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THE QUASI-CANONICAL SOLUTION OPERATOR TO $\overline{\partial}$ RESTRICTED TO THE FOCK-SPACE

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Abstract. We consider the solution operator $S\colon \mathscr{F}_{\mu,(p,q)} \to L^2(\mu)_{(p,q)}$ to the $\overline{\partial}$-operator restricted to forms with coefficients in $\mathscr{F}_\mu = \bigl\{f\colon f$ is entire and $\int_{\mathbb{C}^n} |f(z)|^2 \,\mathrm{d}\mu(z) < \infty\bigr\}$. Here $\mathscr{F}_{\mu,(p,q)}$ denotes (p,q)-forms with coefficients in $\mathscr{F}_\mu$, $L^2(\mu)$ is the corresponding L^2-space and μ is a suitable rotation-invariant absolutely continuous finite measure. We will develop a general solution formula S to $\overline{\partial}$. This solution operator will have the property $Sv \perp \mathscr{F}_{(p,q)} \,\forall\, v \in \mathscr{F}_{(p,q+1)}$. As an application of the solution formula we will be able to characterize compactness of the solution operator in terms of compactness of commutators of Toeplitz-operators $[T_{\overline{z_i}}, T_{z_i}] = [T^*_{z_i}, T_{z_i}]\colon \mathscr{F}_\mu \to L^2(\mu)$.

Keywords: Fock-space, Hankel-operator, reproducing kernel

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1. Preliminaries

In many cases non-compactness of the solution operator already happens when the solution operator is restricted to the corresponding subspace of holomorphic functions. (See [11], [12], [18], [15] and [19].)

It is pointed out in [19] that compactness of the solution operator for $\overline{\partial}$ on $(0,1)$-forms implies that the boundary of Ω—in this case Ω is a bounded convex domain—does not contain any analytic variety of dimension greater or equal to 1. The proof uses the fact that there is a compact solution operator to $\overline{\partial}$ on $(0,1)$-forms with holomorphic coefficients. In this case compactness of the solution operator restricted to $(0,1)$-forms with holomorphic coefficients implies already compactness of the solution operator on general $(0,1)$-forms.

A similar situation appears in [15] where the Toeplitz C^*-algebra $\mathscr{T}(\Omega)$ is considered and the relation between the structure of $\mathscr{T}(\Omega)$ and the $\bar{\partial}$-Neumann problem is discussed.

Our work is motivated by the fact that in some cases the solution operator can be interpreted as a Hankel-operator. See for example [4], [5], [6], [7], [8], [9], [10], [11] and [12].

In [11] the canonical solution operator restricted to spaces of entire function is investigated. For $(0,0)$-forms it is shown that the canonical solution operator $S\colon \mathscr{F}_m = \big\{f\colon f \text{ is entire and } \int_{\mathbb{C}^1} |f(z)|^2 \mathrm{e}^{-|z|^m}\,\mathrm{d}\lambda(z) < \infty\big\} \to \big\{f\colon f \text{ is measurable and } \int_{\mathbb{C}^1} |f(z)|^2 \mathrm{e}^{-|z|^m}\,\mathrm{d}\lambda(z) < \infty\big\}$ is compact for $m > 2$ and that it is not compact for $m = 2$. Here λ denotes the Lebesgue-measure. In [17] it is shown that the canonical solution operator $S\colon \mathscr{F}_{m,(0,1)} \to L^2_m$ is not compact for all m. Here m corresponds to the measure μ with $\mathrm{d}\mu/\mathrm{d}\lambda = \mathrm{e}^{-|z|^m} := \mathrm{e}^{-(|z_1|^m+\ldots+|z_n|^m)}$. In both cases the solution operator has a quite simple form. In this paper we develop a solution operator for (p,q)-forms. In this general case the solution operator is much more complicated. We will characterize compactness of the solution operator as an application and will therefore be able to generalize the results from [11] and [17].

The question of compactness of the solution operator is of interest for various reasons; see [2] for an excellent survey.

