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# AN EXAMPLE OF A FIBER IN FIBRATIONS WHOSE SERRE SPECTRAL SEQUENCES COLLAPSE

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*Abstract.* We give an example of a space $X$ with the property that every orientable fibration with the fiber $X$ is rationally totally non-cohomologous to zero, while there exists a nontrivial derivation of the rational cohomology of $X$ of negative degree.

*Keywords*: Sullivan minimal model, orientable fibration, TNCZ, negative derivation

*MSC 2000*: 55P62

## 1. Introduction

An orientable fibration $X \xrightarrow{i} E \to B$ is said to be *TNCZ* (rationally totally non-cohomologous to zero) if the induced map in rational cohomology $i^*\colon H^*(E;Q) \to H^*(X;Q)$ is surjective. It is equivalent to the fact that the Serre spectral sequence $(E_r, d_r)$ collapses at the $E_2$-level, i.e., $H^*(E;Q) \cong H^*(B;Q) \otimes H^*(X;Q)$ as $H^*(B;Q)$-modules. In this paper everything is considered over the rationals.

A simply connected space $X$ is said to be elliptic if the rank of homotopy group $\pi_*(X)$ and the dimension of $H^*(X;Q)$ are both finite. S. Halperin formulated a conjecture saying that every orientable fibration with fiber an elliptic space $X$ with evenly graded cohomology (equivalently, with positive Euler characteristic) is TNCZ. This conjecture has been neither proved nor disproved yet. It is well-known that the Halperin conjecture can be equivalently formulated as saying that there are no negative-degree derivations of $H^*(X;Q)$ ([5, Theorem A]).

In [4, page 154], M. Markl denoted by (†) the property of certain simply connected spaces $X$ that every orientable fibration with the fiber $X$ is TNCZ. We construct an explicit example of a space $X$ satisfying (†), while there exists a nontrivial derivation of $H^*(X;Q)$ of negative degree. Since the Euler characteristic of $X$ is zero, it does not contradict the Halperin conjecture. This suggests to study general spaces satisfying (†), not only those fulfilling the assumptions of the Halperin conjecture. They may often exist even in non-elliptic spaces.

It is natural to propose the following problem

**Problem.** Find a necessary and sufficient condition for spaces or models satisfying (†).

## 2. Example

Let $M(X) = (\Lambda V, d)$ be the Sullivan minimal model ([7]) of a simply connected space $X$ such that $\dim H^*(X;Q) < \infty$. Let $\mathrm{Der}_i\, M(X)$ be the set of $Q$-derivations of $M(X)$ decreasing the degree by $i$ with $\sigma(xy) = \sigma(x)y + (-1)^{i\cdot\deg(x)}x\sigma(y)$ for $x, y \in M(X)$. The boundary operator $\delta\colon \mathrm{Der}_i\, M(X) \to \mathrm{Der}_{i-1}\, M(X)$ is defined by $\delta(\sigma) = d \circ \sigma - (-1)^i \sigma \circ d$ for $\sigma \in \mathrm{Der}_i\, M(X)$. We denote $\bigoplus_i \mathrm{Der}_i\, M(X)$ by $\mathrm{Der}_*\, M(X)$. In the following, $Q\{*\}$ means the $Q$-graded vector space of basis $*$ and the symbol $(p, q)$ means the derivation which sends $p$ to $q$ and other generators to zero ([7, page 314]).

Let the Sullivan minimal model of the space $X$ be given by

$$M(X) = (\Lambda(x, y, z, a, b, c), d)$$

with the degrees $|x| = 2$, $|y| = 3$, $|z| = 3$, $|a| = 4$, $|b| = 5$, $|c| = 7$ and the differentials $d(x) = d(y) = 0$, $d(z) = x^2$, $d(a) = xy$, $d(b) = xa + yz$, $d(c) = a^2 + 2yb$ [2, p. 439].

