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ON THE UNDERLYING LOWER ORDER BUNDLE FUNCTORS

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Abstract. For every bundle functor we introduce the concept of subordinated functor. Then we describe subordinated functors for fiber product preserving functors defined on the category of fibered manifolds with m-dimensional bases and fibered manifold morphisms with local diffeomorphisms as base maps. In this case we also introduce the concept of the underlying functor. We show that there is an affine structure on fiber product preserving functors.

Keywords: bundle functor, Weil bundle, natural transformation

MSC 2000: 58A05, 58A20

It is well known that the theory of Weil algebras plays an important role in differential geometry. The principal result from this field is that every product preserving bundle functor on the category $\mathcal{M}f$ of smooth manifolds and smooth maps is a Weil functor T^A for some Weil algebra A, [4]. Then the natural transformations $T^A \to T^B$ of two such functors are in a canonical bijection with algebra homomorphisms $A \to B$ and the iteration $T^A \circ T^B$ corresponds to the tensor product $A \otimes B$ of Weil algebras. Further, I. Kolář and W. M. Mikulski [5] have recently described fiber product preserving functors on the category $\mathcal{FM}_m$ of fibered manifolds with m-dimensional bases and fibered manifold morphisms with local diffeomorphisms as base maps, in terms of Weil algebras. In particular, all such functors are in bijection with triples (A, H, t), where A is a Weil algebra, H is a group homomorphism and t is an equivariant algebra homomorphism (see below). The iteration of fiber preserving functors on $\mathcal{FM}_m$ was studied by I. Kolář and the author in [1].

Our starting point was the paper [2] by I. Kolář, who introduced the concept of the underlying lower order Weil functor. He also proved that there is an affine structure on Weil bundles. Our aim is to introduce underlying lower order functors for every

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fiber product preserving functor on $\mathcal{FM}_m$. In this connection we show that it is useful to study underlying functors from a more general point of view.

In Section 1 we introduce the subordinated functor for every bundle functor F. This is a functor G such that there is a surjective natural transformation $F \to G$. Then we describe subordinated Weil functors and subordinated Weil algebras. Taking into account the order of subordinated functors, we generalize the concept of underlying Weil functor by I. Kolář. Such an approach will also be used in Section 4, where we introduce the concept of the underlying functor for fiber product preserving functors on $\mathcal{FM}_m$.

In Section 2 we recall the Weil characterization of fiber product preserving functors on $\mathcal{FM}_m$ by means of the triples (A, H, t). We prove that the order of such a functor can be determined from its Weil description (A, H, t). In Section 3 we characterize the subordinated fiber product preserving functors on $\mathcal{FM}_m$ in an algebraic way. Section 5 is devoted to some examples of subordinated and underlying functors on $\mathcal{FM}_m$. Finally, in Section 6 we show that there is an affine structure on the fiber product preserving functors on $\mathcal{FM}_m$.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from the book [4].

1. Subordinated and underlying Weil functors

First we recall the concept of the underlying lower order Weil functor, which has been introduced recently by I. Kolář, [2]. Consider a Weil algebra $A = \mathbb{R} \times N_A$, where N_A is the ideal of all nilpotent elements of A. We say that A is of order r, $\operatorname{ord}(A) = r$, if $N_A^{r+1} = 0$ and $N_A^r \neq 0$. It is well known that the order of the corresponding Weil functor T^A coincides with the order of A, [4]. According to [2], the factor algebra $A_k = A/N_A^{k+1}$ is called the underlying algebra of order k and the Weil functor T^{A_k} is said to be the underlying kth order functor of T^A.

Clearly, the algebra epimorphism $A \to A_k = A/N_A^{k+1}$ induces a surjective submersion $T^A M \to T^{A_k} M$ for every manifold M. This leads us to a more general concept.

Definition. We say that G is a *subordinated functor* of a bundle functor F (or that G *is dominated by* F), if there exists a surjective natural transformation $t\colon F \to G$.

Notice that the concept of a subordinated functor is independent of the admissible category and also of the order of bundle functors in question.

If T^A and $T^{\tilde{A}}$ are two Weil functors, then surjective natural transformations $T^A \to T^{\tilde{A}}$ are in bijection with algebra epimorphisms $\mu\colon A \to \tilde{A}$. In such a case we have an exact sequence

$$0 \to I \to A \xrightarrow{\mu} \tilde{A} = A/I \to 0 \tag{1}$$

for some ideal $I \subset A$.

Definition. Let T^A and $T^{\tilde{A}}$ be two Weil functors. We say that *the Weil algebra $\tilde{A}$ is dominated by A* if the Weil functor $T^{\tilde{A}}$ is dominated by T^A.

Clearly, we have

Lemma 1. *A Weil algebra $\tilde{A}$ is dominated by A if and only if $\tilde{A} = A/I$ for some ideal $I \subset A$.*

In particular, for $I = N_A^{k+1}$ we obtain the underlying algebra $A_k = A/N_A^{k+1}$ of order k from [2].

