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BROWNIAN REPRESENTATIONS OF CYLINDRICAL LOCAL MARTINGALES, MARTINGALE PROBLEM AND STRONG MARKOV PROPERTY OF WEAK SOLUTIONS OF SPDES IN BANACH SPACES

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Abstract. The paper deals with three issues. First we show a sufficient condition for a cylindrical local martingale to be a stochastic integral with respect to a cylindrical Wiener process. Secondly, we state an infinite dimensional version of the martingale problem of Stroock and Varadhan, and finally we apply the results to show that a weak existence plus uniqueness in law for deterministic initial conditions for an abstract stochastic evolution equation in a Banach space implies the strong Markov property.

Keywords: Brownian representations, martingale problem, strong Markov property

MSC 2000: 60H15

The first part of this paper is devoted to the study of Brownian representations of cylindrical local martingales. If W is a (cylindrical) Wiener process and g a progressively measurable integrable process with values in the space of linear bounded operators then the stochastic integral $M = \int g \,\mathrm{d}W$ is a cylindrical local martingale with the quadratic variation process $\int gg^* \,\mathrm{d}s$. The question is whether, provided M is a cylindrical local martingale with the above quadratic variation, there exists a Wiener process W such that M is the stochastic integral of the process g with respect to W. The affirmative answer is given in Theorem 2 for the general cylindrical case, and in Corollary 6 for the particular case when we consider the representations in 2-smoothable Banach spaces which seem to be the most general spaces where stochastic integrals with respect to Wiener processes exist. In particular, every Hilbert space is 2-smoothable. We refer to [1] or [14] for details on stochastic integration in these

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spaces. The contribution of Theorem 2 is that we do not consider only continuous local martingales in Hilbert or Banach spaces (as in [3] or [4]) but we represent cylindrical local martingales over a Banach space. The motivation for this generalization is that solutions of stochastic evolution equations in Banach spaces are no more semimartingales since generators of C_0-semigroups are unbounded operators. Yet, the solutions are still cylindrical semimartingales and hence, as we will see in the second and the third part of the present paper, this generalization permits to develop the Stroock and Varadhan theory of martingale problems for a rather general class of abstract stochastic evolution equations.

The second part deals with a stochastic evolution equation

$$(\dagger) \qquad \mathrm{d}u = \{Au + F(t, u(t))\}\,\mathrm{d}t + G(t, u(t))\,\mathrm{d}W$$

in a 2-smoothable separable Banach space X. In general, it is not known whether solutions of this abstract equation are or are not norm continuous and so we cannot consider the space of X-continuous functions on $[0, \infty)$ as the state space for the paths of the solutions on which we would like to formulate some sort of a martingale problem. Yet, by a theorem of Chojnowska-Michalik ([2] or [14]), the solutions are continuous for a countably generated Hausdorff topology on X which can be naturally embedded in the Fréchet space of real sequences $\mathbb{R}^{\mathbb{N}}$ (equipped with the product topology), and so we consider the path space $\overline{\Omega}$ of continuous functions from $[0, \infty)$ to $\mathbb{R}^{\mathbb{N}}$. This, in fact, rather complicates the exposition since the embedding is not a homeomorphism. Anyway, we establish Theorem 14 which states sufficient and necessary conditions for a Borel probability measure on $\overline{\Omega}$ to be a law of a solution of the equation ($\dagger$). The advantage of this theorem is that we work on one fixed stochastic base with one (canonical) process and the objects of study are Borel probability measures which correspond in one-to-one way to martingale solutions (weak solutions in the probabilistic sense).

Having established Theorem 14 we formulate an infinite dimensional version of the martingale problem using the ideas of Stroock and Varadhan [17]. We show that if the equation ($\dagger$) is well-posed, i.e. weak existence and uniqueness in law hold for deterministic initial conditions, then the transition function is measurable (Corollary 23) and defines a Markov process (Theorem 24). As a by-product we get weak existence and uniqueness in law for general initial conditions for the equation ($\dagger$) (Corollary 22). On this occasion we refer to [14] where it is shown that provided the equation ($\dagger$) is well-posed, the joint uniqueness in law holds for deterministic initial conditions (i.e. the joint law of the solution and the driving Wiener process is unique).

In the third part of the present paper we consider the equation ($\dagger$) supposing that its solutions have norm continuous trajectories. This additional assumption enables us to return to the path space Ω of continuous functions from $[0,\infty)$ to X considered with the natural evaluation process $Y_t(\omega)=\omega(t)$, $\omega\in\Omega$. And again, under the assumption of well-posedness of the equation, the process Y, together with the probability measures $(\mathbb{P}^{a,x}\colon\ a\geqslant 0, x\in X)$ corresponding to the laws of solutions departing at time a from $x\in X$, is a strong Markov family (Theorem 27).