For the rational fibration ([3]) $(\Lambda(x, y, z), d) \to M(X) \to (\Lambda(a, b, c), \bar{d})$ with $\bar{d}(a) = \bar{d}(b) = 0$ and $\bar{d}(c) = a^2$, the cohomologies of fiber and base are finite dimensional. So $\dim H^*(X;Q) < \infty$. The non-zero rank of $H^i(X;Q)$ is 1 for $i = 0, 2, 3, 11, 12, 14$ and 2 for $i = 7$. The generators of a $Q$-algebra are $S = \{x, y, e = [ya], f = [xb - za], g = [x^2c - xab + yzb], h = [3xyc + a^3]\}$. It is a Poincaré duality algebra and the products of elements of $S$ are all trivial except for $xh$, $yg$ and $ef$ in $H^{14}(X;Q)$. Note that $X$ is realized by a 14-dimensional manifold ([7, Theorem 13.2]).

Let $\mathrm{Der}_+\, H^*(X;Q)$ be the set of derivations decreasing the degrees of $H^*(X;Q)$, namely the negative derivations. We see that there is a non-zero derivation $(g, y)$ in $\mathrm{Der}_8\, H^*(X;Q)$. Thus $\mathrm{Der}_+\, H^*(X;Q) \neq 0$.

**Lemma 2.1.** *$H_+(\operatorname{Der} M(X)) = Q\{(c,x),(c,1)\}$ as a vector space.*

Proof. Since $\delta\colon \operatorname{Der}_1 M(X) \to \operatorname{Der}_0 M(X)$ is given by

$$\begin{aligned}
\delta(y,x) &= (a,x^2) + (b,xz) + 2(c,xb)\\
\delta(a,z) &= (a,x^2) + (b,xz) + 2(c,za)\\
\delta(z,x) &= -(b,xy)\\
\delta(a,y) &= (b,xy) + 2(c,ya)\\
\delta(b,a) &= (b,xy) - 2(c,ya)\\
\delta(b,x^2) &= -2(c,x^2y)\\
\delta(c,xa) &= (c,x^2y)\\
\delta(c,x^3) &= 0,
\end{aligned}$$

we have $\operatorname{Ker}\delta = Q\{\alpha_1 = 2(z,x) + (a,y) + (b,a),\ \beta_1 = (b,x^2) + 2(c,xa),\ (c,x^3)\}$. Since $\delta\colon \operatorname{Der}_2 M(X) \to \operatorname{Der}_1 M(X)$ is given by

$$\begin{aligned}
\delta(x,1) &= -2(z,x) - (a,y) - (b,a) = -\alpha_1\\
\delta(a,x) &= -(b,x^2) - 2(c,xa) = -\beta_1\\
\delta(c,xz) &= (c,x^3)\\
\delta(b,z) &= (b,x^2) - 2(c,yz)\\
\delta(c,b) &= (c,xa) + (c,yz)\\
\delta(b,y) &= 0\\
\delta(c,xy) &= 0,
\end{aligned}$$

we have $H_1(\operatorname{Der} M(X)) = 0$ and $\operatorname{Ker}\delta = Q\{\alpha_2 = (a,x) + (b,z) + 2(c,b),\ (b,y),\ (c,xy)\}$. Since $\delta\colon \operatorname{Der}_3 M(X) \to \operatorname{Der}_2 M(X)$ is given by

$$\begin{aligned}
\delta(y,1) &= (a,x) + (b,z) + 2(c,b) = \alpha_2\\
\delta(z,1) &= (b,y)\\
\delta(b,x) &= -2(c,xy)\\
\delta(c,a) &= (c,xy)\\
\delta(c,x^2) &= 0,
\end{aligned}$$

we have $H_2(\operatorname{Der} M(X)) = 0$ and $\operatorname{Ker}\delta = Q\{\alpha_3 = (b,x) + 2(c,a),\ (c,x^2)\}$. Since $\delta\colon \operatorname{Der}_4 M(X) \to \operatorname{Der}_3 M(X)$ is given by

$$\begin{aligned}
\delta(a,1) &= -(b,x) - 2(c,a) = -\alpha_3\\
\delta(c,z) &= (c,x^2)\\
\delta(c,y) &= 0,
\end{aligned}$$

we have $H_3(\mathrm{Der}\, M(X)) = 0$ and $\mathrm{Ker}\,\delta = Q\{(c,y)\}$. Since $\delta\colon \mathrm{Der}_5\, M(X) \to \mathrm{Der}_4\, M(X)$ is given by