Proposition 1. *Let T^A and $T^{\tilde{A}}$ be two Weil functors such that $T^{\tilde{A}}$ is dominated by T^A. If the order of T^A is r, then the order of $T^{\tilde{A}}$ is at most r.*

Proof. Let $\mu\colon A \to \tilde{A}$ be the algebra epimorphism from (1). Then we have $\mu(N_A) = N_{\tilde{A}}$, so that $(N_{\tilde{A}})^{r+1} = \mu(N_A^{r+1}) = 0$. □

On the other hand, suppose that the Weil functor $T^{\tilde{A}}$ of order s is dominated by T^A. If $\mu\colon A \to \tilde{A} = A/I$ is the algebra epimorphism from (1), then the condition $(N_{\tilde{A}})^{s+1} = 0$ yields $\mu(N_A^{s+1}) = 0$, so that we have $N_A^{s+1} \subset I$. This defines an algebra epimorphism

$$A_s = A/N_A^{s+1} \to \tilde{A} = A/I.$$

Hence every sth order Weil functor $T^{\tilde{A}}$, which is dominated by T^A, is also dominated by T^{A_s}. This property of the functor T^{A_s} can be generalized in the following way.

Definition. A Weil functor F_s is said to be the *underlying sth order functor* of a Weil functor F, if

(1) F_s is dominated by F,
(2) F_s has order s,
(3) Every sth order Weil functor $\tilde{F}$ which is dominated by F, is also dominated by F_s.

Thus, we have deduced

Proposition 2. *The underlying sth order functor of a Weil functor $F = T^A$ is of the form $F_s = T^{A_s}$, where $A_s = A/N_A^{s+1}$.*

We can see that in the case of Weil functors, the underlying sth order functor F_s is exactly the underlying Weil functor T^{A_s} defined by I. Kolář, [2]. However, our general approach to the concept of the underlying functor can also be applied for bundle functors which are defined on another admissible category. The case of fiber product preserving functors on $\mathcal{FM}_m$ will be discussed in Section 4.

Example 1. (a) Let $F = TT$ be the second order iterated tangent functor. Then the functors G and H defined by $GM = TM \times_M TM$, $HM = TM$, are dominated by F. Moreover, $F_1 = G$.

(b) Let T_m^r be the velocities functor defined by $T_m^r M = J_0^r(\mathbb{R}^m, M)$. Then we have $(T_m^r)_{r-1} = T_m^{r-1}$.

(c) Consider the iterated velocities functor $F\colon = T_p^r(T_q^s)$. Clearly, both functors T_p^r and T_q^s are dominated by F. By [2], the underlying $(r+s-1)$th order functor of F is the fiber product

$$(T_p^r(T_q^s)M)_{r+s-1} = T_p^{r-1}T_q^s M \times_{T_p^{r-1}T_q^{s-1}M} T_p^r T_q^{s-1} M.$$

2. Fiber product preserving functors on $\mathcal{FM}_m$

We recall that the definition of the order of a functor on $\mathcal{FM}_m$ is based on the concept of (q,s,r)-jet, $s \geqslant q \leqslant r$, [4]. Consider two fibered manifolds $p\colon Y \to M$, $q\colon Z \to N$ and two $\mathcal{FM}_m$-morphisms $f, g\colon Y \to Z$ with base maps $\underline{f}, \underline{g}\colon M \to N$. We say that f and g determine the same (q,s,r)-jet at $y \in Y$, $j_y^{q,s,r} f = j_y^{q,s,r} g$, if

$$j_y^q f = j_y^q g, \quad j_y^s(f|Y_x) = j_y^s(g|Y_x) \quad \text{and} \quad j_x^r \underline{f} = j_x^r \underline{g}, \quad x = p(y).$$

Let F be a bundle functor on the category $\mathcal{FM}_m$. We say that F is of order (q,s,r), if $j_y^{q,s,r} f = j_y^{q,s,r} g$ implies $Ff|F_yY = Fg|F_yY$.

Definition. The integer r is called the *base order* of F, s is called the *fiber order* and q is called the *total order* of F, $s \geqslant q \leqslant r$.

Denote by $\mathbb{D}_m^r = J_0^r(\mathbb{R}^m, \mathbb{R})$ the algebra of all r-jets of $\mathbb{R}^m$ into $\mathbb{R}$ with source $0 \in \mathbb{R}^m$ and by $G_m^r = \operatorname{inv} J_0^r(\mathbb{R}^m, \mathbb{R}^m)_0$ the rth jet group in dimension m. By I. Kolář and W. M. Mikulski [5], all fiber product preserving bundle functors on $\mathcal{FM}_m$ of the base order r are in bijection with the triples (A, H, t), where A is a Weil algebra, $H\colon G_m^r \to \operatorname{Aut}(A)$ is a group homomorphism of G_m^r into the group of all automorphisms of A and $t\colon \mathbb{D}^r_m \to A$ is an equivariant algebra homomorphism. Further, denote by $P^rM[T^AY]$ the associated bundle to the rth order frame bundle P^rM and by $t_M\colon T^r_mM \to T^AM$ the natural transformation induced by t. Taking into account the inclusion $P^rM \subset T^r_mM$, we have

$$FY = \{\{u, Z\} \in P^rM[T^AY], \quad t_M(u) = T^Ap(Z)\}. \tag{2}$$

Moreover, $Ff\colon FY \to FZ$ is the restriction and corestriction of $P^r\underline{f}[T^Af]\colon P^rM[T^AY] \to P^rN[T^AZ]$.

