1}^{m} Q_i(t) z(\sigma_i(t)), \quad \text{for } \ t \geqslant t_7. \tag{11}$$

Let us set

$$w(t) = \frac{z^{(n-1)}(t)}{z\big(\delta^{-1}(\sigma_i(t))\big)}. \tag{12}$$

We know from (7) that there is a $t_8 \geqslant t_7$ such that for sufficiently large $t \geqslant t_8$, $w(t) > 0$. Therefore, derivativing (12) we obtain

$$\begin{aligned} w'(t) &= \frac{z\big(\delta^{-1}\sigma_i(t)\big) z^{(n)}(t) - \delta^{-1}\sigma_i'(t) z'\big(\delta^{-1}\sigma_i(t)\big) z^{(n-1)}(t)}{z^2\big(\delta^{-1}\sigma_i(t)\big)} \\ &= \frac{z^{(n)}(t)}{z\big(\delta^{-1}\sigma_i(t)\big)} - \frac{1}{\delta} w(t) \frac{z'\big(\delta^{-1}\sigma_i(t)\big)}{z\big(\delta^{-1}\sigma_i(t)\big)} \sigma_i'(t). \end{aligned} \tag{13}$$

We know from (7) that for $t \geqslant t_9$ we have $z'(t) > 0$ and $z^{(n-1)}(t) > 0$. Since $z(t) > 0$ is increasing, $z(\sigma_i(t)) > z\big(\delta^{-1}\sigma_i(t)\big) > 0$ for $i = 1, 2, \ldots, m$. Therefore, by Lemma 2.2, for $\lambda = \delta^{-1}$ and $z'(t)$ there exist a constant $M > 0$ and a $t_{10} \geqslant t_9$ such that for $t \geqslant t_{10}$ we have

$$z'\Big(\frac{1}{\delta}\sigma_i(t)\Big) \geqslant \delta(n-1) M \sigma_i^{n-2}(t) z^{(n-1)}(\sigma_i(t)).$$

Since $z^{(n-1)}(t)$ is decreasing and $\sigma_i(t) \leqslant t$, we obtain

$$z'\Big(\frac{1}{\delta}\sigma_i(t)\Big) \geqslant N \sigma_i^{n-2}(t) z^{(n-1)}(t) \tag{14}$$

where $N = \delta(n-1)M > 0$. Hence, by (11), (13) and (14) we have

$$w'(t) \leqslant -\alpha \sum_{i=1}^{m} Q_i(t) - (n-1) M w^2(t) \sigma_i^{n-2}(t) \sigma_i'(t). \tag{15}$$

From (15) we have

$$\alpha \sum_{i=1}^{m} Q_i(t) \leqslant -w'(t) - (n-1) M w^2(t) \sigma_i^{n-2}(t) \sigma_i'(t) \quad (t \geqslant t_{10}). \tag{16}$$

If we multiply (16) by $\varphi(t)$ and integrate it from t_{10} to t, we obtain

$$\begin{aligned}
\alpha \int_{t_{10}}^{t} \varphi(s) \sum_{i=1}^{m} Q_i(s)\,\mathrm{d}s \leqslant & -\int_{t_{10}}^{t} \varphi(s) w'(s)\,\mathrm{d}s \\
& -(n-1)M \int_{t_{10}}^{t} \varphi(s) w^2(s) \sigma_i^{n-2}(s) \sigma_i'(s)\,\mathrm{d}s \\
= & -\varphi(t)w(t) + \varphi(t_{10})w(t_{10}) + \int_{t_{10}}^{t} \varphi'(s) w(s)\,\mathrm{d}s \\
& -(n-1)M \int_{t_{10}}^{t} \varphi(s) w^2(s) \sigma_i^{n-2}(s) \sigma_i'(s)\,\mathrm{d}s \\
\leqslant & \ \varphi(t_{10})w(t_{10}) - (n-1)M \int_{t_{10}}^{t} \varphi(s) \sigma_i^{n-2}(s) \sigma_i'(s) \\
& \times \left[w(s) - \frac{\varphi'(s)}{2(n-1)M\varphi(s)\sigma_i^{n-2}(s)\sigma_i'(s)} \right]^2 \mathrm{d}s \\
& + \int_{t_{10}}^{t} \frac{[\varphi'(s)]^2}{4(n-1)M\varphi(s)\sigma_i^{n-2}(s)\sigma_i'(s)}\,\mathrm{d}s \\
\leqslant & \ \varphi(t_{10})w(t_{10}) + \int_{t_{10}}^{t} \frac{[\varphi'(s)]^2}{4(n-1)M\varphi(s)\sigma_i^{n-2}(s)\sigma_i'(s)}\,\mathrm{d}s.
\end{aligned}$$

Therefore, by (C_3)

$$\begin{aligned}
+\infty = & \ \alpha \limsup_{t\to\infty} \int_{t_{10}}^{t} \varphi(s) \sum_{i=1}^{m} Q_i(s)\,\mathrm{d}s \\
\leqslant & \ \varphi(t_{10})w(t_{10}) + \frac{1}{4(n-1)M} \int_{t_{10}}^{t} \frac{[\varphi'(s)]^2}{\varphi(s)\sigma_i^{n-2}(s)\sigma_i'(s)}\,\mathrm{d}s < +\infty
\end{aligned}$$

for $i = 1, 2, \ldots$.

