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# CHARACTERIZATIONS OF SUB-SEMIHYPERGROUPS BY VARIOUS TRIANGULAR NORMS

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*Abstract.* We investigate the structure and properties of $TL$-sub-semihypergroups, where $T$ is an arbitrary triangular norm on a given complete lattice $L$. We study its structure under the direct product and with respect to the fundamental relation. In particular, we consider $L = [0, 1]$ and $T = \min$, and investigate the connection between $TL$-sub-semihypergroups and the probability space.

*Keywords*: semihypergroup, complete lattice, triangular norm, fundamental relation, probability space

*MSC 2000*: 20N20

## 1. Introduction

The theory of algebraic hyperstructures which is a generalization of the concept of algebraic structures was first introduced by Marty [13]. Now they are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics; for example, semihypergroups are the simplest algebraic hyperstructures which possess the properties of closure and associativity. They are very important in the theory of sequential machines, formal language, and in certain applications. A comprehensive review of the theory of hyperstructures appears in [4] and [18].

The theory of fuzzy sets was first introduced by Zadeh [20]. Since its inception, the theory has developed in many directions and found applications in a wide variety of fields. The study of fuzzy algebraic structures started with the introduction of the concept of the fuzzy subgroupoid (the subgroup) of a groupoid (a group) in the pioneering paper of Rosenfeld [16]. In [15], Negoita and Ralescu considered a generalization of Rosenfeld's definition in which the unit interval $[0, 1]$ was replaced by an appropriate lattice structure. Since then many researchers have been engaged in extending the concepts and results of abstract algebra to broader framework of the fuzzy setting, for example see [1], [11], [12], [14], [19]. In 1979, Anthony and Sherwood [2] redefined the fuzzy subgroup using the statistical triangular norm. This notion was introduced by Schweizer and Sklar [17] in order to generalize the ordinary triangle inequality in a metric space to the more general probabilistic metric space.

In [8], the present author applied the fuzzy set theory in the theory of algebraic hyperstructures and defined the concept of the fuzzy subhypergroup (and $T$-fuzzy subhypergroup) of a hypergroup which is a generalization of the concept of the fuzzy subgroup. This has been further investigated in [5], [6], [7], [9].

## 2. Preliminaries

Zadeh [20] introduced the concept of a fuzzy set. Rosenfeld [16] applied this concept to the theory of groups. Let $S$ be a semigroup and $\mu\colon S \longrightarrow [0,1]$ a fuzzy set, then $\mu$ is called a fuzzy subsemigroup if it satisfies $\min\{\mu(x), \mu(y)\} \leqslant \mu(xy)$ for all $x, y \in S$. Since then many papers concerning various fuzzy algebraic structures have appeared in literature.

A triangular norm (cf. Schweizer and Sklar [17]) is a function $T\colon [0,1] \times [0,1] \longrightarrow [0,1]$ satisfying for every $x$, $y$, $z$ in $[0,1]$:

i) $T(x,y) = T(y,x)$ (commutative),

ii) $T(x,y) \leqslant T(x,z)$ if $y \leqslant z$ (monotone in the right factor),

iii) $T(x, T(y,z)) = T(T(x,y),z)$ (associative),

iv) $T(x,1) = x$ (having 1 as a right identity).

These four axioms are independent in the sense that none of them can be deduced from the other three. Obviously, the function min defined on $[0,1] \times [0,1]$ is a $t$-norm. Other $t$-norms which are frequently encountered in the study of probabilistic spaces are $T^m$ and prod defined by $T^m(a,b) = \max\{a+b-1, 0\}$, $\operatorname{prod}(a,b) = ab$ for every $a, b \in [0,1]$.

Since a triangular norm $T$ is a generalization of the minimum function, Anthony and Sherwood in [2] replaced the axiom $\min\{\mu(x), \mu(y)\} \leqslant \mu(xy)$ occuring in the definition of a fuzzy subgroup by the inequality $T(\mu(x), \mu(y)) \leqslant \mu(xy)$.