The proofs are given separately in the last section.

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0. Notation, conventions, remarks

Throughout this paper:

- Unless anything contradictory is stated about a stochastic base $(\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P})$ we will suppose that $\mathcal{F}_0$ contains all $\mathbb{P}$-negligible sets of $\mathcal{F}$.
- If $f\colon\ \mathbb{R}\to\mathbb{R}$ is a differentiable function then we denote by $\dot f$ and $\ddot f$ the first and the second derivative, respectively.
- If (Z,ϱ) is a metric space then we equip the space $C([0,\infty);Z)$ with the topology of locally uniform convergence which is metrized e.g. by

$$(f,g)\mapsto\sum_{n=1}^{\infty}2^{-n}\min\Big\{\sup_{0\leqslant t\leqslant n}\{\varrho(f(t),g(t))\},1\Big\},\qquad f,g\in C([0,\infty);Z).$$

- We denote by U a separable Hilbert space, by X a separable Banach space, by Q a symmetric nonnegative operator on U, $U_0=\operatorname{Rng}Q^{1/2}$ the reproducing kernel space for Q with the inner product defined via

$$\langle u,v\rangle_{U_0}=\big\langle Q^{-1/2}u,Q^{-1/2}v\big\rangle,\qquad u\in U_0,\ v\in U_0,$$

  where $Q^{-1/2}$ is the inverse mapping to the restriction of $Q^{1/2}$ onto the orthogonal complement of $\operatorname{Ker}Q^{1/2}$ in U.
- Given a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P})$, following [12], *a cylindrical Q-$(\mathcal{F}_t)$-Wiener process* W is a family $(W(u)\colon\ u\in U)$ of $(\mathcal{F}_t)$-Wiener processes $W(u)$ on $[0,\infty)$ for every $u\in U$ and

$$\mathbb{E}\,W_s(u)W_t(v)=\langle Qu,v\rangle_U\min\{t,s\}$$

  for every $s\geqslant 0$, $t\geqslant 0$, $u\in U$ and $v\in U$.

- We denote by $L(U, X)$ the space of linear bounded operators from U to X equipped with the σ-algebra generated by the family of the mappings $B \mapsto Bu$ for every $u \in U$.
- We denote by $R(U, X)$ the space of γ-radonifying operators from U to X and recall that $R(U, X)$ is a separable Banach space whose Borel σ-algebra coincides with the σ-algebra generated by the mappings $B \mapsto Bu$ for every $u \in U$ (e.g. [13] or [14]). Hence if $(\Omega, \mathcal{F})$ is a measurable space, a mapping ξ from Ω to either $L(U, X)$ or $R(U, X)$ is measurable if and only if $\xi u\colon\ \Omega \to X$ is Borel measurable for every $u \in U$.

Remark. If W is a cylindrical Q-$(\mathcal{F}_t)$-Wiener process and the covariance operator Q is nuclear then there exists a U-valued $(\mathcal{F}_t)$-Wiener process $\mathcal{W}$ with the covariance Q such that $W_t(u) = \langle \mathcal{W}_t, u \rangle_U$ a.s. for every $t \geqslant 0$ and $u \in U$ (see e.g. [12]).

Remark. If X is a Hilbert space then $R(U, X)$ coincides with the space of the Hilbert-Schmidt operators from U to X and the respective norms are equivalent.

1. Brownian representations of cylindrical martingales

Representations of continuous local martingales with an absolutely continuous quadratic variation as stochastic integrals go back to Doob [5] who showed a representation theorem for one-dimensional martingales. Its finite-dimensional version can be found in Theorem 3.4.2 in [7], and a generalization to Hilbert space valued martingales in [10], [15] and in Theorem 8.2 in [3]. A representation of Banach valued continuous local martingales M with quadratic variation process

$$\int_0^{\cdot} gg^* \,\mathrm{d}s$$

where g is a progressively measurable process in the space of radonifying operators is due to [4]. Our version generalizes all the above cited results in the sense that M may be even a cylindrical local continuous martingale indexed by a subset of the dual of the Banach space X and g may take values in the space of linear bounded operators if X is reflexive. If X is non-reflexive then g must take values in the space of compact operators (hence in a wider class than the radonifiyng operators considered in [4]). The proof of the representation theorem in [4] relies on finite dimensional approximations while our proof is based on a functional calculus.

Definition 1. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space, $a \geqslant 0$ and $(M_t)_{t \geqslant a}$ an adapted process with continuous paths. Then we say that $(M_t)_{t \geqslant a}$ is a local martingale provided there exist stopping times τ_n with values in $[a, \infty]$ such that $\lim \tau_n = \infty$ almost surely and $(M_{t\wedge\tau_n})_{t\geqslant a}$ is a bounded martingale on $[a,\infty)$ for every $n \in \mathbb{N}$.