Let μ be a suitable rotation-invariant absolutely continuous finite measure with density g_μ such that $0 < C < g_\mu(z)^{-1} < \infty$ for all z in arbitrary compact sets. Furthermore let the monomials be an orthogonal system. Recall that we understand under the generalized Fock-space $\mathscr{F}_\mu$ the space of holomorphic functions that are square integrable with respect to the measure μ. That is

$$\mathscr{F}_\mu = \left\{f\colon f \text{ is entire and } \int_{\mathbb{C}^n} |f(z)|^2\,\mathrm{d}\mu(z) < \infty\right\}.$$

We will abbreviate $\mathscr{F}_\mu = \mathscr{F}$. Let us consider the following notations:

$$c_k^2 = \int_{\mathbb{C}^n} |z|^{2k}\,\mathrm{d}\mu(z).$$

Here $k = (k_1,\ldots,k_n)$ is a multi-index. We will call c_k^2 moments. The reproducing kernel is given by

$$K(z,w) = \sum_{k=1}^{\infty} \varphi_k(z)\overline{\varphi_k(w)},$$

where $\{\varphi_k\}_{k=1}^{\infty}$ is a complete orthonormal system of $\mathscr{F}$. It is known [1], that

$$K(z,\omega) = \overline{K}(\omega,z),$$
$$f(z) = \int_{\mathbb{C}^n} K(z,\omega)f(\omega)\,\mathrm{d}\mu(\omega) \quad \forall\, f \in \mathscr{F}.$$

It follows from our assumptions about the measure μ that $\{z^m/c_m\colon m \in \mathbb{N}^n\}$ constitutes an orthonormal system of $\mathscr{F}$. Inserting this special orthonormal system one can see, that

$$K(z,w) = \sum \frac{z^m}{c_m}\,\frac{\overline{w}^m}{c_m}.$$

Let $L^2_{(p,q)}(\mu)$ be the space of (p,q)-forms with coefficients in

$$L^2(\mu) = \left\{ f\colon\ f \text{ is measurable and } \int_{\mathbb{C}^n} |f(z)|^2\,\mathrm{d}\mu(z) < \infty \right\}.$$

That is

$$L^2_{(p,q)}(\mu) = \left\{ {\sum_{I,J}}' f_{I,J}\,\mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J;\ f_{I,J} \in L^2(\mu) \right\}.$$

Here, the prime denotes summation over strictly increasing q-tuples J and p-tuples I. Furthermore

$$\mathrm{d}\overline{z}_J = \mathrm{d}\overline{z}_{j_1} \wedge \ldots \wedge \mathrm{d}\overline{z}_{j_q}.$$

The norm is

$$\left\| {\sum_{I,J}}' f_{I,J}\,\mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J \right\|^2 = {\sum_{I,J}}' \int_{\mathbb{C}^n} |f_{I,J}|^2\,\mathrm{d}\mu.$$

Recall that the $\overline{\partial}$-operator acts via

$$\overline{\partial}\left({\sum_{I,J}}' f_{I,J}\,\mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J \right) = {\sum_{j=1}^{n}}' \sum_J \frac{\partial f_{I,J}}{\partial \overline{z}_j}\,\mathrm{d}\overline{z}_j \wedge \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J.$$

The derivatives are taken in the distribution sense, and the domain of $\overline{\partial}$ consists of those $(0,q)$-forms where the right-hand side is in $L^2_{(0,q+1)}(\mu)$. For a global survey of the $\overline{\partial}$-operator on the Bergman space see [2], [3].

2. The quasi-canonical solution operator

Several solution formulas to $\overline{\partial}$ are known at the moment. See for instance [14] for a solution formula restricted to $(0,q)$-forms with Bergman-space coefficients on a bounded pseudoconvex domain. Furthermore solution formulas can be found as well in [12]. There the special case of $(n,n-1)$-forms is considered and n is the number of different variables on which the coefficients depend and only the restriction to Bergman-space coefficients is considered. The aim of this paper is to develop a solution formula to $\overline{\partial}$ restricted to (p,q)-forms with generalized Fock-space coefficients.

It will have the property

$$(*) \qquad Sv \perp \mathscr{F}_{(p,q)}$$

and

$$\bar\partial Sv = v \quad \forall\, v \in \mathscr{F}_{(p,q+1)}.$$

Here S denotes our solution formula and the second condition just means that S is a solution formula to $\bar\partial$ restricted to (p,q)-forms with Fock-space coefficients. Condition $(*)$ is quite similar to the condition that the operator is the canonical solution operator—but it is not quite the same.

We will need some facts about Hankel operators with unbounded symbols.