Goguen in [10] generalized the fuzzy subsets of $X$ to $L$-subsets, as functions from $X$ to a lattice $L$. From now, in this paper $L$ is a complete lattice (see [3]), i.e., there is a partial order $\leqslant$ on $L$ such that, for any $S \subseteq L$, infimum of $S$ and supremum of $S$ exist and they will be denoted by $\bigwedge\limits_{s\in S} \{s\}$ and $\bigvee\limits_{s\in S} \{s\}$, respectively. In particular, for any elements $a, b \in L$, $\inf\{a,b\}$ and $\sup\{a,b\}$ will be denoted by $a \wedge b$ and $a \vee b$, respectively. Also, $L$ is a distributive lattice with a least element 0 and a greatest element 1. If $a, b \in L$, we write $a \geqslant b$ if $b \leqslant a$, and $a > b$ if $a \geqslant b$ and $a \neq b$.

Now, we adopt the following definition of a triangular norm on a lattice. Note that, as a lattice, the real interval $[0,1]$ is a complete lattice. A binary composition $T$ on the lattice $(L, \leqslant, \vee, \wedge)$ which contains 0 and 1 is a triangular norm on $L$ if the four axioms of the above definition are satisfied for all $x, y, z \in L$. Let $S$ and $T$ be triangular norms on $L$. If $S(x,y) \leqslant T(x,y)$ for all $x, y \in L$, one writes $S \leqslant T$. The meet $\wedge$ is a triangular norm on $L$. Now, let $I_T = \{x \in L \mid T(x,x) = x\}$ which is the set of all $T$-idempotent elements of $L$. Under the partial ordering induced by the partial ordering $\leqslant$ of $L$, $I_T$ is a complete lattice with join $\vee$ and meet $T$.

We consider $L$-subsets of $X$ in the sense of Goguen [10]. Accordingly, an $L$-subset of $X$ is a mapping of $X$ into $L$. If $L$ is the unit interval $[0,1]$ of real numbers, these are the usual fuzzy subsets of $X$. For a non-empty set $X$, let $F^L(X) = \{\mu \mid \mu \text{ is an } L\text{-subset of } X\}$. Let $\mu$, $\lambda$, $\mu_i$ $(i \in I)$ be in $F^L(X)$. Then the inclusion $\mu \subseteq \lambda$, the intersection $\mu \cap \lambda$ and the union $\mu \cup \lambda$ are defined in $F^L(X)$ as follows: for all $x \in X$, $\mu \subseteq \lambda \Longleftrightarrow \mu(x) \leqslant \lambda(x)$; $(\mu \cap \lambda)(x) = \mu(x) \wedge \lambda(x)$; $(\mu \cup \lambda)(x) = \mu(x) \vee \lambda(x)$. One defines an arbitrary intersection and an arbitrary union in $F^L(X)$ as follows:

$$\Big(\bigcap_{i \in I} \mu_i\Big)(x) = \bigwedge\{\mu_i(x) \mid i \in I\}, \quad \Big(\bigcup_{i \in I} \mu_i\Big)(x) = \bigvee\{\mu_i(x) \mid i \in I\}.$$

Let $f$ be a mapping from a set $X$ into a set $Y$, and let $\mu \in F^L(X)$ and $\lambda \in F^L(Y)$. Then the fuzzy subsets $f(\mu)$ and $f^{-1}(\lambda)$ are defined by

$$f(\mu)(y) = \begin{cases} \bigvee\limits_{x \in f^{-1}(y)} \{\mu(x)\} & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise} \end{cases} \qquad \text{for all } y \in Y$$

and $f^{-1}(\lambda)(x) = \lambda(f(x))$ for all $x \in X$.

