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ON $k$-SPACES AND $k_R$-SPACES

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Abstract. In this note we study the relation between $k_R$-spaces and $k$-spaces and prove that a $k_R$-space with a $\sigma$-hereditarily closure-preserving $k$-network consisting of compact subsets is a $k$-space, and that a $k_R$-space with a point-countable $k$-network consisting of compact subsets need not be a $k$-space.

Keywords: $k_R$-spaces, $k$-spaces, $k$-networks, $\sigma$-hereditarily closure-preserving collections, point-countable collections

MSC 2000: 54D50, 54C30

1. Introduction

Suppose $X$ is a topological space and $\mathcal{P}$ is a collection of subsets of $X$. A space $X$ is determined by $\mathcal{P}$ if $U \subset X$ is open (closed) in $X$ if and only if $U \cap P$ is open (closed) in $P$ for every $P \in \mathcal{P}$. A space $X$ is a $k$-space, if it is determined by the cover consisting of all compact subsets of $X$. A space $X$ is called a $k_R$-space, if $X$ is completely regular and the necessary and sufficient condition for a real valued function $f$ on $X$ to be continuous is that the restriction of $f$ on each compact subset is continuous. Obviously, every completely regular $k$-space is a $k_R$-space. The converse is false, as was first shown by an example of Katětov which appeared in a paper by V. Pták (see [1]). What additional properties of a $k_R$-space ensure it is a $k$-space has attracted considerable attention, and some noticeable results have been obtained in [2]–[5]. Such conditions were formulated using the notion of a $k$-network. $\mathcal{P}$ is a $k$-network for $X$ if whenever $K \subset U$ with $K$ compact and $U$ open in $X$, then $K \subset \bigcup \mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$ (see [6]). Suppose $\mathcal{P}$ is a $k$-network for $X$, then $\mathcal{P}$ is a compact $k$-network if $P$ is compact in $X$ for every $P \in \mathcal{P}$. A family $\mathcal{P}$

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of subsets in $X$ is called locally countable (locally finite), if for each point $x$ there is a neighborhood of $x$ which meets at most countably many (finite many) members of $\mathcal{P}$. A family $\mathcal{P}$ of subsets in $X$ is called star countable, if each member of $\mathcal{P}$ meets at most countably many other members of $\mathcal{P}$. A family $\mathcal{P}$ of subsets in $X$ is called point-countable, if each single point meets at most countably many members of $\mathcal{P}$. A family $\{A_\alpha : \alpha \in I\}$ of subsets of a space $X$ is said to be hereditarily closure-preserving (briefly, HCP) if $\bigcup_{\alpha \in J} \overline{B_\alpha} = \bigcup_{\alpha \in J} B_\alpha$ whenever $J \subset I$ and $B_\alpha \subset A_\alpha$ for each $\alpha \in J$. A collection $\mathcal{P}$ in $X$ is $\sigma$-locally countable (locally finite, HCP) if it is a collection that is the union of countably many locally countable (locally finite, HCP) families. Let $\mathcal{P}$ be a $k$-network consisting of compact subsets in a regular space $X$. Then $\mathcal{P}$ is locally countable $\Rightarrow \mathcal{P}$ is $\sigma$-locally countable $\Rightarrow \mathcal{P}$ is star countable $\Rightarrow \mathcal{P}$ is point-countable. But the inverse implications are not true.

In 1973, Michael constructed an example of a $kR$-space which is not a $k$-space, but has a countable $k$-network (see [2]). In 1991, S. Lin showed that a $kR$-space with a star countable compact $k$-network is a $k$-space (see [3]), which answered affirmatively a question posed in [4]. In 2000, Z. Yun proved in [5] that the following statements are equivalent for a $kR$-space with a $k$-network $\mathcal{P}$ of compact subsets, and each of them implies that $X$ is a $k$-space:

(a) $\mathcal{P}$ is star countable.
(b) $\mathcal{P}$ is locally countable.
(c) $\mathcal{P}$ is $\sigma$-locally countable.

