Czechoslovak Mathematical Journal

Enrico Jabara

A note on a class of factorized p-groups

Czechoslovak Mathematical Journal, Vol. 55 (2005), No. 4, 993–996

Persistent URL: http://dml.cz/dmlcz/128040

A NOTE ON A CLASS OF FACTORIZED p-GROUPS

Enrico Jabara, Venezia

(Received March 14, 2003)

Abstract. In this note we study finite p-groups $G = AB$ admitting a factorization by an Abelian subgroup A and a subgroup B. As a consequence of our results we prove that if B contains an Abelian subgroup of index p^{n-1} then G has derived length at most $2n$.

Keywords: factorizable groups, products of subgroups, p-groups

MSC 2000: 20D40, 20D15

1. Introduction

A group G is called (properly) factorizable if it contains two (proper) subgroups A and B such that $G = AB$, namely $G = \{ab \mid a \in A,\ b \in B\}$. A classical problem in the theory of factorizable groups is to determine how the structure of the factors A and B determines that of the whole group G. If, for example, A and B are finite and nilpotent, a well-known result by Wielandt and Kegel (see [1, Theorem 2.4.3]) states that such a group is solvable. Several examples show that the Wielandt-Kegel theorem cannot be extended to infinite groups; indeed, a more satisfactory result on factorizable groups is Itô's theorem (see [1, Theorem 2.1.1]): if A and B are Abelian then G is metabelian.

In the light of the previous and several other results the following conjecture has been stated:

Conjecture. Let $G = AB$ where A and B are finite and nilpotent of class, α and β, respectively. Then, there exists a function f depending only on α and β such that the derived length of G is bounded by $f(\alpha, \beta)$.

By Itô's theorem $f(1, 1) = 2$; moreover it has been conjectured by some authors that $f(\alpha, \beta) = \alpha + \beta$. This conjecture has been disproved by some examples constructed by Cossey and Stonehewer in [2].

In [5] Pennington has proved that the conjecture holds whenever A and B have coprime orders, and in fact $f(\alpha, \beta) = \alpha + \beta$ (see [1, Theorem 2.5.3]). As a consequence, it is enough to consider p-groups in order to bound the derived length of G. Recently Morigi [4] and Mann [3] show that if $G = AB$ is a p-group, A Abelian and $|B'| = p^m$, then the derived length of G is bounded by a function of m ($m + 2$ and $2 \cdot \log_2(m+2)+3$, respectively). In this paper we continue the study of finite p-groups with a factorization where one of the factors is Abelian. In particular we study the case in which B has an Abelian subgroup of *small* index (in a certain sense, a dual of the situation considered in [4] and [3]). Then we define, for every natural number n, the class $\mathscr{A}_n$ of finite p-groups as follows. Let $\mathscr{A}_1$ be the class of Abelian p-groups, and $B \in \mathscr{A}_n$ if and only if for every principal series

$$\{1\} = K_0 < K_1 < \ldots < K_r = B \qquad (|B| = p^r)$$

there exists an Abelian term K_i with $B/K_i \in \mathscr{A}_{n-1}$.

We will prove:

Theorem. *If $G = AB$ is a finite p-group, where A is Abelian and $B \in \mathscr{A}_n$, then G has derived length at most $2n$.*

Corollary. *Let $G = AB$ be a finite p-group. If A is Abelian and B contains an Abelian subgroup of index p^{n-1}, then G has derived length at most $2n$.*

2. Notations and preliminary results

All groups considered will be finite p-groups where p is a fixed prime number; if B is a group and $\{1\} = K_0 < K_1 < \ldots < K_r = B$ is a principal series for B, we shall denote by K_* the largest Abelian term of the series.

The rest of the notation will be standard (see, for example, [1]).

It is clear that, in order to prove $B \in \mathscr{A}_n$, it suffices to show that, for every principal series of B, $B/K_* \in \mathscr{A}_{n-1}$.

The following lemma from [4] is very useful.

Lemma 1. *Let $\{1\} \neq G = AB$ where A is Abelian. Then $A_G \neq \{1\}$ or $B_G \neq \{1\}$.*

P r o o f ([4]). Let ab be a nontrivial element of $Z(G)$, $a \in A$, $b \in B$. Without loss of generality we may assume $a \neq 1 \neq b$ since otherwise the result is trivial. Then for every $x \in A$ we have $1 = [ab, x] = [a, x]^b[b, x] = [b, x]$ and then $[A, b] = 1$. Therefore $\langle b\rangle^G = \langle b\rangle^{AB} = \langle b\rangle^B \leqslant B$ is a nontrivial normal subgroup of G contained in B. $\square$

We will also use the following two observations:

Lemma 2. *The class $\mathscr{A}_n$ is closed under homomorphic images.*

Lemma 3. *If B contains a subgroup E of index p^k such that $E \in \mathscr{A}_n$, then $B \in \mathscr{A}_{n+k}$.*

Proof. We argue by induction on k.

I) Let $k = 1$ and $1 = K_0 < K_1 < \ldots < K_r = B$ be a principal series of B; it suffices to show that $B/K_* \in \mathscr{A}_n$.

We can distinguish three cases:

a) $K_{*+1} \leqslant E$. We prove this point arguing by induction on r.