## 3. Semihypergroups and the fundamental relation

By a hypergroupoid we mean a non-empty set $H$ together with a map $\circ\colon H \times H \longrightarrow \mathscr{P}^*(H) = \mathscr{P}(H) - \emptyset$ called a hyperoperation. The image of the pair $(x,y)$ is denoted by $x \circ y$. If $A, B \subseteq H$ then we define $A \circ B = \bigcup\{a \circ b \mid a \in A, b \in B\}$. Notation, $x \circ B$ is used for $\{x\} \circ B$ and $A \circ x$ for $A \circ \{x\}$. Generally, the singleton $\{x\}$ is identified with its member $x$. A hypergroupoid $(H, \circ)$ is called a semihypergroup if $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in H$. A motivating example is the following: Let $S$ be a semigroup and $K$ any subsemigroup of $S$. Then $S/K = \{xK \mid x \in S\}$ becomes a semihypergroup where the hyperoperation is defined in the usual manner by $\bar{x} \circ \bar{y} =$

$\{\bar{z} \mid z \in \bar{x} \cdot \bar{y}\}$ where $\bar{x} = xK$. The associativity condition on a semihypergroup $(H, \circ)$ implies associativity for sets, that is, $(A \circ B) \circ C = A \circ (B \circ C)$ for all $A, B, C \subseteq H$. By a sub-semihypergroup of $H$ we mean a non-empty subset $K$ of $H$ such that $K \circ K \subseteq K$. Suppose $(H_1, \circ)$ and $(H_2, *)$ are two semihypergroups. A function $f\colon H_1 \longrightarrow H_2$ is called an inclusion homomorphism if all $x, y \in H_1$ satisfy the condition $f(x \circ y) \subseteq f(x) * f(y)$; $f$ is a strong homomorphism if $f(x \circ y) = f(x) * f(y)$. A strong homomorphism is called isomorphism if it is bijective.

The most powerful tool for obtainning a stricter structure from a given one is the quotient out procedure. To use this method in ordinary algebraic domains one needs special equivalence relations. Let $(H, \circ)$ be a semihypergroup. We define a relation $\beta^*$ as the smallest equivalence relation on $H$ such that the quotient $H/\beta^*$ is a semigroup. In this case $\beta^*$ is called the fundamental equivalence relation and $H/\beta^*$, the set of all equivalence classes, is called the fundamental semigroup. This relation was studied by Corsini [4] concerning hypergroups and by Vougiouklis concerning weak hyperstructures [18]. The product $\odot$ in $H/\beta^*$ is defined as follows: Suppose $\beta^*(a)$ is the equivalence class containing $a \in H$, then the product $\odot$ on $H/\beta^*$ is $\beta^*(a) \odot \beta^*(b) = \beta^*(c)$ for all $c \in \beta^*(a) \cdot \beta^*(b)$. Let us denote by $\mathscr{U}$ the set of all finite products of elements of $H$ and define a relation $\beta$ on $H$ as follows: $x\beta y$ iff $\{x, y\} \subseteq u$ for some $u \in \mathscr{U}$. Let us denote by $\overline{\beta}$ the transitive closure of $\beta$. Then we can rewrite the definition of $\overline{\beta}$ on $H$ as follows: $a\overline{\beta}b$ if and only if there exist $z_1, z_2, \ldots, z_{n+1} \in H$ with $z_1 = a$, $z_{n+1} = b$ and $u_1, \ldots, u_n \in \mathscr{U}$ such that $\{z_i, z_{i+1}\} \subseteq u_i$ $(i = 1, \ldots, n)$. According to [18], one can prove that the fundamental relation $\beta^*$ is the transitive closure of the relation $\beta$. If we consider the semihypergroup $H = \{x, y, z, t\}$ where $x \circ x = \{y, z\}$ and $a \circ b = \{y, t\}$ for all $(a, b) \in H \times H$ with $(a, b) \neq (x, x)$, then it is easy to see that $z\beta^* t$ but not $z\beta t$.

Suppose that $(H_1, \cdot)$ and $(H_2, *)$ are two semihypergroups. We know $(H_1 \times H_2, \circ)$ is a semihypergroup where $(x_1, y_1) \circ (x_2, y_2) = \{(x, y) \mid x \in x_1 \cdot x_2,\ y \in y_1 * y_2\}$. If $\beta_1^*$, $\beta_2^*$ and $\beta^*$ are fundamental equivalence relations on $H_1$, $H_2$ and $H_1 \times H_2$, respectively, then it is easy to see that $(x_1, y_1)\beta^*(x_2, y_2)$ if and only if $x_1\beta_1^*$, $y_1\beta_2^* y_2$ for all $(x_i, y_i) \in H_1 \times H_2$ $(i = 1, 2)$. Also we have $(H_1 \times H_2)/\beta^* \cong H_1/\beta_1^* \times H_2/\beta_2^*$.