Notice that if $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a filtered probability space, g is a progressively measurable process in $L(U_0, X)$ with $\|g\|_{L(U_0,X)} \in L^2_{\mathrm{loc}}[0,\infty)$, W is a cylindrical Q-$(\mathcal{F}_t)$-Wiener process then

$$M_t(x^*) = \int_0^t x^* \circ g_s \,\mathrm{d}W, \qquad t \geqslant 0 \tag{1}$$

is a continuous local $(\mathcal{F}_t)$-martingale starting from zero for every $x^* \in X^*$ with the cross-variation

$$\langle M(x^*), M(y^*)\rangle_t = \int_0^t \langle g_s^* x^*, g_s^* y^*\rangle_{U_0} \,\mathrm{d}s, \qquad t \geqslant 0, \quad x^*,\, y^* \in X^*. \tag{2}$$

The next theorem says that every family of continuous local martingales indexed by an X-separating subset of X^* with quadratic variation (2) is representable as the stochastic integral (1).

Theorem 2. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space, g a progressively measurable $L(U_0, X)$-valued process such that*

$$\int_0^T \|g_s\|^2_{L(U_0,X)} \,\mathrm{d}s < \infty \qquad \textit{almost surely for every } T > 0$$

and $(M(x^*)\colon\, x^* \in D)$ a family of continuous local $(\mathcal{F}_t)$-martingales starting from zero where $D \subseteq X^*$ separates points of X. Suppose that

$$\langle M(x^*), M(y^*)\rangle_t = \int_0^t \langle g_s^* x^*, g_s^* y^*\rangle_{U_0} \,\mathrm{d}s, \qquad t \geqslant 0 \tag{3}$$

holds for every $x^*, y^* \in D$ and let at least one of the following conditions be satisfied:

- ▷ *the operator $g(t,\omega)$ is injective for $dt \otimes \mathbb{P}$-almost all (t,ω),*
- ▷ *$(\Omega, \mathcal{F}, \mathbb{P})$ supports an infinite number of independent standard real $(\mathcal{F}_t)$-Wiener processes which are, altogether, independent of the process M.*

Let either X be reflexive or let the operator $g(t,\omega)$ be compact for $dt \otimes \mathbb{P}$-almost all (t,ω). Then there exists a Q-$(\mathcal{F}_t)$-Wiener process W such that

$$M_t(x^*) = \int_0^t x^* \circ g \,\mathrm{d}W_s$$

for every $t \geqslant 0$ and $x^* \in D$.

Remark 3. If D is a group for the binary operation $+$ then (3) is equivalent to

$$\langle M(x^*)\rangle_t = \int_0^t \|g_s^* x^*\|_{U_0}^2 \,\mathrm{d}s, \qquad t \geqslant 0,\ x^* \in D.$$

Definition 4. A Banach space X is 2-smoothable provided that there exists a constant c and an equivalent norm $\|\cdot\|_1$ on X such that

$$\|x+y\|_1^2 + \|x-y\|_1^2 \leqslant 2\|x\|_1^2 + c\|y\|_1^2, \qquad x \in X,\ y \in X.$$

Remark 5. Every 2-smoothable Banach space X admits, by definition, an equivalent uniformly convex norm. Hence, X is isomorphic to a uniformly convex Banach space an so X is reflexive by the Milman-Pettis theorem.

Theorem 2 has an immediate corollary if X is a 2-smoothable Banach space or, in particular, a Hilbert space, and the process g satisfies an additional integral condition. Then we have an integral representation of the cylindrical process M in terms of an X-valued process.

Corollary 6. *If, in addition to the assumptions in Theorem* 2, *the Banach space* X *is* 2*-smoothable and*

$$\mathbb{P}\left[\int_0^T \|g_s\|_{R(U_0,X)}^2 \,\mathrm{d}s < \infty\right] = 1$$

holds for every $T > 0$ *then*

$$M_t(x^*) = \left\langle x^*, \int_0^t g \,\mathrm{d}W \right\rangle$$

for every $t \geqslant 0$ *and* $x^* \in D$.

2. The stochastic evolution equation and the settings

This preparatory section is devoted to definitions of the mathematical environment of the stochastic equation in question, to the settings for the martingale problem that will be introduced in the sequel, and, eventually, to examples of the objects defined.

Notation and conventions. From now on

- X will be, in addition, 2-smoothable.
- $\mathbb{B}(Z)$ stands for the Borel σ-algebra over a topological space Z and $\mathbb{R}^{\mathbb{N}}$ is the Fréchet space of real sequences equipped with the product topology.