Let $F = (A, H, t)$ be a fiber product preserving bundle functor on $\mathscr{FM}_m$ and denote by $\beta_M\colon T^r_mM \to T^q_mM$ the jet projection. We are going to show that the order (q, s, r) of F can be determined from the triple (A, H, t) in the following way.

Proposition 3. *The base order r corresponds to $H\colon G^r_m \to \operatorname{Aut}(A)$, the fiber order s is of the form $s = \operatorname{ord}(A)$ and the total order is the greatest integer $q \leqslant r$ such that the algebra homomorphism $t\colon \mathbb{D}^r_m \to A$ is projectable over $\bar{t}\colon \mathbb{D}^q_m \to A$, i.e.*

$$t_M = \bar{t}_M \circ \beta_M. \tag{3}$$

P r o o f. The condition for the base order r is obvious and the condition for the fiber order s follows from (2). By locality, it suffices to restrict ourselves to the case of a product fibered manifold $Y = M \times N$. Then $T^Ap(M \times N) = T^AM$ and we have an identification

$$F(M \times N) = P^rM \times T^AN, \quad (u, W) \mapsto \{u, (t_M(u), W)\}, \quad u \in P^rM, \ W \in T^AN.$$

Further, an $\mathscr{FM}_m$-morphism $f\colon M \times N \to \overline{M} \times \overline{N}$ is of the form

$$f(x, y) = (\underline{f}(x), \tilde{f}(x, y)), \quad \underline{f}\colon M \to \overline{M}, \quad \tilde{f}\colon M \times N \to \overline{N}.$$

Denote by $\tilde{f}_1\colon M \to \overline{N}$ and $\tilde{f}_2\colon N \to \overline{N}$ the horizontal and the vertical restriction of $\tilde{f}$ at (x, y), respectively. Obviously, we have

$$Ff = P^r\underline{f}[T^Af], \quad T^Af = (T^A\underline{f}, T^A\tilde{f})$$

and if $\overline{N} = \mathbb{R}^n$, then $T^A\overline{N} = A^n$. Since $t_M(u) \in T^AM$ is horizontal and $W \in T^AN$ is vertical, we obtain for $\overline{N} = \mathbb{R}^n$

$$Ff(u, W) = \{P^r\underline{f}(u), (T^A\underline{f}(t_M(u)), T^A\tilde{f}_1(t_M(u)) + T^A\tilde{f}_2(W))\}. \tag{4}$$

Consider another $\mathscr{FM}_m$-morphism $g = (\underline{g}, \tilde{g})\colon M \times N \to \overline{M} \times \overline{N}$. The condition $j^{q,s,r}_{(x,y)} f = j^{q,s,r}_{(x,y)} g$ reads

$$j^r_x \underline{f} = j^r_x \underline{g}, \quad j^s_{(x,y)} \tilde{f}_2 = j^s_{(x,y)} \tilde{g}_2, \quad j^q_{(x,y)} \tilde{f}_1 = j^q_{(x,y)} \tilde{g}_1. \tag{5}$$

Moreover, the naturality on $h\colon M \to \overline{M}$ yields the commutative diagram

$$\begin{array}{ccc} T^r_m M & \xrightarrow{T^r_m h} & T^r_m \overline{M} \\ {\scriptstyle t_M}\big\downarrow & & \big\downarrow{\scriptstyle t_{\overline{M}}} \\ T^A M & \xrightarrow{T^A h} & T^A \overline{M}. \end{array}$$

Hence we have

$$T^A \underline{f}(t_M(u)) = t_{\overline{M}}(P^r \underline{f}(u)), \quad T^A \tilde{f}_1(t_M(u)) = t_{\overline{N}}(T^r_m(\tilde{f}_1(u))$$

and (4) is of the form

$$Ff(u, W) = \{P^r \underline{f}(u), (t_{\overline{M}}(P^r \underline{f}(u)), t_{\overline{N}}(T^r_m(\tilde{f}_1(u)) + T^A \tilde{f}_2(W))\}. \tag{6}$$

By (3), (5) and (6), $Fg(u, W)$ coincides with (4). □

Example 2: Holonomic jet functors. Write J^r for the r-jet functor defined on $\mathscr{M}f_m \times \mathscr{M}f$ and $J^r_h\colon \mathscr{FM}_m \to \mathscr{FM}$ for the functor of the r-jet prolongation of fibered manifolds. Further, let J^r_v be the vertical r-jet functor on $\mathscr{FM}_m$ defined by

$$J^r_v(Y) = \bigcup_{x \in M} J^r_x(M, Y_x)$$

and analogously for morphisms. It holds $\mathrm{Aut}(\mathbb{D}^r_m) = G^r_m$, which induces the canonical action C of G^r_m on the algebra $\mathbb{D}^r_m$ given by the composition of jets. Then the triples (A, H, t) of the functors J^r_h and J^r_v are of the form

$$J^r_h = (\mathbb{D}^r_m, \mathrm{id}_{G^r_m}, \mathrm{id}_{\mathbb{D}^r_m}), \quad J^r_v = (\mathbb{D}^r_m, \mathrm{id}_{G^r_m}, 0_{\mathbb{D}^r_m})$$

where $0_{\mathbb{D}^r_m} : \mathbb{D}^r_m \to \mathbb{D}^r_m$ is the zero homomorphism transforming the nilpotent part into zero. This implies that J^r_h and J^r_v have the orders (r, r, r) and $(0, r, r)$, respectively.