Therefore, the following question is raised naturally:

(1) If a $kR$-space $X$ has a point-countable compact $k$-network, then is $X$ a $k$-space?

It is known that locally finite families are HCP. Hence $\sigma$-locally finite families are $\sigma$-HCP. Further, $\sigma$-locally finite families of compact sets are easily seen to be star countable. Thus $\sigma$-HCP is a generalization of $\sigma$-locally finite in another direction than star countable.

Therefore, the following question seems to be of some interest.

(2) If a $kR$-space $X$ has a $\sigma$-HCP compact $k$-network, then is $X$ a $k$-space?

In this paper, we show that question 1 has negative answer by the example below, and question 2 has affirmative answer. In fact, a stronger result is proved—with a $k$-cover instead of a $k$-network. A family $\mathcal{P}$ of subsets in $X$ is a $k$-cover if for any compact subset $K$, $K \subset \bigcup \mathcal{P}'$ for some finite $\mathcal{P}' \subset \mathcal{P}$ (see [7]).

In this paper, all spaces are Hausdorff spaces, and $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{Q}$ denote the set of natural numbers, real numbers and rational numbers, respectively.
2. Results

The following Lemma 1 is easy to show.

**Lemma 1.** Let $X$ be a topological space, $\mathcal{P}$ an HCP-cover of $X$ by closed sets.

1. If $P$ is a $k$-space for each $P \in \mathcal{P}$, then so is $X$.
2. If $P$ is normal for each $P \in \mathcal{P}$, then so is $X$.

**Lemma 2.** Suppose that $X$ is a $k_R$-space and $X = \bigcup \mathcal{P}$, where $\mathcal{P} = \{X_n : n \in \mathbb{N}\}$ and $X_n$ is a closed normal $k$-space. If $\mathcal{P}$ is a $k$-cover for $X$, then $X$ is a $k$-space.

**Proof.** First we shall show that $X$ is determined by $\mathcal{P}$. Suppose not. There is a set $A$ which is not closed in $X$ such that for any $n \in \mathbb{N}$, $A \cap X_n$ is closed in $X$. Taking $a \in A \setminus A$, we have $a \in X_m$ for some $m \in \mathbb{N}$. For $i \in \mathbb{N}$, let $Y_i = \bigcup\{X_n : n \leq m + i - 1\}$, then $a \in Y_1 \subset Y_1 \subset Y_1$. $Y_1$ is a normal $k$-subspace by Lemma 1, and $A \cap Y_i$ is closed in $X$. We can assume that $A \cap Y_1 \neq \emptyset$. Since $a \notin A \cap Y_1$, there is a continuous function $f_1$ on $Y_1$ such that $f_1(a) = 1$, and $f_1(A \cap Y_1) = \{0\}$. We define $g_1 : A \cap Y_2 \to \mathbb{R}$ such that $g_1(A \cap Y_2) = \{0\}$. Since $Y_1$ and $A \cap Y_2$ are closed in $X$, $f_1$ is continuous on $Y_1$, $g_1$ is continuous on $A \cap Y_2$ and $f_1 = g_1$ on $Y_1 \cap (A \cap Y_2) = A \cap Y_1$, we can define a real valued function $h_1 : Y_1 \cup (A \cap Y_2) \to \mathbb{R}$ such that $h_1(x) = f_1(x)$ if $x \in Y_1$; $h_1(x) = g_1(x)$ if $x \in A \cap Y_2$. So $h_1$ is continuous on $Y_1 \cup (A \cap Y_2)$. Since $Y_2$ is a normal space and $Y_1 \cup (A \cap Y_2)$ is closed in $Y_2$, $h_1$ can be expanded continuously to $Y_2$, that is, we can define $f_2 : Y_2 \to \mathbb{R}$ such that $f_2$ is continuous on $Y_2$ with the restriction of $f_2$ on $Y_1$ being $f_1$, i.e. $f_2|Y_1 = f_1$, and $f_2(A \cap Y_2) = \{0\}$. By induction, we can define a sequence of real valued continuous functions $f_n : Y_n \to \mathbb{R}$ such that $f_n(A \cap Y_n) = \{0\}$ and $f_n|Y_{n-1} = f_{n-1}$. Define $f : X \to \mathbb{R}$ by $f|Y_n = f_n$, then $f(A) = \{0\}$ and $f(a) = 1$. From the fact $a \in A$ we know that $f(A) \subset f(a)$, and hence $f$ is not continuous on $X$. On the other hand, for any compact subset $K \subset X$ there exists $n \in \mathbb{N}$ such that $K \subset Y_n$. $f$ is continuous on $K$ because $f$ is continuous on $Y_n$. Since $X$ is a $k_R$-space, $f$ is continuous on $X$. This is a contradiction. Hence $X$ is determined by $\mathcal{P}$. Next, let $F \subset X$ be such that $F \cap K$ is closed in $K$ for each compact set $K \subset X$. As each $X_n$ is a $k$-space, $(F \cap X_n) \cap K = (F \cap K) \cap X_n = F \cap K$ is closed in $K$ for each compact set $K \subset X_n$, so $F \cap X_n$ is closed in $X_n$ for each $n \in \mathbb{N}$. Since $X$ is determined by $\mathcal{P}$, $F$ is closed in $X$. Hence $X$ is a $k$-space. \hfill \Box