Definition 1. A linear operator

$$S = S_q\colon\ L^2(\mu)_{(p,q)} \cap \operatorname{Ker}(\bar\partial_q) \to L^2(\mu)_{(p,q-1)}$$

is called a solution operator to $\bar\partial_{q-1}$ if

$$\bar\partial_{q-1} S_q f = f,\ \ \forall\, f \in L^2(\mu)_{(p,q)} \cap \ker(\bar\partial_q).$$

If we have

$$S_q f \perp \mathscr{F}_{(p,q-1)}$$

we call S_q a quasi-canonical solution operator.

Definition 2. Let $0 < \varrho < 1$ and $k \in \mathbb{N}$. Then we define

$$f_\varrho(z) := f(\varrho z)$$

and

$$\tilde f_{l,\varrho}(z) = \bar z_1^l f_\varrho(z).$$

The following Propositions are the more-dimensional analogue of [16].

Proposition 1. *Let $f \in \mathscr{F}$. Furthermore assume that $\{c_{i+l_1}^2 c_i^{-2} \varrho^{2i}\}$ is bounded and $0 < \varrho < 1$. Then we have*

$$\tilde f_{l,\varrho}(z) = \bar z_1^l f_\varrho(z) \in L^2(\mathbb{C}^n, \mu).$$

Remark 1. Generally multiplication operators with unbounded symbols are not globally defined.

Proposition 2. *Let $n \in \mathbb{N}$. Furthermore assume that the sequences $c_{k+l_1}^2 c_k^{-2} - c_k^2 c_{k-l_1}^{-2}$ and $c_{i+l}^2 c_i^{-2} \varrho^{2i}$ $\forall\, 0 < \varrho < 1$ are bounded. Then the Hankel-operator with symbol $\bar{z}_1^l$*

$$H_{\bar{z}_1^l}\colon \mathscr{F} \to \mathscr{F}^{\perp}$$

is bounded.

Proof. We only carry out the proof for $l = 1$. Let $k = (k_1, \ldots, k_n)$ a multi-index. Then $k + 1_1 = (k_1 + 1, k_2, \ldots, k_n)$ and $\sum\limits_{k=0}^{\infty}$ means summation over all positive multi-indices. $\sum\limits_{k=1}^{\infty}$ means summation over all positive multi-indices except those with $k_1 = 0$. For $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k$ it follows by the methods from [16] that

$$\begin{aligned}\int_{\mathbb{C}^n} &|\tilde{f}_\varrho(z) - P(\tilde{f}_\varrho)(z)|^2 \,\mathrm{d}\mu(z)\\ &= |a_0|^2 c_{k+1_1}^2 \varrho^{2k} + \sum_{k=1}^{\infty} |a_k|^2 c_k^2 \varrho^{2k} \left(\frac{c_{k+1_1}^2}{c_k^2} - \frac{c_k^2}{c_{k-1_1}^2}\right).\end{aligned}$$

It follows from the boundedness of the sequence $c_{k+1_1}^2 c_k^{-2} - c_k^2 c_{k-1_1}^{-2}$ that

$$\begin{aligned}\int_{\mathbb{C}^n} \lim_{\varrho \to 1} |\tilde{f}_\varrho(z) - P(\tilde{f}_\varrho)(z)|^2 \,\mathrm{d}\mu(z) &\leqslant \sup_{0<\varrho<1} \int_{\mathbb{C}^n} |\tilde{f}_\varrho(z) - P(\tilde{f}_\varrho)(z)|^2 \,\mathrm{d}\mu(z)\\ &\leqslant C \int_{\mathbb{C}^n} |f(z)|^2 \,\mathrm{d}\mu(z).\end{aligned}$$

Hence

$$\int_{\mathbb{C}^n} |H_{\bar{z}_1} f(z)|^2 \,\mathrm{d}\mu(z) \leqslant C \int_{\mathbb{C}^n} |f(z)|^2 \,\mathrm{d}\mu(z)$$

and therefore the Hankel-operator is bounded. □

Remark 2. The above result is still valid if one replaces the multiplication with $\bar{z}_1$ by the one with $\bar{z}_i$.