Example 3: Vertical functors. (a) *Vertical A-prolongation functor V^A.* Let A be a Weil algebra of order s and $V^A\colon \mathcal{FM}_m \to \mathcal{FM}$ be the functor defined by

$$V^A(Y) = \bigcup_{x\in M} T^A Y_x$$

and analogously for morphisms. Clearly, the order of V^A is $(0,s,0)$. Since $G^0_m = \{e\}$ is the one-element group and $\mathbb{D}^0_m = \mathbb{R}$, we have

$$V^A = (A, H_e, i_{\mathbb{R}})$$

where $H_e\colon \{e\} \to \mathrm{Aut}(A)$ is the trivial homomorphism and $i_{\mathbb{R}}\colon \mathbb{R} \to A$ is the canonical injection.

(b) *General vertical functor $V^{A,H}$.* Consider an arbitrary group homomorphism $H\colon G^r_m \to \mathrm{Aut}(A)$ and define a functor $F^{A,H}$ on $\mathcal{M}f_m \times \mathcal{M}f$ by

$$F^{A,H}(M,N) = P^r M[T^A N, H_N]$$

where H_N is the action induced by H. Then we can introduce a functor $V^{A,H}$ on $\mathcal{FM}_m$ by

$$V^{A,H}(Y) = \bigcup_{x\in M} F^{A,H}_x(M, Y_x), \quad V^{A,H}(f) = \bigcup_{x\in M} F^{A,H}_x(\underline{f}, f_x).$$

We have

$$V^{A,H} = (A, H, 0_A)$$

where $0_A\colon \mathbb{D}^r_m \to A$ is the zero homomorphism. Obviously, $V^{A,H}$ has the order $(0,s,r)$. For example, $V^{A,H_e} = V^A$ and $V^{\mathbb{D}^r_m, \mathrm{id}_{G^r_m}} = J^r_v$. From (2) it follows directly that

$$(A,H,t) = V^{A,H} \quad \text{if and only if } \ t = 0_A.$$

Since $V^A = V^{A,H_e}$, we have $(A,H,t) = V^A$ if and only if $t = 0_A$ and $H = H_e$.

Example 4: Nonholonomic and general jet functors. Denote by $\tilde{J}^r\colon \mathcal{M}f_m \times \mathcal{M}f \to \mathcal{FM}$ the nonholonomic r-jet functor and by $\tilde{J}^r_h$ the functor of nonholonomic r-jet prolongation of fibered manifolds. Then $\tilde{J}^r_h$ coincides with the r-fold iteration of J^1_h, i.e.

$$\tilde{J}^r_h = J^1_h \circ \ldots \circ J^1_h.$$

By [1], the Weil algebra $\tilde{\mathbb{D}}^r_m = \tilde{J}^r_0(\mathbb{R}^m, \mathbb{R})$ of $\tilde{J}^r_h$ is of the form

(7) $$\tilde{\mathbb{D}}^r_m = \underbrace{\mathbb{D}^1_m \otimes \ldots \otimes \mathbb{D}^1_m}_{r-\text{times}}.$$

The composition of nonholonomic jets defines an action of the group

$$\tilde{G}^r_m = \operatorname{inv} \tilde{J}^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$$

on $\tilde{\mathbb{D}}^r_m$ and its restriction to $G^r_m \subset \tilde{G}^r_m$ yields the action $\tilde{C}$ of G^r_m on $\tilde{\mathbb{D}}^r_m$. By [3], we have

$$\tilde{J}^r_h = (\tilde{\mathbb{D}}^r_m, \tilde{C}, \tilde{i})$$

where $\tilde{i}\colon \mathbb{D}^r_m \to \tilde{\mathbb{D}}^r_m$ is the canonical inclusion. I. Kolář has recently introduced the general concept of an rth order jet functor on $\mathcal{FM}_m$, [3]. This is a subfunctor

$$J^r_h \subset F \subset \tilde{J}^r_h$$

that preserves fiber products. According to [3], these functors are in bijection with the G^r_m-invariant Weil subalgebras A of $\tilde{\mathbb{D}}^r_m$ satisfying $\mathbb{D}^r_m \subset A \subset \tilde{\mathbb{D}}^r_m$. Denoting by C_A the restriction of the action $\tilde{C}$ to A and by $i_A\colon A \to \tilde{\mathbb{D}}^r_m$ the injection, a general rth order jet functor F on $\mathcal{FM}_m$ is of the form

$$F = (A, C_A, i_A).$$

3. Subordinated functors on $\mathcal{FM}_m$

Consider two fiber product preserving bundle functors $F = (A, H, t)$ and $G = (B, D, \tau)$ on $\mathcal{FM}_m$ and suppose that the base order of F and G is r and s, respectively, $s \leqslant r$. Write $\beta\colon \mathbb{D}^r_m \to \mathbb{D}^s_m$, $\pi\colon G^r_m \to G^s_m$ and $\varrho\colon P^rM \to P^sM$ for the jet projections. From [5] it follows easily

Lemma 2. *Surjective natural transformations* $(A, H, t) \to (B, D, \tau)$ *are in bijection with* π*-equivariant epimorphisms* $\mu\colon A \to B$ *of Weil algebras satisfying* $\mu \circ t = \tau \circ \beta$.