- If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\mathcal{G} \subseteq \mathcal{F}$ then $\mathcal{G}^{\mathbb{P}}$ is the smallest σ-algebra containing $\mathcal{G}$ and every $\mathbb{P}$-negligible set in $\mathcal{F}$ and we call $\mathcal{G}^{\mathbb{P}}$ *the* $\mathbb{P}$*-augmentation of* $\mathcal{G}$ *in* $\mathcal{F}$. A $\mathbb{P}$-augmentation of a filtration $(\mathcal{G}_t)$ in $\mathcal{F}$ is the filtration $(\mathcal{G}_t^{\mathbb{P}})$.
- If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and ξ a measurable mapping from Ω to some measurable space Y then we denote by $\mathfrak{Law}_{\mathbb{P}}(\xi)$ the law of ξ under $\mathbb{P}$.

Moreover, we write

▷ $\pi_s\colon C([0,\infty), \mathbb{R}^{\mathbb{N}}) \to \mathbb{R}^{\mathbb{N}} : f \mapsto f(s)$ for $s \in [0,\infty)$,

▷ $\overline{\Omega} = C([0,\infty), \mathbb{R}^{\mathbb{N}})$,

▷ $\overline{\mathcal{F}} = \mathbb{B}(C([0,\infty), \mathbb{R}^{\mathbb{N}})) = \sigma(\pi_s\colon s < \infty)$,

▷ $\overline{\mathcal{F}}_t = \sigma(\pi_s\colon s \leqslant t)$ for $t \in [0,\infty)$,

▷ $\overline{\mathcal{F}}_{a,b} = \sigma(\pi_s\colon a \leqslant s \leqslant b)$ or $\overline{\mathcal{F}}_{a,\infty} = \sigma(\pi_s\colon s \geqslant a)$ for $0 \leqslant a \leqslant b < \infty$.

Further, we consider an infinitesimal generator A of a C_0-semigroup $(S_t\colon t \geqslant 0)$ on X. The adjoint operators $(S_t^*\colon t \geqslant 0)$ on the dual space X^* form a C_0-semigroup as well, and the adjoint operator A^* is its infinitesimal generator by Corollary 1.10.6 in [16] since X is reflexive. Thus we can fix a countable vector space over rational numbers $D = \{h_n\colon n \in \mathbb{N}\}$ in the domain $D(A^*)$ of the adjoint operator A^* such that the set $\{(h_n, A^*h_n)\colon n \in \mathbb{N}\}$ is dense in $\{(x^*, A^*x^*)\colon x^* \in D(A^*)\}$ for the topology of $X^* \times X^*$ since X^* is separable. In particular, D is dense in X^*.

Definition 7. Each element of $x \in X$ can be mapped into $\mathbb{R}^{\mathbb{N}}$ by $\vec{x} = (\langle h_n, x\rangle : n \in \mathbb{N})$ and defines a mapping $e\colon X \to \mathbb{R}^{\mathbb{N}} : x \mapsto \vec{x}$.

Proposition 8. *The mapping e is injective, continuous and maps Borel sets of X into Borel sets of $\mathbb{R}^{\mathbb{N}}$.*

P r o o f. Clearly, e is injective and continuous with respect to the weak topology in X. Furthermore, the system of subsets of X whose image under e is a Borel set in $\mathbb{R}^{\mathbb{N}}$ is closed under countable unions and complements and contains closed balls and X as these are weakly compact and weakly σ-compact, respectively. Hence e maps Borel subsets of X into Borel subsets of $\mathbb{R}^{\mathbb{N}}$. □

Corollary 9. *If we denote by $e^{-1}\colon \mathbb{R}^{\mathbb{N}} \to X$ the inverse of e extended by zero to $\mathbb{R}^{\mathbb{N}} \setminus \operatorname{Rng} e$ then $e^{-1}\colon \mathbb{R}^{\mathbb{N}} \to X$ is Borel measurable.*

We also consider some measurable nonlinearities $F\colon [0,\infty) \times X \to X$, $G\colon [0,\infty) \times X \to L(U_0, X)$ and an auxiliary measurable mapping $J\colon [0,\infty) \times X \to [0,\infty]$ with the following property that we will refer to as the *J-property*:

Whenever $a \geqslant 0$ and $v\colon [a,\infty) \to [0,\infty)$, $w\colon [a,\infty) \to X$ are measurable such that

$$\int_a^t J(v_s, w_s)\,\mathrm{d}s < \infty, \qquad t \geqslant a$$

then

$$\int_a^t \{\|w_s\|_X + \|F(v_s,w_s)\|_X + \|G(v_s,w_s)\|^2_{L(U_0,X)} + \|S_{t-s}G(v_s,w_s)\|^2_{R(U_0,X)}\}\,\mathrm{d}s < \infty$$

for every $t \geqslant a$.