$$\delta(b,1) = -2(c,y)$$
$$\delta(c,x) = 0,$$

we have $H_4(\mathrm{Der}\, M(X)) = 0$ and $\mathrm{Ker}\,\delta = Q\{(c,x)\}$. We have $H_5(\mathrm{Der}\, M(X)) = Q\{(c,x)\}$ since $\mathrm{Der}_6\, M(X) = 0$. Finally, since $\delta\colon \mathrm{Der}_7\, M(X) \to \mathrm{Der}_6\, M(X)$ is given by $\delta(c,1) = 0$, we have

$$H_+(\mathrm{Der}\, M(X)) = Q\{(c,x),(c,1)\}$$

as a vector space. □

**Theorem 2.2.** *The manifold $X$ satisfies* (†).

Proof. By Lemma 2.1, the KS-extension ([3]) of an orientable fibration $X \xrightarrow{i} E \to B$,

$$(A^*(B), d_B) \to (A^*(B) \otimes \Lambda V, D) \xrightarrow{i^*} M(X) = (\Lambda V, d)$$

is equivalent ([6]) to that given by $D(v) = d(v)$ for $v = x, y, z, a, b$ and $D(c) = d(c) + xb_6 + b_8$ for some $d_B$-cocycle $b_k \in A^k(B)$, the $k$-dimensional rational polynomial forms of $B$ ([7]). We see that $H^*(E;Q) = H^*(A^*(B) \otimes \Lambda V, D)$ contains as a part the generators of the $Q$-algebra

$$\{x, y, e, f, \bar{g} = g - xzb_6 - zb_8, \bar{h} = h - 3zyb_6 - 3ab_8\}.$$

Then $i^*\colon H^*(E;Q) \to H^*(X;Q)$ is identity for $x, y, e, f$, and $i^*(\bar{g}) = g$, $i^*(\bar{h}) = h$. Thus $i^*$ is surjective. □

**Remark 1.** Even if $M(\tilde{X})$ is isomorphic to $M(X)$ as a differential (not graded) algebra, it need not satisfy (†). For example, let $\tilde{X}$ be a space such that the degrees of generators of $M(\tilde{X})$ are given by $|x| = 4, |y| = 3, |z| = 7, |a| = 6, |b| = 9, |c| = 11$ and $d(x) = d(y) = 0$, $d(z) = x^2$, $d(a) = xy$, $d(b) = xa + yz$, $d(c) = a^2 + 2yb$. Then $(z,y) \in H_4(\mathrm{Der}\, M(\tilde{X}))$ and there is a fibration with a base 5-dimensinal sphere $\tilde{X} \to E \to S^5$ whose KS-extension is given by $D(v) = d(v)$ for $v = x, y, a, b, c$ and $D(z) = d(z) + y \cdot s$ for the fundamental class $s$ of $S^5$. Since $D(f) = -ysa = -e \cdot s$, it is not TNCZ.

**Remark 2.** Let $m\colon H_+(\mathrm{Der}\, M(X)) \to \mathrm{Der}_+\, H^*(X;Q)$ be the natural graded Lie algebra homomorphism. I. Belegradek and V. Kapovitch searched in [1] examples that $\mathrm{Im}(m) = 0$ but $\mathrm{Der}_+\, H^*(X;Q) \neq 0$. Our example is a response to it. The auther does not know whether or not the condition "$\mathrm{Im}(m) = 0$" is sufficient for (†) in general (not only over a sphere).

*References*

[1] *I. Belegradek and V. Kapovitch*: Obstructions to nonnegative curvature and rational homotopy theory. J. Amer. Math. Soc. *16* (2003), 259–284.
[2] *Y. Félix, S. Halperin and J. C. Thomas*: Rational Homotopy Theory. Springer GTM, 205, New York, 2001.
[3] *S. Halperin*: Rational fibrations, minimal models, and fibrings of homogeneous spaces. Trans. A.M.S. *244* (1978), 199–244.
[4] *M. Markl*: Towards one conjecture on collapsing of the Serre spectral sequence. Rend. Circ. Mat. Palermo (2) Suppl. *22* (1990), 151–159.
[5] *W. Meier*: Rational universal fibrations and flag manifolds. Math. Ann. *258* (1982), 329–340.
[6] *M. Schlessinger and J. Stasheff*: Deformation theory and rational homotopy type. Preprint.
[7] *D. Sullivan*: Infinitesimal computations in topology. Publ. I.H.E.S. *47* (1977), 269–331.



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