So the following diagram commutes

$$\begin{array}{ccc} G^r_m \times A & \xrightarrow{H} & A \\ {\scriptstyle\pi}\downarrow \quad \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mu} \\ G^s_m \times B & \xrightarrow{D} & B. \end{array}$$

By (2), $GY = \{\{\bar{u}, \overline{Z}\} \in P^sM[T^BY],\ \tau_M(\bar{u}) = T^Bp(\overline{Z})\}$. Denoting by $\mu_Y\colon T^AY \to T^BY$ the natural transformation induced by the algebra epimorphism μ, we can construct the induced map of the associated bundles

$$[\mu]_Y\colon P^rM[T^AY] \to P^sM[T^BY].$$

Proposition 4. *The natural transformations $F \to G$ are of the form*

$$(8) \qquad [\mu]_Y(\{u, Z\}) = \{\varrho(u), \mu_Y(Z)\}, \quad u \in P^r M, \quad Z \in T^A Y.$$

P r o o f. Taking any $g \in G^r_m$ we have

$$[\mu]_Y(\{ug, g^{-1}Z\}) = \{\varrho(ug), \mu_Y(g^{-1}Z)\} = \{\varrho(u)\pi(g), \pi(g^{-1})\mu_Y(Z)\}$$

so that (8) is independent of the choice of (u, Z). Further, the naturality of μ on $p\colon\ Y \to M$ yields $T^B p \circ \mu_Y = \mu_M \circ T^A p$. Then we have

$$\tau_M(\beta_M(u)) = \mu_M(t_M(u)) = \mu_M(T^A p(Z)) = T^B p(\mu_Y(Z)).$$

Hence (8) maps FY into GY. □

In what follows, a functor F with the base order r will be shortly called an r-functor. By Lemma 2, if an s-functor $G = (B, D, \tau)$ is dominated by an r-functor $F = (A, H, t)$, then $B = A/I$, where $I \subset A$ is an ideal. Write $\mathbb{D}^r_m = \mathbb{R} \times N^r_m$ and define $G^{r,s}_m$ and $N^{r,s}_m$ by the exact sequences

$$e \to G^{r,s}_m \to G^r_m \xrightarrow{\ \pi\ } G^s_m \to e, \quad 0 \to N^{r,s}_m \to N^r_m \xrightarrow{\ \beta\ } N^s_m \to 0.$$

If $\mu\colon\ A \to B = A/I$ is an equvariant epimorphism, then

$$\mu(H(k)(a)) = D(e)(\mu(a)) = \mu(a), \quad a \in A, \ \ k \in G^{r,s}_m.$$

In particular, $H(k)(I) \subset I$ and for all $a \in A$ we have

$$(9) \qquad H(k)(a + I) = a + I, \quad k \in G^{r,s}_m.$$

Definition. We say that an ideal $I \subset A$ is *strongly $G^{r,s}_m$-invariant*, if (9) holds for all $a \in A$, $k \in G^{r,s}_m$.

Clearly, the kernel I of an equivariant epimorphism $\mu\colon\ A \to B$ is a strongly $G^{r,s}_m$-invariant ideal.

Lemma 3. *If I and $\tilde{I}$ are two strongly $G^{r,s}_m$-invariant ideals, then $I \cap \tilde{I}$ is also a strongly $G^{r,s}_m$ invariant ideal.*

P r o o f. For $s \in I \cap \tilde{I}$ we have $H(k)(a + s) = a + \bar{s}$, which yields $\bar{s} \in I \cap \tilde{I}$. □

From Lemma 2 it follows directly that if an s-functor $G = (A/I, D, \tau)$ is dominated by an r-functor $F = (A, H, t)$, then $t(N^{r,s}_m) \subset I$.

Definition. An ideal $I \subset A$ is called *s-admissible* if it is strongly $G^{r,s}_m$-invariant and satisfies the condition $t(N^{r,s}_m) \subset I$.

Thus, we have

Proposition 5. *If an s-functor (B, D, τ) is dominated by an r-functor (A, H, t), then $B = A/I$, where I is an s-admissible ideal in A.*

Remark. By Proposition 3, the fiber order $\bar{s}$ of a functor $G = (A/I, D, \tau)$ which is dominated by $F = (A, H, t)$, is less than or equal to the fiber order s of F. On the other hand, the base order of a subordinated functor G can be greater than the base order of the original functor F. In fact, consider two Weil algebras A and B such that there is an epimorphism $A \to B$. Let $D\colon G_m^r \to \mathrm{Aut}(B)$ be an arbitrary homomorphism. Write $F = V^A$, $G = V^{B,D}$. One evaluates directly that the functor G is dominated by F. On the other hand, the base order of F is 0, while the base order of G is r.