Let $a \geqslant 0$. We say that a 6-tuple $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, u)$ consisting of a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, a Q-$(\mathcal{F}_t)$-Wiener process and a progressively measurable X-valued process $(u(t)\colon\ t \geqslant a)$ is *a solution* of the equation (with the coefficients F, G and the function J)

$$\mathrm{d}u = \{Au + F(t,u(t))\}\,\mathrm{d}t + G(t,u(t))\,\mathrm{d}W \tag{4}$$

on $[a,\infty)$ provided that

$$\mathbb{P}\left[\int_a^t J(s,u(s))\,\mathrm{d}s < \infty\right] = 1, \qquad t \geqslant a \tag{5}$$

and

$$\mathbb{P}\left[u(t) = S_{t-a}u(a) + \int_a^t S_{t-s}F(s,u(s))\,\mathrm{d}s + \int_a^t S_{t-s}G(s,u(s))\,\mathrm{d}W\right] = 1 \tag{6}$$

holds for every $t \geqslant a$.

Remark 10. The equality (6) holds if and only if

$$\begin{aligned}\langle h_n, u(t)\rangle = \langle h_n, u(a)\rangle + \int_a^t \langle A^* h_n, u(s)\rangle\,\mathrm{d}s + \int_a^t \langle h_n, F(s,u(s))\rangle\,\mathrm{d}s \\ + \int_a^t h_n \circ G(s,u(s))\,\mathrm{d}W\end{aligned}$$

holds almost surely for every $t \geqslant a$ and every $n \in \mathbb{N}$ by the theorem of Chojnowska-Michalik ([2] or [14]) since $\{(h_n, A^* h_n)\}_n$ is dense in $\{(x^*, A^* x^*)\colon\ x^* \in D(A^*)\}$ by the definition of the set D.

Remark 11. In particular, the $\mathbb{R}^{\mathbb{N}}$-valued process $(\vec{u}(t \vee a))_t$ (Definition 7) has a continuous modification whose paths belong to $\overline{\Omega}$.

Remark 12. The class of processes where a solution must live is restricted in the present paper by the assumption (5). Actually, in general, it is sufficient to assume

$$\mathbb{P}\left[\int_a^t \|S_{t-s}F(s,u(s))\|_X + \|S_{t-s}G(s,u(s))\|^2_{R(U_0,X)}\,\mathrm{d}s < \infty\right] = 1 \tag{7}$$

to hold for every $t > a$ in order that the integrals in (6) exist. But such a class of processes $(u_t)_t$ is delimited by a continuum of conditions (one must verify whether (7) holds for every $t > a$) and, therefore, too wide and inconvenient for measure theoretical considerations where only countable operations are admitted—unlike the class delimited by a countable number of conditions in (5) where only $t > a$, $t \in \mathbb{N}$ may be taken into account. Moreover, if (5) holds then the equivalent countable characterization of a solution in Remark 10 is available and frequently used throughout this paper.

Now we will give three examples of choices of the function J if the semigroup (S_t), the covariance Q and the diffusion G, respectively, satisfy additional hypotheses.

First example of a choice of J. Let $p \geqslant 2$ and $1/p < \alpha < 1$. Suppose that

$$\int_0^t \frac{\|S_s\|^2_{R(X)}}{s^{2\alpha}} < \infty, \qquad t \geqslant 0.$$

Then

$$J(s,x) = 1 + \|x\|_X + \|F(s,x)\|_X + \|G(s,x)\|^p_{L(U_0,X)}, \qquad s \geqslant 0,\ x \in X$$

has the J-property.

Proof. Since J dominates $\|x\|_X + \|F(s,x)\|_X + \|G(s,x)\|^2_{L(U_0,X)}$ it suffices to prove that

$$\int_a^t \|S_{t-s}G(v_s, w_s)\|^2_{R(U_0,X)}\,\mathrm{d}s < \infty$$

whenever $t > a \geqslant 0$ and $v\colon [a,\infty) \to [0,\infty)$, $w\colon [a,\infty) \to X$ are measurable such that

$$\int_a^t \|G(v_r, w_r)\|^p_{L(U_0,X)}\,\mathrm{d}r < \infty.$$

Towards this end,

$$\left[\int_0^t \left(\int_0^t v(s,r)\,\mathrm{d}r\right)^2 \mathrm{d}s\right]^{1/2} \leqslant \int_0^t \left(\int_0^t v^2(s,r)\,\mathrm{d}s\right)^{1/2} \mathrm{d}r$$