Remark 3. It can be shown that in our case the Hankel-operator $H_{\bar{z}_1}$ has the form

$$f \mapsto F$$

where

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

and

$$F(z) = \lim_{N\to\infty} \left(\overline{z}_1 \sum_{k=0}^{N} a_k z^k - \sum_{k=1}^{N} a_k \frac{c_k^2}{c_{k-1_1}^2} z^{k-1_1} \right).$$

Remark 4. Let us define $H_{\overline{z}_i}\colon \mathscr{F}_{(p,q)} \to L^2(\mu)_{(p,q)}$ by

$$H_{\overline{z}_i} \sum_{|I|=p,|K|=q}{}' f_{I,K}\,\mathrm{d}\overline{z_K} \wedge \mathrm{d}z_I = \sum_{|I|=p,|K|=q}{}' H_{\overline{z}_i} f_{I,K}\,\mathrm{d}\overline{z_K} \wedge \mathrm{d}z_I.$$

It is clear from the proof above that even $H_{\overline{z}_i}\colon \mathscr{F}_{(p,q)} \to L^2(\mu)_{(p,q)}$ is bounded if the assumptions about the moments are fulfilled.

Remark 5. The assumptions about the moments are fulfilled in the case where $\mathrm{d}\mu/\mathrm{d}\lambda = \mathrm{e}^{-|z|^m} := \mathrm{e}^{-(|z_1|^m+\ldots+|z_n|^m)}$. Here λ is the Lebesgue-measure and $m \in \mathbb{N}$.

Theorem 1. *Let* $f = \sum_{I,J}{}' f_{I,J}\,\mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J \in \mathscr{F}_{(p,q+1)}$. *Then*

$$Sf = (-1)^p \frac{1}{q} \sum_{I,|J^\sim|=q-1}{}' \sum_{J\sim J^\sim}{}' \operatorname{sgn}(J \sim J^\sim) \times \left(\int_{\mathbb{C}^n} K(z,\omega)(\overline{z+\omega})_{|J\sim J^\sim|} f_{IJ})\,\mathrm{d}\mu(\omega) \right) \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_{J^\sim}$$

is a quasi-canonical solution operator to $\overline{\partial}$. *Here* $J^\sim = (j_1^\sim,\ldots,j_{q-1}^\sim) \sim J = (j_1,\ldots,j_q)$ *if* $\{j_1^\sim,\ldots,j_{q-1}^\sim\} \cup \{i\} = \{j_1,\ldots,j_q\}$ *for some* $i \in \{1,\ldots,n\}$. *Furthermore* $|J^\sim \sim J| = j_r$ *if* $(j_1^\sim,\ldots,j_{r-1}^\sim, i, j_{r+1}^\sim,\ldots,j_{q-1}^\sim) = (j_1,\ldots,j_q)$. *In this case we define* $\operatorname{sgn}(J^\sim \sim J)$ *as* $(-1)^{r-1}$. *We will use the following definitions:*

$$g_{I,J^\sim} = \sum_{J\sim J^\sim}{}' (-1)^p \operatorname{sgn}(J \sim J^\sim)(\overline{z}_{|J\sim J^\sim|} f_{IJ})$$

and

$$\widetilde{g}_{I,J^\sim} = \sum_{J\sim J^\sim}{}' (-1)^p \operatorname{sgn}(J \sim J^\sim)(H_{\overline{z}_{|J\sim J^\sim|}} f_{IJ}).$$

Then S *can be written as*

$$Sf = \frac{1}{q} \sum_{I,|J^\sim|=q-1}{}' g_{I,J^\sim} \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_{J^\sim}.$$

P r o o f. Making the ansatz

$$u = \widetilde{S}f = \frac{1}{q} \sum_{I,J}{}' \sum_{r=1}^{q} (-1)^{r+p-1} (\overline{z_{j_r}} f_{IJ}) \, \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_{j_1} \wedge \ldots \wedge [\mathrm{d}\overline{z}_{j_r}] \wedge \ldots \wedge \mathrm{d}\overline{z}_{j_q}$$