Suppose now that $F = (A, H, t)$ is an arbitrary r-functor and $I \subset A$ is an s-admissible ideal. In the rest of this section we construct an s-functor

$$F_I := (A/I, H_I, t_I)$$

which is dominated by F. First, define the action H_I of the group $G_m^s = G_m^r/G_m^{r,s}$ on A/I by

$$H_I(gG_m^{r,s})(a + I) = H(g)(a) + I, \quad g \in G_m^r, \;\; a \in A.$$

By $G_m^{r,s}$-invariance of I we have

$$H(gk)(a + I) = H(g)(H(k)(a + I)) = H(g)(a + I).$$

This implies that the definition of H_I is correct. Further, from the condition $t(N_m^{r,s}) \subset I$ it follows that there exists a factor homomorphism

$$t_I\colon \mathbb{D}_m^s = \mathbb{D}_m^r/N_m^{r,s} \to A/I, \quad t_I(x + N_m^{r,s}) = t(x) + I, \quad x \in N_m^r.$$

It remains to show that t_I is G_m^s-equivariant. In fact, G_m^r-equivariancy of t means

$$t(gn) = H(g)(t(n)) \quad \text{for} \quad g \in G_m^r, \quad n \in N_m^r.$$

Then we have

$$\begin{aligned} t_I((gG_m^{r,s})(n + N_m^{r,s})) &= t_I(gn + N_m^{r,s}) = t(gn) + I = H(g)(t(n)) + I \\ &= H_I(gG_m^{r,s})(t(n) + I) = H_I(gG_m^{r,s})(t_I(n + N_m^{r,s})). \end{aligned}$$

We have proved

Proposition 6. *The s-functor $F_I = (A/I, H_I, t_I)$ constructed above is dominated by (A, H, t).*

4. Underlying functors on $\mathcal{FM}_m$

In Section 1 we have defined the underlying sth order functor F_s of a Weil functor F. However, the order of a functor on $\mathcal{FM}_m$ is determined by three integers (q, s, r). That is why we can define several types of underlying functors on $\mathcal{FM}_m$.

Definition. Let F be a functor on $\mathcal{FM}_m$. A functor F_s^b is said to be the *underlying functor with the base order* s, if

(1) F_s^b is dominated by F,

(2) the base order of F_s^b is s,

(3) every functor $\tilde{F}$ on $\mathcal{FM}_m$ with the base order s, which is dominated by F, is also dominated by F_s^b.

If we replace the base order by the fiber order, we obtain the concept of an *underlying functor F_s^f with the fiber order s.*

By Lemma 3, the intersection $I \cap \tilde{I}$ of two s-admissible ideals is an s-admissible ideal as well. Denote by I_s the minimal s-admissible ideal in A and write

$$F_s := F_{I_s} = (A/I_s, H_{I_s}, t_{I_s}).$$

Proposition 7. *Let F be a fiber product preserving functor on $\mathcal{FM}_m$. The underlying functor F_s^b of F is of the form $F_s^b = F_s$.*

Proof. Suppose that $\tilde{F} := F_I$ is another functor of the base order s, which is dominated by F. We have to show that there is a surjective natural transformation $F_s \to \tilde{F}$. Since I_s is minimal, we have $I_s \subset I$. This defines a factor epimorphism μ: $A/I_s \to A/I$, which is $\mathrm{id}_{G_m^s}$-equivariant. Finally, one evaluates easily that μ satisfies $\mu \circ t_{I_s} = t_I$. □

Proposition 8. *Let $F = (A, H, t)$ be an r-functor and suppose that the ideal $N_A^{s+1} \subset A$ is s-admissible. Then we have*

$$F_s = (A_s, H_s, t_s) \quad \text{where } A_s = A/N_A^{s+1},\ H_s = H_{N_A^{s+1}},\ t_s = t_{N_A^{s+1}}.$$

Moreover, it holds $F_s^b = F_s^f = F_s$.

P r o o f. It suffices to show that the ideal N_A^{s+1} is minimal. Suppose that $I \subset A$ is another s-admissible ideal and write $\mu\colon A \to \tilde{A} = A/I$ for the algebra epimorphism. The condition $(N_{\tilde{A}})^{s+1} = 0$ yields $(\mu(N_A))^{s+1} = 0$, which implies $N_A^{s+1} \subset I$. Further, by Proposition 1 we have $F_s^f = F_s$. $\square$

5. Examples of subordinated and underlying functors on $\mathscr{FM}_m$

I. Subordinated 0-functors and r-functors.

Proposition 9. *Let $F = (A, H, t)$ be an r-functor and $B = A/I$ be a Weil algebra dominated by A. We have*

(1) *If $t(N_m^r) \subset I$, then the 0-functor V^B is dominated by F,*

(2) *If $t(N_m^r) = 0$, then $F_0 = V^A$.*

P r o o f. Clearly, $G_m^{r,0} = G_m^r$ and $N_m^{r,0} = N_m^r$. One evaluates directly that each ideal $I \subset A$ is strongly $G_m^{r,0}$-invariant. Moreover, $I = 0$ is the minimal 0-admissible ideal. $\square$