## 4. $TL$-sub-semihypergroups

**Definition 4.1.** Let $T$ be a triangular norm on a complete lattice $(L, \leqslant, \vee, \wedge)$. An $L$-subset $\mu \in F^L(H)$ of the semihypergroup $H$ is a $TL$-sub-semihypergroup of $H$ if the following axioms hold:

1) $\mathrm{Im}(\mu) \subseteq I_T$,
2) $T(\mu(x), \mu(y)) \leqslant \bigwedge\limits_{\alpha \in x \circ y} \mu(\alpha)$ for all $x, y \in H$.

**Theorem 4.2.** *Let $T$ be a triangular norm on the complete lattice $(L, \leqslant, \vee, \wedge)$ and let $\mu$ be an $L$-subset of $H$ such that $\operatorname{Im}(\mu) \subseteq I_T$ and $b = \bigvee \operatorname{Im}(\mu)$. Then the following two statements are equivalent:*

i) *$\mu$ is a $TL$-sub-semihypergroup of $H$,*

ii) *$\mu^{-1}[a, b]$ is a sub-semihypergroup of $H$ whenever $a \in I_T$ and $0 < a \leqslant b$.*

Proof. (i) $\Longrightarrow$ (ii) Suppose $a \in I_T$ and $0 < a \leqslant b$. If $x, y \in \mu^{-1}[a, b]$, then $\bigwedge\limits_{\alpha \in x \circ y} \mu(\alpha) \geqslant T(\mu(x),\ \mu(y)) \geqslant T(a, a) = a$, which implies $x \circ y \subseteq \mu^{-1}[a, b]$, and so $\mu^{-1}[a, b]$ is a sub-semihypergroup of $H$.

(ii) $\Longrightarrow$ (i) Let $x, y \in H$. Since $\operatorname{Im}(\mu) \subseteq I_T$, both $\mu(x)$ and $\mu(y)$ are in $I_T$. We have

$$\begin{aligned} T(T(\mu(x), \mu(y)), T(\mu(x), \mu(y))) &= T(T(\mu(x), T(\mu(y), \mu(x))), \mu(y)) \\ &= T(T(\mu(x), T(\mu(x), \mu(y))), \mu(y)) \\ &= T(T(\mu(x), \mu(x)), T(\mu(y), \mu(y))) \\ &= T(\mu(x), \mu(y)), \end{aligned}$$

and so $T(\mu(x), \mu(y)) \in I_T$. Assume that $a = T(\mu(x), \mu(y))$. If $a = 0$ then $T(\mu(x), \mu(y)) = 0 \leqslant \bigwedge\limits_{\alpha \in x \circ y} \mu(\alpha)$. So, let $0 < a = T(\mu(x), \mu(y)) \leqslant \mu(x) \wedge \mu(y) \leqslant \mu(x) \leqslant b$. Hence $x, y \in \mu^{-1}[a, b]$, which implies $x \circ y \subseteq \mu^{-1}[a, b]$. Therefore $T(\mu(x), \mu(y)) \leqslant \bigwedge\limits_{\alpha \in x \circ y} \mu(\alpha)$. □

**Corollary 4.3.** *Let $A \subseteq H$. Then the characteristic function $\chi_A$ is a $TL$-sub-semihypergroup of $H$ if and only if $A$ is a sub-semihypergroup of $H$.*

**Corollary 4.4.** *Let $T$ be a triangular norm on the complete lattice $(L, \leqslant, \vee, \wedge)$, and let $\{\mu_i\}_{i \in I}$ be a family of $TL$-sub-semihypergroups of $H$. Then $\bigcap\limits_{i \in I} \mu_i$ is a $TL$-sub-semihypergroup of $H$.*

**Corollary 4.5.** *Let $f\colon H_1 \longrightarrow H_2$ be a strong homomorphism, $\mu$ any $TL$-sub-semihypergroup of $H_1$ and $\lambda$ any $TL$-sub-semihypergroup of $H_2$. Then $f(\mu)$ and $f^{-1}(\lambda)$ are $TL$-sub-semihypergroup of $H_2$ and $H_1$, respectively.*

**Definition 4.6.** Let $H_1$, $H_2$ be semihypergroups and let $\mu$, $\lambda$ be $TL$-subsemihypergroups of $H_1$, $H_2$, respectively. The product of $\mu$, $\lambda$ is defined to be the $TL$-subset $\mu \times \lambda$ of $H_1 \times H_2$ with $(\mu \times \lambda)(x, y) = T(\mu(x), \lambda(x))$ for all $(x, y) \in H_1 \times H_2$.

