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AFFINE COMPLETENESS AND LEXICOGRAPHIC PRODUCT DECOMPOSITIONS OF ABELIAN LATTICE ORDERED GROUPS

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Abstract. In this paper it is proved that an abelian lattice ordered group which can be expressed as a nontrivial lexicographic product is never affine complete.

Keywords: Abelian lattice ordered group, lexicographic product decomposition, affine completeness

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1. Introduction

The problem proposed by Kaarli and Pixley (cf. [5, Problem 5.6.19]) on the existence of a nontrivial affine complete lattice ordered group remains open. We remark that this problem was formulated already in [2].

Let \( \mathcal{G}_0 \) be the class of all nonzero lattice ordered groups. In order to arrive nearer to the solution of the problem mentioned it seems to be useful to describe “large” areas \( S \) in \( \mathcal{G}_0 \) such that no affine complete lattice ordered group can exist in \( S \).

Some types of lattice ordered groups which fail to be affine complete have been described by Kaarli and Pixley [5], the author and Csontóová [4], and the author [2], [3].

In [2] it has was proved that if \( G \) is an abelian lattice ordered group which can be expressed as a nontrivial direct product, then \( G \) is not affine complete. In [3], this result was generalized to lattice ordered groups which need not be abelian. The corresponding result of [5] also deals with a certain form of direct product decompositions. Let \( G \) be a nonzero lattice ordered group; in [2] it was shown that if

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G is complete, then it is not affine complete. An analogous result was proved in [4] for the case when G is abelian and projectable.

Assume that G is an abelian lattice ordered group which can be expressed as a nontrivial lexicographic product. In this paper we define the notion of a regular ℓ-subgroup of G. We prove that if H is a regular ℓ-subgroup of G, then H is not affine complete. In particular, G is not affine complete.

2. Preliminaries

Throughout the paper, G denotes an abelian lattice ordered group. Let P(G) be the set of all polynomials over G. If for each mapping f : G^n → G such that n ∈ N and f is compatible with all congruence relations on G we have f ∈ P(G), then G is called affine complete.

For the sake of completeness and for fixing the notation we recall the definition of the lexicographic product decomposition of G (cf., e.g., Fuchs [1]).

Let I be a linearly ordered set and for each i ∈ I let G_i be a lattice ordered group such that, whenever i fails to be the greatest element of I, then G_i is linearly ordered. The direct product of groups G_i will be denoted by G^0. The elements of G^0 are written in the form g = (g_i)i∈I; g_i is the component of g in G_i. We put

\[ s(g) = \{i \in I : g_i \neq 0\} \]

If s(g) ≠ ∅, then s(g) is linearly ordered (as a subset of I).

Let G be the set of all g ∈ G^0 such that either g = 0 or the linearly ordered set s(g) is well-ordered. Then G is a subgroup of the group G^0.

For g ∈ G we put g > 0 if g ≠ 0 and g_{i(0)} > 0, where i(0) is the least element of s(g). Then G turns out to be a lattice ordered group. We denote

\[ G = \Gamma_{i \in I} G_i \]

G is said to be the lexicographic product of lattice ordered groups G_i.

Assume that G ≠ {0}. Then all G_i with G_i = {0} can be omitted in (1). Hence without loss of generality we can suppose that G_i ≠ {0} for each i ∈ I. If this is satisfied and card I > 1, then we say that the lexicographic product decomposition (1) of G is nontrivial.

Let i(1) ∈ I and let g^{i(1)} be a fixed element of G_{i(1)}. We denote by g^{(1)} the element of G^0 such that

\[ (g^{(1)})_i = \begin{cases} g^{i(1)} & \text{if } i = i(1), \\ 0 & \text{otherwise.} \end{cases} \]

Then we clearly have g^{(1)} ∈ G.
Let $H$ be an $\ell$-subgroup of $G$ such that $g_{i(1)} \in H$ whenever $i(1) \in I$ and $g_{i(1)} \in G_{i(1)}$. Under this assumption we say that $H$ is a regular $\ell$-subgroup of $G$.

2.1. Lemma (Cf. [4, Lemma 1.1]). Let $G$ be an abelian lattice ordered group and let $p(x) \in P(G)$ be such that $p(x)$ fails to be a constant. There exist $a, x_0 \in G$ and an integer $n$ such that, whenever $x_1 \in G$ and $x_1 \geq x_0$, then $p(x_1) = a + nx_1$.

3. Regular $\ell$-subgroups

In this section we assume that $G \neq \{0\}$ is an abelian lattice ordered group. Further, we suppose that the relation (1) from Section 2 is valid and that $H$ is a regular $\ell$-subgroup of $G$.

3.1. Lemma. Let $0 \neq x \in H$, $i(0) = \min s(x)$. Then

\begin{equation}
-2|x| < \varpi_{i(0)} < 2|x|.
\end{equation}

Proof. a) At first suppose that $i(0)$ is not the maximal element of $I$. Then either $\varpi_{i(0)} > 0$ or $\varpi_{i(0)} < 0$.