**Theorem 3.** A $k_R$-space with a $\sigma$-HCP $k$-cover consisting of compact subsets is a $k$-space.

**Proof.** Suppose $X$ is a $k_R$-space and has a $\sigma$-HCP $k$-cover consisting of compact subsets. Let $\mathcal{P} = \bigcup \{\mathcal{P}_n : n \in \mathbb{N}\}$ be a $\sigma$-HCP $k$-cover, where each $\mathcal{P}_n$ is a
HCP collection consisting of compact subsets. For any \( n \in \mathbb{N} \), put \( X_n = \bigcup \mathcal{P}_n \). Clearly each \( X_n \) is closed in \( X \). By Lemma 1, each \( X_n \) is a normal \( k \)-space. By Lemma 2, \( X \) is a \( k \)-space.

**Corollary 4.** A \( k_{R} \)-space with a \( \sigma \)-HCP compact \( k \)-network is a \( k \)-space.

As for point-countable compact \( k \)-networks, we have

**Example 5.** There exists a \( k_{R} \)-space \( X \) with a point-countable compact \( k \)-network, such that \( X \) is not a \( k \)-space.

Let \( X \) be the plane and \( \tau_0 \) its usual topology. Let \( A \subset X \) be the \( x \)-axis. For each \( x \in A \), let \( U(x) \) be the vertical line through \( x \); also let \( \mathcal{V}(x) \) be the collection of all \( V \subset X \) of the form \( V = B(x, \delta) - H(x) \), where \( B(x, \delta) \) is an open disc centered at \( x \) with radius \( \delta \) and \( H(x) \) is a \( \tau_0 \)-closed subspace of \( X - \{x\} \) which is disjoint from \( U(x) \). Let \( \land \) be the topology on \( X \) with the following open neighborhood system: an open neighborhood of a point \( p \in X - A \) is an open disc centered at \( p \); an open neighborhood of a point \( q \in A \) is a set which results from picking a \( V(x) \in \mathcal{V}(x) \), for each \( q_1 - \varepsilon < x_1 < q_1 + \varepsilon \), and forming the union of these \( V(x) \); it will be denoted by \( B(q, \varepsilon, \{V(x)\}) \). In [8], R. Borges proved that \((X, \land)\) is homeomorphic to the space \((X, \tau)\) of Example 1.1 in [2]. Recall that \( \tau \) is the coarsest topology on \( X \) which makes every function \( f : X \to \mathbb{R} \) (the real line) \( \tau_0 \)-continuous on \( X - A \) and \( \tau_0 \)-separately continuous at each \( x \in A \), (i.e., for each \( x \in A \), \( f|U(x) \) and \( f|x \)-axis are continuous). In [2], Michael showed that \((X, \tau)\) is a \( \sigma \)-space and a cosmic \( k_{R} \)-space which is not a \( k \)-space. By the construction of the topological space \((X, \tau)\), the subspaces \( A \) and \( X \setminus A \) of \( X \) have their usual topology, and so they have a countable \( k \)-network consisting of compact subsets in \( A \) and \( X \setminus A \), which are denoted by \( \alpha \), \( \beta \), respectively. For every \( x = (x_1, 0) \in A \) and every \( p, q \in \mathbb{Q} \), we denote \( F(x, p, q) = \{(x_1, y_2) \in X : p \leq y_2 \leq q\} \). Since the space \( \{x_1\} \times \mathbb{R} \) has its usual topology, \( F(x, p, q) \) is compact in \( X \). Let \( \mathcal{P} = \alpha \cup \beta \cup \{F(x, p, q) : x = (x_1, 0) \in A, p, q \in \mathbb{Q}\} \).