**Corollary 4.7.** *In the above definition, $\mu \times \lambda$ is a $TL$-subsemihypergroup of $H_1 \times H_2$.*

P r o o f. Let $(x_1, x_2), (y_1, y_2) \in H_1 \times H_2$. For every $(\alpha_1, \alpha_2) \in (x_1, x_2) \circ (y_1, y_2)$ we have

$$\begin{aligned}
(\mu \times \lambda)(\alpha_1, \alpha_2) &= T(\mu(\alpha_1), \lambda(\alpha_2)) \\
&\geqslant T(T(\mu(x_1), \mu(y_1)), T(\lambda(x_2), \lambda(y_2)) \\
&= T(T(T(\mu(x_1), \mu(y_1)), \lambda(x_2), \lambda(y_2))) \\
&= T(T(\lambda(x_2), T(\mu(x_1), \mu(y_1)), \lambda(y_2))) \\
&= T(T(T(\lambda(x_2), \mu(x_1)), \mu(y_1), \lambda(y_2))) \\
&= T(\lambda(y_2), T(\mu(y_1), T(\lambda(x_2), \mu(x_1))) \\
&= T(T(\mu(x_1), \lambda(x_2)), T(\mu(y_1), \lambda(y_2)) \\
&= T((\mu \times \lambda)(x_1, x_2), (\mu \times \lambda)(y_1, y_2)).
\end{aligned}$$

Taking the infimum in the complete lattice $(L, \leqslant, \vee, \wedge)$ over all $(\alpha_1, \alpha_2) \in (x_1, x_2) \circ (y_1, y_2)$ we get

$$\bigwedge_{(\alpha_1,\alpha_2)\in(x_1,x_2)\circ(y_1,y_2)} (\mu \times \lambda)(\alpha_1, \alpha_2) \geqslant T((\mu \times \lambda)(x_1, x_2), (\mu \times \lambda)(y_1, y_2)).$$

□

**Definition 4.8.** Let $H$ be a semihypergroup and $\mu$ an $L$-subset of $H$. The $L$-subset $\mu_{\beta^*}$ on $H/\beta^*$ is defined as follows:

$$\begin{gathered}
\mu_{\beta^*}\colon\ H/\beta^* \longrightarrow L, \\
\mu_{\beta^*}(\beta^*(x)) = \bigvee_{\alpha\in\beta^*(x)} \{\mu(a)\}.
\end{gathered}$$

**Theorem 4.9.** *Let $H$ be a semihypergroup and $\mu$ a $TL$-sub-semihypergroup of $H$. Then $\mu_{\beta^*}$ is a $TL$-subsemigroup of $H/\beta^*$.*

P r o o f. We regard $H/\beta^*$ as a semihypergroup (since every semigroup is a semihypergroup). Since the canonical map $\varphi\colon\ H \longrightarrow H/\beta^*$ is an epimorphism, the proof is completed by using Corollary 4.5. □

**Theorem 4.10.** *Let $H_1$ and $H_2$ be two semihypergroups, $\beta_1^*$, $\beta_2^*$ and $\beta^*$ fundamental equivalence relations on $H_1$, $H_2$ and $H_1 \times H_2$, respectively. Let $T$ be a triangular norm on the complete lattice $(L, \leqslant, \vee, \wedge)$. If $\mu$, $\lambda$ are $TL$-sub-semihypergroups of $H_1$, $H_2$, respectively, then $(\mu \times \lambda)_{\beta^*} = \mu_{\beta_1^*} \times \lambda_{\beta_2^*}$.*