holds for every nonnegative function v by Fubini's theorem. Now we fix some $t > 0$ and we set $v(s,r) = (t-r)^{\alpha-1}(r-s)^{-\alpha}\|S_{t-s}G(v_s,w_s)\|_{R(U_0,X)}$ for $0 < s < r < t$

and zero otherwise, to the above inequality. Then

$$
\begin{aligned}
&\frac{\pi}{\sin(\pi\alpha)}\left(\int_0^t \|S_{t-s}G(v_s,w_s)\|^2_{R(U_0,X)}\,\mathrm{d}s\right)^{1/2}\\
&\leqslant \int_0^t \frac{\|S_{t-r}\|_{L(X)}}{(t-r)^{(1-\alpha)}}\left(\int_0^r \frac{\|S_{r-s}\|^2_{R(X)}}{(r-s)^{2\alpha}}\|G(v_s,w_s)\|^2_{L(U_0,X)}\,\mathrm{d}s\right)^{1/2}\mathrm{d}r\\
&\leqslant \left(\int_0^t \frac{\|S_r\|^{p'}_{L(X)}}{r^{(1-\alpha)p'}}\,\mathrm{d}r\right)^{1/p'}\left[\int_0^t\left(\int_0^r \frac{\|S_{r-s}\|^2_{R(X)}}{(r-s)^{2\alpha}}\|G(v_s,w_s)\|^2_{L(U_0,X)}\,\mathrm{d}s\right)^{p/2}\mathrm{d}r\right]^{1/p}\\
&\leqslant \left(\int_0^t \frac{\|S_r\|^{p'}_{L(X)}}{r^{(1-\alpha)p'}}\,\mathrm{d}r\right)^{1/p'}\left(\int_0^t \frac{\|S_r\|^2_{R(X)}}{r^{2\alpha}}\,\mathrm{d}r\right)^{1/2}\left(\int_0^t \|G(v_r,w_r)\|^p_{L(U_0,X)}\,\mathrm{d}r\right)^{1/p}
\end{aligned}
$$

by Hölder's and Young's inequalities. □

Second example of a choice of J. Let Q be a trace class operator on U and denote by V the orthogonal complement of $\operatorname{Ker} Q^{1/2}$ in U. Let us equip the vector space V with the norm of U. Then

$$J(s,x)=\|x\|_X+\|F(s,x)\|_X+\|G(s,x)\|^2_{L(V,X)},\qquad s\geqslant 0,\ x\in X$$

has the J-property.

P r o o f. Since J dominates $\|x\|_X+\|F(s,x)\|_X$ it suffices to prove that it dominates $\|S_{t-s}G(s,x)\|^2_{R(U_0,X)}$ and $\|G(s,x)\|^2_{L(U_0,X)}$ as well. However

$$\|G(s,x)\|_{L(U_0,X)}\leqslant \|G(s,x)\|_{L(V,X)}\|Q^{1/2}\|_{L(U)}$$

and

$$
\begin{aligned}
&\|S_{t-s}G(s,x)\|_{R(U_0,X)}=\|S_{t-s}G(s,x)Q^{1/2}\|_{R(V,X)}\\
&\leqslant \|S_{t-s}\|_{L(X)}\|G(s,x)\|_{L(V,X)}\|Q^{1/2}\|_{R(V)}=\|S_{t-s}\|_{L(X)}\|G(s,x)\|_{L(V,X)}(\operatorname{Tr} Q)^{1/2}.
\end{aligned}
$$

□

Third example of a choice of J. Let the diffusion operator G take values in the space of γ-radonifying operators $R(U_0,X)$. Then

$$J(s,x)=\|x\|_X+\|F(s,x)\|_X+\|G(s,x)\|^2_{R(U_0,X)},\qquad s\geqslant 0,\ x\in X$$

has the J-property.

P r o o f. Since J dominates $\|x\|_X + \|F(s,x)\|_X$ it suffices to prove that the same is true for $\|S_{t-s}G(s,x)\|^2_{R(U_0,X)}$ and $\|G(s,x)\|^2_{L(U_0,X)}$. Now

$$\|G(s,x)\|_{L(U_0,X)} \leqslant \|G(s,x)\|_{R(U_0,X)}$$

and

$$\|S_{t-s}G(s,x)\|_{R(U_0,X)} \leqslant \|S_{t-s}\|_{L(X)}\|G(s,x)\|_{R(U_0,X)}.$$

□

3. Martingale problem

In this section we will show a sufficient and necessary condition for the existence of a solution $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, u)$ of the equation (4). Namely, we formulate an infinite dimensional version of the martingale problem of Stroock and Varadhan.