Assume that the first of these possibilities is valid. Then $x > 0$, whence $|x| = x$ and $2|x| = 2x$. Further, $i(0) \in \min s(2x)$ and $\varpi_{i(0)} = \varpi_{i(0)}$. Since

\[-2\varpi_{i(0)} < \varpi_{i(0)} < 2\varpi_{i(0)}, \quad (\varpi_{i(0)})_{i(0)} = \varpi_{i(0)},\]

we get

\[-2|x| < \varpi_{i(0)} < 2|x|.\]

The case $\varpi_{i(0)} < 0$ can be treated analogously.

b) Now suppose that $i(0)$ is the greatest element of $I$. Then we have $\varpi_{i(0)} = x$ and then it suffices to apply the well-known relation

\[-2|x| < x < 2|x|.\]
3.2. Lemma. Let $0 \neq x \in H$, $i(1) \in I$. Then

\begin{align*}
(2) & \quad -2|x| < \varphi_{i(1)} < 2|x|, \\
(3) & \quad -2|x| < -\varphi_{i(1)} < 2|x|.
\end{align*}

Proof. If $i(1) = i(0) = \min s(x)$, then from (1) we conclude that both (2) and (3) are valid.

Let $i(1) < i(0)$. Then $\varphi_{i(1)} = 0$, whence (2) and (3) hold. Finally, let $i(1) > i(0)$. Then we have

\begin{align*}
-|\varphi_{i(0)}| & < \varphi_{i(1)} < |\varphi_{i(0)}|, \\
-|\varphi_{i(0)}| & < -\varphi_{i(1)} < |\varphi_{i(0)}|,
\end{align*}

whence according to 3.1, the relations (2) and (3) are satisfied. 

3.3. Lemma. Let $A$ be an $\ell$-ideal of $H$ and $x \in A$. Then for each $i(1) \in I$ we have $\varphi_{i(1)} \in A$.

Proof. From $x \in A$ we obtain $|x| \in A$ and $2|x| \in A$, $-2|x| \in A$. Since $A$ is a convex subset of $H$, in view of 3.2 we conclude that $\varphi_{i(1)} \in A$. 

Let $i(1) \in I$. Define a mapping $f : H \to H$ by putting

$$f(x) = \varphi_{i(1)} \quad \text{for each } x \in H.$$ 

3.4. Lemma. The mapping $f$ is compatible with all congruence relations on $H$.

Proof. Let $\equiv$ be a congruence relation on $H$. There exists an $\ell$-ideal $A$ of $H$ such that $\equiv$ is determined by $A$.

Let $x, y \in H$. Suppose that $x \equiv y$. This means that $x - y \in A$. Put $x - y = z$. In view of 3.3 we get $\varphi_{i(1)} \in A$. Clearly

$$\varphi_{i(1)} = (x - y)_{i(1)} = \varphi_{i(1)} - \varphi_{i(1)} = f(x) - f(y).$$

Hence $f(x) - f(y) \in A$ and thus $f(x) \equiv f(y)$. 

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3.5. **Theorem.** Let $i(1) \in I$ and let $f(x)$ be as above. Then $f(x) \not\in P(H)$.

**Proof.** By way of contradiction, assume that there exists $p(x) \in P(H)$ such that $p(x) = f(x)$. Then $p(x)$ fails to be a constant. We apply Lemma 2.1 for $H$. Let $a$, $x_0$ and $n$ be as in 2.1.

Since the lexicographic product decomposition (1) is nontrivial, there exists $i(2) \in I$ with $i(2) \neq i(1)$. It is easy to verify that there exists $z \in G$ with $z_{i(2)} > 0$; then $0 < z_{i(2)} \in H$.

Further, choose $x_1 \in H$ with $x_1 \geq x_0 \lor 0$. If $(x_1)_{i(2)} = 0$, then we replace $x_1$ by the element $x_2 = x_1 + z_{i(2)}$. We have $x_1 < x_2 \in H$ and $(x_2)_{i(2)} \neq 0$.

Thus without loss of generality we can suppose that $(x_1)_{i(2)} \neq 0$. We obtain

$$(x_1)_{i(1)} = a + nx_1.$$ 

Similarly, taking $2x_1$ instead of $x_1$ we get

$$(2x_1)_{i(1)} = a + n \cdot 2x_1.$$ 

Since $(2x_1)_{i(1)} = 2(x_1)_{i(1)}$, we have

$$(x_1)_{i(1)} = nx_1.$$ 

But

$$0 = (x_1)_{i(1)}_{i(2)}, \quad 0 \neq (nx_1)_{i(2)},$$

and thus we have arrived at a contradiction. \(\Box\)

3.6. **Theorem.** Let $G$ be an abelian lattice ordered group which can be expressed as a nontrivial lexicographic product. Assume that $H$ is a regular $\ell$-subgroup of $G$. Then $H$ fails to be affine complete.

**Proof.** This is a consequence of 3.4 and 3.5. \(\Box\)

3.7. **Corollary.** Let $G$ be an abelian lattice ordered group which can be expressed as a nontrivial lexicographic product. Then $G$ is not affine complete.

We conclude by remarking that if $G$ is a nontrivial lexicographic product, then

(i) $G$ is not complete;
(ii) $G$ is not projectable;
(iii) $G$ is directly indecomposable.

Hence the affine incompleteness of $G$ does not follow from the results of [2], [3], [4].
The condition applied in [5] when investigating affine incompleteness of a lattice ordered group $G$ was as follows:

(α) $G$ is a direct product of a nonzero subdirectly irreducible lattice ordered group and any lattice ordered group.

It is easy to construct a lattice ordered group $G$ such that $G$ is a nontrivial lexicographic product and $G$ fails to be subdirectly irreducible. Therefore 3.7 is not a consequence of the mentioned result of [5].

References


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