Proposition 10. *Let $F = (A, H, t)$ be an r-functor and $B = A/I$ be a Weil algebra dominated by A. Then every r-functor of the form $G = (B, D, \tau)$ is dominated by F. Further, we have $F_r = F$.*

P r o o f. Since $N_m^{r,r} = 0$, every ideal $I \subset A$ satisfies $t(N_m^{r,r}) \subset I$. Further, we have $G_m^{r,r} = e$, so that I is r-admissible. Finally, $I = 0$ is a minimal r-admissible ideal. $\square$

Example 5. (a) By Proposition 9, if the algebra B is dominated by A, then the 0-functor V^B is dominated by V^A and also by $V^{A,H}$ for an arbitrary group homomorphism $H\colon G_m^r \to \mathrm{Aut}(A)$. Further, we have $(V^A)_0 = (V^{A,H})_0 = V^A$.

(b) Clearly, the vertical jet functor $J_v^r = V^{\mathbb{D}_m^r, \mathrm{id}_{G_m^r}}$ has a subordinated functor V^B, where B is an arbitrary Weil algebra dominated by $\mathbb{D}_m^r$. Moreover, $(J_v^r)_0 = V^{\mathbb{D}_m^r}$. On the other hand, $(J_h^r)_0 = \mathrm{id}_{\mathscr{FM}_m}$.

II. The iterated jet functor. By [1], $J_h^r(J_h^s) = (A, H, t)$, where the Weil algebra A is the tensor product $A = \mathbb{D}_m^r \otimes \mathbb{D}_m^s$. Consider now the iterated velocities functor $T_m^r(T_m^s)$, which is a Weil functor of order $r + s$. Write

$$T^B M := (T_m^r(T_m^s M))_{r+s-1}$$

for the underlying $(r+s-1)$th order Weil functor, which has been described in Example 1. Then the Weil algebra B is of the form

$$B = A_{r+s-1} = A/I, \quad \text{where } I = (N_m^r \otimes N_m^s)^{r+s}.$$

Clearly, the $(r+s)$-functor $J_h^r(J_h^s)$ has the subordinated $(r+s-1)$-functor G, where

$$GY = J_h^{r-1}(J_h^s Y) \times_{J_h^{r-1} J_h^{s-1} Y} J_h^r(J_h^{s-1} Y).$$

Then the Weil algebra of G is A/I. Since G is dominated by $J_h^r(J_h^s)$, the ideal I is $(r+s-1)$-admissible. By Proposition 8 we have

$$(J_h^r(J_h^s))_{r+s-1} = G.$$

III. Underlying functors of a general jet functor.

Proposition 11. *Let $F = (A, C_A, i_A)$ be an rth order jet functor on $\mathcal{FM}_m$. Then we have*

$$F_s = (A_s, (C_A)_s, (i_A)_s) \quad \text{where } A_s = A/N_A^{s+1}.$$

P r o o f. By Proposition 8, it suffices to show that the ideal N_A^{s+1} is s-admissible. Clearly, this is true in the case of the holonomic jet functor J_h^r for the ideal $(N_m^r)^{s+1} \subset \mathbb{D}_m^r$. The s-admissibility of $(N_m^r)^{s+1}$ means that $N_m^{r,s} \subset (N_m^r)^{s+1}$. For $i_A \colon \mathbb{D}_m^r \to A$ we have

$$i_A(N_m^r)^{s+1} \subset N_A^{s+1},$$

so that also $i_A(N_m^{r,s}) \subset N_A^{s+1}$. Finally, since C_A is the restriction of the action given by composition of jets, the ideal N_A^{s+1} is also strongly $G_m^{r,s}$-invariant. □

Corollary. *We have*

$$(\tilde{J}_h^r)_s = ((\tilde{\mathbb{D}}_m^r)_s, \tilde{C}_s, \tilde{i}_s).$$

In the rest of this section we describe the functor $(\tilde{J}_h^r)_{r-1}$. The formula (7) defines r projections $\tilde{\mathbb{D}}_m^r \to \tilde{\mathbb{D}}_m^{r-1}$. Denoting by

$$\tilde{T}_m^r M = \tilde{J}_0^r(\mathbb{R}^m, M)$$

the nonholonomic velocities functor, we have r projections $q_i \colon \tilde{T}_m^r M \to \tilde{T}_m^{r-1} M$, $i = 1, \dots, r$. Let

$$B_m^r M = \tilde{T}_m^r M \times_{\tilde{T}_m^{r-1} M} \dots \times_{\tilde{T}_m^{r-1} M} \tilde{T}_m^r M$$

be a generalized fiber product with respect to r projections q_i, $i = 1, \dots, r$, see [6]. This is a generalization of the bundle of boundaries, which was introduced by J. E. White in [7]. By M. Kureš [6], we have $(\tilde{T}^r_m)_{r-1} = B^{r-1}_m$, where the Weil algebra of B^{r-1}_m is

$$(\tilde{\mathbb{D}}^r_m)_{r-1} = \tilde{\mathbb{D}}^r_m/(\tilde{N}^r_m)^r.$$

Write $q_Y\colon J^1_h Y \to Y$ for the jet projection and

$$\beta_r = q_{\tilde{J}^{r-1}_h Y}\colon \tilde{J}^r_h Y \to \tilde{J}^{r-1}_h Y.$$