Proof. Using Corollary 4.7, we conclude that $\mu \times \lambda$ is a $TL$-sub-semihypergroup of $H_1 \times H_2$. Then by Theorem 4.9, $(\mu \times \lambda)_{\beta^*}$ is a $TL$-subsemigroup of $(H_1 \times H_2)/\beta^*$. Now, let $x \in H_1$ and $y \in H_2$. Then

$$\begin{aligned}
(\mu \times \lambda)_{\beta^*}(\beta^*(x,y)) &= \vee \{\mu \times \lambda(a,b) \mid (a,b) \in \beta^*(x,y)\} \\
&= \vee \{T(\mu(a), \lambda(b)) \mid (a,b) \in \beta^*(x,y)\} \\
&= \vee \{T(\mu(a), \lambda(b)) \mid a \in \beta_1^*(x), b \in \beta_2^*(y)\} \\
&= T(\vee\{\mu(a) \mid a \in \beta_1^*(x)\}, \vee\{\lambda(b) \mid b \in \beta_2^*(y)\}) \\
&= T(\mu_{\beta_1^*}(\beta^*(x), \lambda_{\beta_2^*}(\beta_2^*(y)) \\
&= (\mu_{\beta_1^*} \times \lambda_{\beta_2^*}(\beta_1^*(x), \beta_2^*(y)).
\end{aligned}$$

□

## 5. Probabilistic fuzzy semihypergroups

Throughout this section, we consider $L = [0,1]$.

**Definition 5.1.** Let $\mu$ be a fuzzy set. For every $t \in [0,1]$, the set $\mu_t = \{x \in H \mid \mu(x) \geqslant t\}$ is called the level subset of $\mu$.

**Corollary 5.2.** *Let $H$ be a semihypergroup and $\mu$ a fuzzy subset of $H$. Then $\mu$ is a fuzzy sub-semihypergroup of $H$ if and only if for any $t \in [0,1]$, $\mu_t$ $(\neq \emptyset)$ is a sub-semihypergroup of $H$.*

In the theory of probability we start with $(\Omega, \mathscr{A}, P)$, where $\Omega$ is the set of elementary events and $\mathscr{A}$ a $\sigma$-algebra of subsets of $\Omega$ called events. A probability on $\mathscr{A}$ is defined as a countable additive and nonnegative function $P$ such that $P(\Omega) = 1$.

**Definition 5.3.** Let $H$ be a semihypergroup, let $(\Omega, \mathscr{A}, P)$ be a probability space, and let $R\colon \Omega \longrightarrow \mathscr{P}(H)$ be a random set. If for any $\omega \in \Omega$, $R(\omega)$ is a sub-semihypergroup of $H$, then the falling shadow $S$ of the random set $R$, i.e., $S(x) = P(\omega \mid x \in R(\omega))$, is called a $\pi$-fuzzy sub-semihypergroup of $H$.

**Proposition 5.4.** *Let $S$ be a $\pi$-fuzzy sub-semihypergroup of semihypergroup $(H,\circ)$. Then $\inf\{S(z) \mid z \in x \circ y)\} \geqslant T^m(S(x), S(y))$ for all $x, y \in H$.*

Proof. We know $R(\omega)$ is a sub-semihypergroup. Now, let $x \in R(\omega)$ and $y \in R(\omega)$, then $x \circ y \subseteq R(\omega)$. So for every $z \in x \circ y$ we have $\{\omega \mid z \in R(\omega)\} \supseteq \{\omega \mid x \in R(\omega)\} \cap \{\omega \mid y \in R(\omega)\}$. Then

$$\begin{aligned} S(z) &= P(\omega \mid z \in R(\omega)) \\ &\geqslant P(\{\omega \mid x \in R(\omega)\} \cap \{\omega \mid y \in R(\omega)\}) \\ &\geqslant P(\omega \mid x \in R(\omega)) + P(\omega \mid y \in R(\omega)) - P(\omega \mid x \in R(\omega) \text{ or } y \in R(\omega)) \\ &\geqslant S(x) + S(y) - 1. \end{aligned}$$