Clearly \( \mathcal{P} \) is a point-countable cover consisting of compact subsets in \( X \). We shall show \( \mathcal{P} \) is a \( k \)-network for \( X \). Assume that \( C \) and \( U \) are respectively compact and open in \( X \) and such that \( C \subset U \). Since \( C \cap A \subset U \cap A \) and \( C \cap A \) is compact and \( U \cap A \) open in \( A \), there exists a finite \( \alpha' \subset \alpha \) such that \( C \cap A \subset \bigcup \alpha' \subset U \cap A \). By Lemma 3.4 in [2], a compact subset of \( X \) has the following property:

If \( C \) is compact in \( X \), then there are \( \varepsilon > 0 \) and a finite \( A' \subset A \) such that for anything \( y = (y_1, y_2) \in C \) and \( 0 < |y_2| < \varepsilon \), there is \( x = (x_1, x_2) \in A' \) with \( y_1 = x_1 \).

Take \( m \in \mathbb{N} \) with \( 1/m < \varepsilon \). Let \( L = \{(x_1, x_2) \in X : x_1 \in \mathbb{R} \text{ and } |x_2| \leq 1/m\} \).

Then \( L \) is closed in \( X \), \( C \setminus \text{int}(L) \subset U \setminus A \) with \( C \setminus \text{int}(L) \) compact in \( X \setminus A \) and \( U \setminus A \) open in \( X \setminus A \), thus \( C \setminus \text{int}(L) \subset \bigcup \beta' \subset U \setminus A \) for some finite \( \beta' \subset \beta \).

For every \( x = (x_1, 0) \in A' \), since \( F(x_1, -1/m, 1/m) \cap C \subset (\{x_1\} \times \mathbb{R}) \cap U \) and
F(x_1, -/m, 1/m) \cap C are compact and \((\{x_1\} \times \mathbb{R}) \cap U\) is open in \(\{x_1\} \times \mathbb{R}\), and 
\(\{F(x, p, q): p, q \in \mathbb{Q}\}\) is a \(k\)-network for \(\{x_1\} \times \mathbb{R}\), there is a finite \(\gamma_x \subset \{F(x, p, q): p, q \in \mathbb{Q}\}\) such that \(F(x, -1/m, 1/m) \cap C \subset \bigcup x \gamma_x \subset (\{x_1\} \times \mathbb{R}) \cap U\). Clearly \(C \subset \bigcup (\alpha' \cup \beta' \cup \{\gamma_x: x \in A'\}) \subset U\), and \(\alpha' \cup \beta' \cup \{\gamma_x: x \in A'\}\) is a finite subfamily of \(P\). Thus \(P\) is a \(k\)-network for \(X\).

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References


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