Then we have r projections $\tilde{J}^r_h Y \to \tilde{J}^{r-1}_h Y$ of the form

$$\beta_r, J^1_h \beta_{r-1}, \dots, (\underbrace{J^1_h \dots J^1_h}_{r})\beta_1.$$

By Corollary,

$$(\tilde{J}^r_h)_{r-1}(Y) = \tilde{J}^{r-1}_h Y \times_{\tilde{J}^{r-2}_h Y} \dots \times_{\tilde{J}^{r-2}_h Y} \tilde{J}^{r-1}_h Y,$$

where on the right we have a generalized fiber product with respect to r projections $\tilde{J}^r_h Y \to \tilde{J}^{r-1}_h Y$.

6. Affine structure on fiber product preserving functors on $\mathcal{FM}_m$

One verifies directly the following assertion.

Lemma 4. *Let $E = P[S, \ell]$ be a bundle associated to the principal bundle $P(M, G)$. Let the fibre S be an affine space with the associated vector space V and let $\bar{\ell}\colon G \to \mathrm{GL}(V)$ be such a representation of the group G on V, that for arbitrary $a \in G$, $v \in V$, $s \in S$ we have*

$$\ell_a(s) + \bar{\ell}_a(v) = \ell_a(s + v). \tag{10}$$

Then we have the canonical affine bundle structure on E with the associated vector bundle $\overline{E} = P[V, \bar{\ell}]$. Moreover, the addition of two elements $A \in E_x$, $B \in \overline{E}_x$ is of the form $\{u, s\} + \{u, v\} = \{u, s + v\}$.

Consider an arbitrary r-functor $F = (A, H, t)$ such that the ideal N^r_A is $(r-1)$-admissible. By Proposition 8,

$$F_{r-1} = (A_{r-1}, H_{r-1}, t_{r-1}), \quad A_{r-1} = A/N^r_A.$$

By [2], $T^A M \to T^{A_{r-1}} M$ is an affine bundle, whose associated vector bundle is the pullback of $TM \otimes N_A^r$ over $T^{A_{r-1}} M$. Moreover, if $p\colon Y \to M$ is a fibered manifold, then $T^A p\colon T^A Y \to T^A M$ is an affine bundle morphism over $T^{A_{r-1}} p\colon T^{A_{r-1}} Y \to T^{A_{r-1}} M$, whose associated vector bundle morphism is the pullback of $Tp \otimes \mathrm{id}_{N_A^r}\colon TY \otimes N_A^r \to TM \otimes N_A^r$. According to (2), $FY \subset P^r M[T^A Y]$. Clearly, the fibres of this associated bundle are affine spaces. Hence we have

Lemma 5. *$P^r M[T^A Y] \to P^r M[T^{A_{r-1}} Y]$ is an affine bundle, whose associated vector bundle is the pullback of $P^r M[TY \otimes N_A^r]$ over $P^r M[T^{A_{r-1}} Y]$.*

P r o o f. Let $H\colon G_m^r \times T^A Y \to T^A Y$ be the action of G_m^r. We have to define a representation $\overline{H}\colon G_m^r \to \mathrm{GL}(V)$, $V = TY \otimes N_A^r$ satisfying the condition (10). Since $T^A Y$ is an affine space, we have

$$H_a(y + v) = H_a(y) + \bar{v}, \quad y \in T^A Y, \ v, \bar{v} \in V, \ a \in G_m^r.$$

Write $\overline{H}_a(v) = \bar{v}$, so that $H_a(y + v) = H_a(y) + \overline{H}_a(v)$. One evaluates easily that

$$\overline{H}_{ab}(v) = \overline{H}_b(\overline{H}_a(v)).$$

□

Proposition 12. *Let $F = (A, H, t)$ be an r-functor. Then $FY \to F_{r-1} Y$ is an affine bundle, whose associated vector bundle is the pullback of $P^r M[VY \otimes N_A^r]$ over $F_{r-1} Y$.*

P r o o f. Consider the expression (2). If $v \in V := TY \otimes N_A^r$, then $T^A p(Z + v) \in T^A M$. Hence we have

$$T^A p(Z + v) = T^A p(Z) + w, \tag{11}$$

where $w \in TM \otimes N_A^r$ is of the form

$$w = (Tp \otimes \mathrm{id}_{N_A^r})(v).$$

Clearly, $w = 0$ if and only if $v \in VY \otimes N_A^r$. Then (11) reads

$$T^A p(Z + v) = T^A p(Z) = t_M(u).$$

So for $\{u, Z\} \in FY$ and $v \in VY \otimes N_A^r$ we have $\{u, Z + v\} \in FY$, which proves our claim. □

As an example we obtain the well known result that $J^r Y \to J^{r-1} Y$ is an affine bundle whose associated vector bundle is the pullback of $VY \otimes S^r T^* M$ over $J^{r-1} Y$.

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