If $r = 1$ then $|B| = p$ and the initial step is trivial. Since K_{*+1} is not Abelian and $E \in \mathscr{A}_n$ it is clear that $n > 1$. Since $E \in \mathscr{A}_n$ it follows that $E/K_* \in \mathscr{A}_{n-1}$ and in $\overline{B} = B/K_*$ the subgroup $\overline{E}$ has index p. Thus, by induction on r, $\overline{B} \in \mathscr{A}_n$.

b) $K_* \not\leqslant E$. Since E is a maximal subgroup of B we have $B = EK_*$ and so $B/K_* = EK_*/K_* \cong E/(E \cap K_*) \in \mathscr{A}_n$.

c) $K_* \leqslant E$ and $K_{*+1} \not\leqslant E$. Then $K_{*+1} = K_*\langle t\rangle$ where $t \notin E$ and $t^p \in K_*$; in $\overline{B} = B/K_*$, $\bar{t} \in Z(\overline{B})$ and $\bar{t}^p = \overline{1}$ so that $\overline{B} = \overline{E}\langle\bar{t}\rangle = \overline{E} \times \langle\bar{t}\rangle$. Since $\overline{E} \in \mathscr{A}_n$ it is clear that $\overline{B} \in \mathscr{A}_n$.

II) Suppose $k > 1$ and $x \in N_B(E)$, $x \notin E$, $x^p \in E$. Let $E_1 = E\langle x\rangle$; by induction it follows that $E_1 \in \mathscr{A}_{n+1}$. Since $|B : E_1| = p^{k-1}$ the induction hypothesis gives $B \in \mathscr{A}_{(n+1)+(k-1)} = \mathscr{A}_{n+k}$. □

It follows from the previous lemma that if B_0 is an Abelian group and $\langle b\rangle$ a cyclic group of prime order p, then the standard wreath product $B = B_0 \wr \langle b\rangle$ belongs to the class $\mathscr{A}_2$. Note that the nilpotency class of B is not bounded and that $|B'|$ is not bounded, not even as a function of p.

There are groups in $\mathscr{A}_2$ with no Abelian maximal subgroup, as the following example shows:

$$B = \langle x, y \mid x^{p^4} = 1 = y^{p^2},\ x^y = x^{1+p^2}\rangle.$$

3. The proofs

In this section we prove the results stated in the introduction.

Proof of the Theorem. We argue by induction on n, observing that the first induction step follows from Itô's theorem. We can distinguish two cases:

I) $X = A \cap B = \{1\}$.

Let $\{1\} = G_0 < G_1 < \ldots < G_t = G$ be a principal series of G, built up as follows: if in $\overline{G} = G/G_i$ there exists an element $\bar{a} \in Z(\overline{G}) \cap \overline{A}_{\overline{G}}$ of order p, then we define $G_{i+1} = \langle a, G_i \rangle$. Otherwise Lemma 1 shows that $\overline{B}_{\overline{G}} \neq \{1\}$ and we define $G_{i+1} = \langle b, G_i \rangle$, where $\bar{b}$ is some element of order p of $Z(\overline{G}) \cap \overline{B}_{\overline{G}}$.

Since for every $i \in \{1, 2, \ldots, t\}$, we have $\overline{A} \cap \overline{B} = \{\bar{1}\}$ in $\overline{G} = G/G_i$, each G_i is factorized, namely $G_i = (A \cap G_i)(B \cap G_i)$.

Let $G_\star$ be the maximal element of the above series such that $B \cap G_\star$ is Abelian. Then $G_\star = (A \cap G_\star)(B \cap G_\star)$ is metabelian by Itô's theorem. Since in the principal series $K_i = B \cap G_i$ of B, we have $K_* = B \cap G_\star$, then in $\overline{G} = G/G_\star$ we have $\overline{B} = BG_\star/G_\star \cong B/(B \cap G_\star) = B/K_* \in \mathscr{A}_{n-1}$ (clearly $\overline{A}$ is Abelian). The induction hypothesis implies that $\overline{G}$ has derived length at most $2(n-1)$. Therefore G has derived length at most $2n$.

II) $X = A \cap B \neq \{1\}$.

Then $X^G = X^{AB} = X^B \leqslant B$. Therefore X^G is factorized and in $\overline{G} = G/X^G$ we have $\overline{A} \cap \overline{B} = \{\bar{1}\}$. Let $\{1\} = G_0 < G_1 < \ldots < G_k = X^G$ be any principal series with $G_i \lhd G$ for all $i \in \{1, 2, \ldots, k\}$. Such a series can be extended to a principal series of G by constructing a principal series of G/X^G as in the case of $A \cap B = \{1\}$. With the same notation as before, if $G_\star$ contains X^G, then $G_\star$ is factorized and the conclusion follows. Otherwise, if $G_\star < X^G \leqslant B$, then $G_\star$ is an Abelian subgroup of B and, since the term $G_{\star+1} \leqslant X^G \leqslant B$ is nonabelian, we must have $B/G_\star \in \mathscr{A}_{n-1}$. Therefore $G/G_\star$ has derived length at most $2(n-1)$ and G has derived length at most $1 + 2(n-1) < 2n$. □

P r o o f of the Corollary. It is an easy consequence of the Theorem and of our Lemma 3. □

References

[1] *B. Amberg, S. Franciosi and F. de Giovanni*: Products of Groups. Clarendon Press, Oxford, 1992.
[2] *J. Cossey and S. Stonehewer*: On the derived length of finite dinilpotent groups. Bull. London Math. Soc. *30* (1998), 247–250.
[3] *A. Mann*: The derived length of p-groups. J. Algebra *224* (2000), 263–267.
[4] *M. Morigi*: A note on factorized (finite) groups. Rend. Sem. Mat. Padova *98* (1997), 101–105.
[5] *E. A. Pennington*: On products of finite nilpotent groups. Math. Z. *134* (1973), 81–83.

Author's address: Università di Ca' Foscari, Dipartimento di Matematica Applicata, Dorsoduro 3825/E, 30123 Venezia, Italy; e-mail: jabara@unive.it.