Let $0 \leqslant a < \infty$ and denote

$$\tau_m(b) = \tau^a_m(b) = \inf\{s \geqslant a\colon\ |b(s)| \geqslant m\} \tag{8}$$

the first exit time of a function $b\colon [a,\infty) \to \mathbb{R}$ from the interval $(-m,m)$ with the convention that $\inf\emptyset = \infty$ and

$$M_a = \bigcap_{t>a}\left\{\omega \in \overline{\Omega}\colon\ \int_a^t J(s, e^{-1}\pi_s(\omega))\,\mathrm{d}s < \infty\right\}$$

is the set of all trajectories $\omega \in \overline{\Omega}$ for which $J(\cdot, e^{-1}\omega(\cdot))$ is locally integrable on the interval $[a,\infty)$.

Consequently, if $\omega \in M_a$ then, by the definition of the J-function J,

$$\|F(\cdot, e^{-1}\omega(\cdot))\|_X \qquad \text{and} \qquad \|G(\cdot, e^{-1}\omega(\cdot))\|^2_{L(U_0,X)}$$

are locally integrable on $[a,\infty)$, and so we may define real continuous processes defined on $\overline{\Omega}$

$$\begin{aligned} L_n(f)(t,\omega) = L^a_n(f)(t,\omega) &= f(\pi^n_t(\omega)) - f(\pi^n_a(\omega)) \\ &\quad - \int_a^t \dot f(\pi^n_s(\omega))\left(\left\langle A^*h_n, e^{-1}\pi_s(\omega)\right\rangle + \left\langle h_n, F(s, e^{-1}\pi_s(\omega))\right\rangle\right)\,\mathrm{d}s \\ &\quad - \frac{1}{2}\int_a^t \ddot f(\pi^n_s(\omega))\left\|G^*(s, e^{-1}\pi_s(\omega))h_n\right\|^2_{U_0}, \qquad t \geqslant a,\ \omega \in M_a \\ L_n(f)(t,\omega) = L^a_n(f)(t,\omega) &= 0, \qquad t \geqslant a,\ \omega \in \overline{\Omega} \setminus M_a \end{aligned}$$

where $\pi_s(\omega) = (\pi^j_s(\omega)\colon\ j \in \mathbb{N}) \in \mathbb{R}^{\mathbb{N}}$ for $f \in C^2(\mathbb{R})$ and $n \in \mathbb{N}$.

Finally, we stop the process $L_n^a(f)$ after its first exit from the interval $(-m, m)$ which happens at time $\tau_m(L_n^a(f))$, getting a bounded continuous process

$$L_{nm}(f)(t,\omega) = L_{nm}^a(f)(t,\omega) = L_n^a(f)(t \wedge \tau_m(L_n^a(f)(\cdot,\omega)),\omega), \qquad t \geqslant a,\ \omega \in \overline{\Omega}$$

for every $f \in C^2(\mathbb{R})$, $n \in \mathbb{N}$ and $m \in \mathbb{N}$.

Remark 13. The processes $L_n^a(f)$, $L_{nm}^a(f)$ are adapted to the filtration

$$(\sigma(\overline{\mathcal{F}}_{a,t} \cup \{M_a\}))_{t \geqslant a},$$

so if $\overline{\mathbb{P}}(M_a) = 1$ for some probability measure $\overline{\mathbb{P}}$ on $\overline{\mathcal{F}}$ then the processes $L_n^a(f)$ and $L_{nm}^a(f)$ are adapted to the $\overline{\mathbb{P}}$-augmentation of the filtration $(\overline{\mathcal{F}}_{a,t})_{t \geqslant a}$ in $\overline{\mathcal{F}}$ for every $f \in C^2(\mathbb{R})$, $n \in \mathbb{N}$ and $m \in \mathbb{N}$.

In view of Remark 11 and Remark 13 we see that the objects appearing in the next theorem are well defined. See Proposition 5.4.6, p. 315 in [7], or also Theorem 4.5.2, p. 108 in [17] for the finite dimensional case.

Theorem 14. *Let* $0 \leqslant a < \infty$.

1° *If* $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, u)$ *is a solution of the equation* (4) *on* $[a, \infty)$ *then the law* $\overline{\mathbb{P}}$ *of the process* $(\vec{u}(t \vee a))_t$ *on* $\overline{\mathcal{F}}$ *satisfies*

$$\overline{\mathbb{P}}(M_a) = \overline{\mathbb{P}}[\pi_t \in \operatorname{Rng} e] = 1$$

for every $t \in [a, \infty)$ *and the bounded continuous processes* $L_{nm}^a(f)$ *are martingales on* $[a, \infty)$ *under* $\overline{\mathbb{P}}$ *with respect to the* $\overline{\mathbb{P}}$*-augmentation of the filtration* $(\overline{\mathcal{F}}_t)$ *in* $\overline{\mathcal{F}}$ *for every* $f \in C^2(\mathbb{R})$, $m \in \mathbb{N}$ *and* $n \in \mathbb{N}$.