leads to $\overline{\partial} u = f$ since

$$\begin{aligned}
&\overline{\partial}((\overline{z_{j_r}} f_{IJ}) \, \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_{j_1} \wedge \ldots \wedge [\mathrm{d}\overline{z}_{j_r}] \wedge \ldots \wedge \mathrm{d}\overline{z}_{j_q} \\
&\qquad = f_{IJ} \, \mathrm{d}\overline{z}_{j_r} \wedge \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_{j_r} \wedge \mathrm{d}\overline{z}_{j_1} \wedge \ldots \wedge [\mathrm{d}\overline{z}_{j_r}] \wedge \ldots \wedge \mathrm{d}\overline{z}_{j_q} \\
&\qquad = (-1)^{r+p-1} f_{IJ} \, \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_{j_1} \wedge \ldots \wedge \mathrm{d}\overline{z}_{j_q}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\bar{\partial} u &= \frac{1}{q} \sum_{I,J}{}' \sum_{r=1}^{q} (-1)^{r+p-1} \bar{\partial} \left((\overline{z_{j_r}} f_{IJ}) \, \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_{j_1} \wedge \ldots \wedge [\mathrm{d}\overline{z}_{j_r}] \wedge \ldots \wedge \mathrm{d}\overline{z}_{j_q} \right) \\
&= \frac{1}{q} \sum_{I,J}{}' \sum_{r=1}^{q} (-1)^{r+p-1} (-1)^{r+p-1} f_{IJ} \, \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_{j_1} \wedge \ldots \wedge \mathrm{d}\overline{z}_{j_q} \\
&= \sum_{I,J}{}' f_{IJ} \, \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_{j_1} \wedge \ldots \wedge \mathrm{d}\overline{z}_{j_q} = f.
\end{aligned}$$

We have

$$\begin{aligned}
\widetilde{S}f &= \frac{1}{q} \sum_{I,|J|=q}{}' \sum_{r=1}^{q} (-1)^{r+p-1} (\overline{z_{j_r}} f_{IJ}) \, \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_{j_1} \wedge \ldots \wedge \mathrm{d}\overline{z}_{j_r} \wedge \ldots \wedge \mathrm{d}\overline{z}_{j_q} \\
&= \frac{1}{q} \sum_{I,|J^{\sim}|=q-1}{}' \sum_{J \sim J^{\sim}}{}' (-1)^p \operatorname{sgn}(J \sim J^{\sim}) (\overline{z}_{|J \sim J^{\sim}|} f_{IJ}) \, \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_{J^{\sim}} \\
&= \frac{1}{q} \sum_{I,|J^{\sim}|=q-1}{}' g_{I,J^{\sim}} \, \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_{J^{\sim}}.
\end{aligned}$$

Since $u = \widetilde{S}f$ it follows from above that

$$\bar{\partial} \left(\frac{1}{q} \sum_{I,|J^{\sim}|=q-1}{}' g_{I,J^{\sim}} \, \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_{J^{\sim}} \right) = f.$$

Since we have $\frac{\partial}{\partial z_i} Pf = 0 \;\; \forall\, i, f \in L^2(\mu)$

$$\begin{aligned}
Sf &= \frac{1}{q} \sum_{I,|J^{\sim}|=q-1}{}' \sum_{J \sim J^{\sim}}{}' (-1)^p \operatorname{sgn}(J \sim J^{\sim}) (H_{\overline{z}_{|J \sim J^{\sim}|}} f_{IJ}) \, \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_{J^{\sim}} \\
&= \frac{1}{q} \sum_{I,|J^{\sim}|=q-1}{}' \widetilde{g}_{I,J^{\sim}} \, \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_{J^{\sim}}
\end{aligned}$$

is a solution operator as well. Using the identity

$$f(z)=\int_{\mathbb{C}^n} K(z,\omega)f(\omega)\,\mathrm{d}\mu(\omega)$$

yields

$$\begin{aligned}
Sf &= \frac{1}{q}\sum_{I,|J^{\sim}|=q-1}{}' \sum_{J\sim J^{\sim}}{}' (-1)^p \operatorname{sgn}(J\sim J^{\sim})(H_{\bar z_{|J\sim J^{\sim}|}} f_{IJ})\,\mathrm{d}z_I\wedge \mathrm{d}\bar z_{J^{\sim}}\\
&= (-1)^p\frac{1}{q}\sum_{I,|J^{\sim}|=q-1}{}' \sum_{J\sim J^{\sim}}{}' \operatorname{sgn}(J\sim J^{\sim})\\
&\quad\times\left(\int_{\mathbb{C}^n} K(z,\omega)(\overline{z+\omega})_{|J\sim J^{\sim}|} f_{IJ})\,\mathrm{d}\mu(\omega)\right)\mathrm{d}z_I\wedge \mathrm{d}\bar z_{J^{\sim}}.
\end{aligned}$$