Now let us consider the case of $y(t) < 0$ for $t \geqslant t_1$. By (1.1) and (1.2) we have

$$z^{(n)}(t) = -\sum_{i=1}^{m} q_i(t) f_i(y(\sigma_i(t))) > 0 \quad (t \geqslant t_1).$$

That is, $z^{(n)}(t) > 0$. It follows that $z^{(j)}(t)$ $(j = 0, 1, 2, \ldots, n-1)$ is strictly monotone and eventually of constant sign. Since $P(t)$ is oscillating function, there exists a $t_2 \geqslant t_1$ such that for $t \geqslant t_2$ we have $z(t) < 0$. Since $y(t)$ is bounded, by (C_1) and (2) there is a $t_3 \geqslant t_2$ such that $z(t)$ is also bounded for $t \geqslant t_3$. Assume that $x(t) = -z(t)$. Then $x^{(n)}(t) = -z^{(n)}(t)$. Therefore, $x(t) > 0$ and $x^{(n)}(t) < 0$ for $t \geqslant t_3$. Hence, we observe that $x(t)$ is bounded. Since n is even, by Lemma 2.1 there exist a $t_4 \geqslant t_3$ and $l = 1$ (otherwise, $x(t)$ is not bounded) such that $(-1)^k x^{(k)}(t) > 0$

for $k = 0, 1, 2, \ldots, n-1$ and $t \geqslant t_4$. That is, $(-1)^k z^{(k)}(t) < 0$ for $k = 0, 1, 2, \ldots, n-1$ and $t \geqslant t_4$. In particular, for $t \geqslant t_4$ we have $z'(t) > 0$. Therefore, $z(t)$ is increasing. For the rest of proof, we can proceed the proof similar to the case of $y(t) > 0$. Hence, the proof is completed. □

References

[1] *R. P. Agarwal, Said R. Grace and Donal O'Regan*: Oscillation Theory for Difference and Functional Differential Equations. Kluwer Academic Publishers, Dordrecht, 2000.

[2] *D. Bainov and D. P. Mishev*: Oscillation Theory of Operator-Differential Equations. World Scientific, Singapore-New Yersey, 1995.

[3] *G. S. Ladde, V. Lakshmikantham and B. G. Zhang*: Oscillation Theory of Differential Equations with Deviating Arguments. M. Dekker, New York, 1987.

[4] *D. Bainov and D. P. Mishev*: Oscillation Theory for Neutral Differential Equations with Delay. Adam Hilger, New York, 1991.

[5] *I. Gyori and G. Ladas*: Oscillation Theory of Delay Differential Equations with Applications. Clarendon Press, Oxford, 1991.

[6] *N. Parhi*: Oscillations of higher order differential equations of neutral type. Czechoslovak Math. J. *50(125)* (2000), 155–173.

[7] *Feng Yuecai*: Oscillatory behavior of higher order nonlinear neutral functional differential equation with oscillating coefficients. J. South Chine Normal Unv. (1999), 6–11. (In Chinese.)

[8] *P. R. Agarwal and S. R. Grace*: The oscillation of higher order differential equations with deviating arguments. Comput. Math. Appl. *38* (1999), 185–199.

[9] *S. R. Grace and B. S. Lalli*: Oscillation theorems for certain neutral differential equations. Czechoslovak Math. J. *38* (1988), 745–783.

[10] *R. P. Agarwal, E. Thandapani and P. J. Y. Wong*: Oscillation of higher order neutral differential equations. Appl. Math. Letters *10* (1997), 71–78.

Author's address: Y. Bolat, Afyon Kocatepe University, Faculty of Science and Arts, Department of Mathematics, ANS Campus, 03200 Afyonkarahisar, Turkey, e-mail: yasarbolat@aku.edu.tr; Ö. Akin, TOBB University of Economics and Technology, Department of Mathematics, Faculty of Science and Arts, Sogutozu Cad. No: 43, 06530 Sogutozu, Ankara, Turkey, e-mail: omerakin@etu.edu.tr.