Hence $\inf\{S(z) \mid z \in g(x,y)\} \geqslant T^m(S(x), S(y))$. □

**Theorem 5.5.** i) *Let $\mathscr{H}$ denote the set of all sub-semihypergroups of $H$. For each $x \in H$, write $H_x = \{A \mid A \in \mathscr{H},\ x \in A\}$. Let $(\mathscr{H}, \sigma)$ be a measurable space where $\sigma$ is a $\sigma$-algebra that contains $\{H_x \mid x \in H\}$, and $P$ a probability measure on $(\mathscr{H}, \sigma)$. We define $\mu\colon H \longrightarrow [0,1]$ as follows: $\mu(x) = P(H_x)$ for $x \in H$. Then $\mu$ is a $T^m$-fuzzy sub-semihypergroup of $H$.*

ii) *Suppose that there exists $\mathscr{A} \in \sigma$ such that $\mathscr{A}$ is a chain with respect to the set inclusion and $P(\mathscr{A}) = 1$. Then $\mu$ is a fuzzy sub-semihypergroup of $H$.*

Proof. i) Suppose $x, y \in H$, then $H_z \supseteq H_x \cup H_y$ for all $z \in x \circ y$, and so

$$\mu(z) = P(H_z) \geqslant P(H_x \cap H_y) \geqslant \max\{P(H_x) + P(H_y) - 1, 0\} = T^m(\mu(x), \mu(y)).$$

Therefore $\inf\{\mu(z) \mid z \in x \circ y\} \geqslant T^m(\mu(x), \mu(y))$.

ii) Since $P$ is a probability measure and $P(\mathscr{A}) = 1$ we have $P(H_x \cap \mathscr{A}) = P(H_x)$ for all $x \in H$. Therefore for every $z \in x \circ y$ we have

$$\mu(z) = P(H_z) \geqslant P(H_x \cap H_y) = P((H_x \cap \mathscr{A}) \cap (H_y \cap \mathscr{A})).$$

Since $\mathscr{A}$ with the set inclusion forms a chain, it follows that either $H_x \cap \mathscr{A} \subseteq H_y \cap \mathscr{A}$ or $H_y \cap \mathscr{A} \subseteq H_x \cap \mathscr{A}$. Therefore

$$\mu(z) \geqslant \min\{P(H_x \cap \mathscr{A}), P(H_y \cap \mathscr{A})\} = \min\{\mu(x), \mu(y)\}$$

and so $\inf\{\mu(z) \mid z \in x \circ y)\} \geqslant \min\{\mu(x), \mu(y)\}$. □

**Theorem 5.6.** *Let $H$ be a semihypergroup and $\mu$ a fuzzy sub-semihypergroup of $H$. Then there exists a probability space $(\Omega, \mathscr{A}, P)$ such that for some $A \in \mathscr{A}$, $\mu(x) = P(A)$.*

Proof. Suppose $\Omega = \mathscr{H}$, the set of all sub-semihypergroups of $H$. We know $([0,1]), \sigma, m)$ is a probability space, where $\sigma$ is the $\sigma$-algebra consisting of all Borel subsets of $[0,1]$ and $m$ is the usual Lebesgue measure on the measurable space $([0,1], \sigma)$. Suppose $R\colon\ [0,1] \longrightarrow \mathscr{H}$ given by $t \longmapsto \mu_t$ ($\mu_t$ is a level sub-semihypergroup of $H$, Definition 5.1) is a random set. Let $\mathscr{A} = \{A \mid A \in \mathscr{H},\ R^{-1}(A) \in \sigma\}$ and $P = m \circ R^{-1}$. It is easy to see that $(\mathscr{H}, \mathscr{A}, P)$ is a probability space. If we put $H_x = \{A \mid A \in \mathscr{H},\ x \in A\}$ then for $x \in H$ we have $\mu_t \in H_x$ for all $t \in [0, \mu(x)]$ and $\mu_s \notin H_x$ for all $s \in (\mu(x), 1]$. So $R^{-1}(H_x) = [0, \mu(x)]$ and hence $H_x \in \mathscr{A}$. Now we get $P(H_x) = m \circ R^{-1}(H_x) = m([0, \mu(x)]) = \mu(x)$. □

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