□

3. Applications of the quasi-canonical solution operator

In this section we will use our solution formula to characterize compactness of the quasi-canonical solution operator in the case of (p,q)-forms with generalized Fock-space coefficients without any restrictions on p and q. As an application we will consider the measures μ where $\mathrm{d}\mu/\mathrm{d}\lambda = \mathrm{e}^{-|z|^m}$. It will turn out, that the quasi-canonical solution operator is not compact in the case of several variables, $\mathrm{d}\mu/\mathrm{d}\lambda = \mathrm{e}^{-|z|^m}$ and p, q without any restrictions. Furthermore we will be able to characterize compactness of the quasi-canonical solution operator restricted to forms with generalized Fock-space coefficients in terms of compactness of commutators of certain Toeplitz-operators.

Corollary 1. *The quasi-canonical solution operator* $S\colon \mathscr{F}_{(p,q+1)} \to L^2(\mu)_{(p,q)}$ *to* $\bar\partial$ *is compact if and only if the commutators* $[T_{\overline{z_i}}, T_{z_i}] = [T_{z_i}^*, T_{z_i}]\colon \mathscr{F} \to L^2(\mu)$ *are compact* $\forall\, i\in\{1,\dots,n\}$. *Here* $T_{\overline{z_i}} = PM_{\bar z_i}$ *is the Toeplitz-operator with the symbol* $\overline{z_i}$.

P r o o f. It is clear, that $[T_{\overline{z_i}}, T_{z_i}]$ is compact if and only if $H_{\overline{z_i}}$ is compact since

$$[T_{\overline{z_i}}, T_{z_i}] = Pz_i(I-P)\overline{z_i}$$

and

$$Pz_i(I-P)\overline{z_i} = H_{\overline{z_i}}^* H_{\overline{z_i}}.$$

So it is clear from Theorem 1 that compactness of the commutators $[T_{\overline{z_i}}, T_{z_i}]$ implies compactness of the quasi-canonical solution operator. Conversely an easy argument shows

$$[I-P]\overline{z_i}f = S\Bigg(\sum_{|I|=p,|K|=q}{}' f_{I,K}\,\mathrm{d}\overline{z_i}\wedge \mathrm{d}\overline{z_K}\wedge \mathrm{d}z_I\Bigg),$$

where $f = \sum_{|I|=p,|K|=q}{}' f_{I,K}\mathrm{d}\overline{z_K}\wedge \mathrm{d}z_I$ has holomorphic coefficients. So the restriction of S to $(p,q+1)$-forms of the form $\sum_{|I|=p,|K|=q}{}' f_{I,K}\mathrm{d}\overline{z_i}\wedge \mathrm{d}\overline{z_K}\wedge \mathrm{d}z_I$ coincides with the Hankel-operator. This finishes the proof. □

Lemma 1. *If the Hankel-operator $H_{\bar z_i^l}\colon \mathscr{F}\to L^2(\mu)$ is compact then the sequence $c^2_{m+l_i}c^{-2}_m - c^2_m c^{-2}_{m-l_i}$ tends to 0 as $|m| = \sum\limits_{i=1}^n m_i \to\infty$.*

P r o o f. With calculations of [16] it follows that

$$H^*_{\bar z_i^n} H_{\bar z_i^n}(u_m)(w) = \left(\frac{c^2_{m+l_i}}{c^2_m} - \frac{c^2_m}{c^2_{m-l_i}}\right) u_m(w)$$

if $m_i > l$. Here $u_m = z^m/c_m$.

Since the set $\{u_m(z)\} = \{z^m/c_m\colon m = (m_1,\dots,m_n), m_1,\dots,m_n\in\mathbb{N}\}$ is orthonormal the result follows easily. □

Corollary 2. *The quasi canonical solution operator $S\colon \mathscr{F}_{(p,q)}\to L^2(\mathrm{e}^{-|z|^m})_{(p,q)}$ is not compact if $n>1$.*

P r o o f. This follows easily. □

Remark 6. For $(0,0)$-forms—$n=0$—it is shown in [11] that the canonical solution operator is compact if $m>2$ and for $(0,1)$-forms it is shown in [17] that the canonical solution operator is not compact for all m.

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