2° *There exists a countable set* B *of* $C^\infty(\mathbb{R})$*-functions with compact supports, independent of* a, *such that whenever* $\overline{\mathbb{P}}$ *is a measure on* $\overline{\mathcal{F}}$ *such that*

$$\overline{\mathbb{P}}(M_a) = \overline{\mathbb{P}}[\pi_a \in \operatorname{Rng} e] = \overline{\mathbb{P}}[\pi_t \in \operatorname{Rng} e] = 1$$

for almost every $t \in [a, \infty)$ *and the bounded continuous processes* $L_{nm}^a(f)$ *are martingales on* $[a, \infty)$ *under* $\overline{\mathbb{P}}$ *with respect to the* $\overline{\mathbb{P}}$*-augmentation of the filtration* $(\overline{\mathcal{F}}_{a,t})$ *in* $\overline{\mathcal{F}}_{a,\infty}$ *for every* $f \in B$, $m \in \mathbb{N}$ *and* $n \in \mathbb{N}$, *then there exists a solution* $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, u)$ *of the equation* (4) *such that* $\overline{\mathbb{P}}$ *coincides with the law of the process* $(\vec{u}(t \vee a))_t$ *on* $\overline{\mathcal{F}}_{a,\infty}$.

The core of this section is primarily in Theorem 21 where conditional laws of solutions of (4) on $\overline{\mathcal{F}}$ are characterized, and secondarily in Theorem 20 which states that weak existence and uniqueness in law for the equation (4) with deterministic initial conditions is sufficient for Borel measurability of laws relative to solutions of (4). In fact, the rest of the paper concerning weak existence and uniqueness in law for the equation (4) with non-deterministic initial conditions (Corollary 22), joint measurability of the transition function relative to solutions of (4) (Corollary 23), the Markov property of solutions of (4) (Theorem 24), the semigroup property of Markov operators (Corollary 25), and the strong Markov property of solutions of (4), is a mere consequence of Theorems 20 and Theorem 21.

Definition 15. We denote by $\mathbb{M}(Z)$ the Polish space of Borel probability measures on a Polish space Z equipped with the narrow topology, i.e. the topology induced by the mappings

$$\mathbb{M}(Z) \to \mathbb{R}\colon\ \mu \mapsto \int_Z h\,\mathrm{d}\mu$$

for every real bounded continuous function h on Z. Moreover, we will write $\mathbb{M}$ instead of $\mathbb{M}(C([0,\infty),\mathbb{R}^{\mathbb{N}}))$ and define a shift operator

$$\theta\colon\ \mathbb{R}\times C([0,\infty),\mathbb{R}^{\mathbb{N}}) \to C([0,\infty),\mathbb{R}^{\mathbb{N}})\colon\ (y,f)\mapsto (f((s+y)^+)\colon\ s\geqslant 0).$$

Definition 16. Let $a\geqslant 0$ and $x\in X$. We say that the equation (4) is (a,x)-unique in law provided that whenever $(\Omega^i,\mathcal{F}^i,(\mathcal{F}^i_t),\mathbb{P}^i,W^i,u^i)$, $i=1,2$ are two solutions of the equation (4) on $[a,\infty)$ such that $\mathbb{P}^1[u^1(a)=x]=\mathbb{P}^2[u^2(a)=x]=1$ then $\mathfrak{Law}_{\mathbb{P}^1}\left(u^1(t_i)\colon\ i\leqslant k\right)=\mathfrak{Law}_{\mathbb{P}^2}\left(u^2(t_i)\colon\ i\leqslant k\right)$ for every finite family of times $\{t_1,\dots,t_k\}$ in $[a,\infty)$.

Remark 17. If (a,x)-uniqueness in law holds for the equation (4) then the laws of $\vec{u}$, $\vec{v}$ coincide on $\overline{\mathcal{F}}$ whenever $(\Omega^1,\mathcal{F}^1,(\mathcal{F}^1_t),\mathbb{P}^1,W^1,u)$, $(\Omega^2,\mathcal{F}^2,(\mathcal{F}^2_t),\mathbb{P}^2,W^2,v)$ are two solutions of the equation (4) on $[a,\infty)$ such that

$$\mathbb{P}^1[u(a)=x]=\mathbb{P}^2[v(a)=x]=1.$$

Definition 18. Let $a\geqslant 0$ and $x\in X$ be given and suppose that there exists a solution $(\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P},W,u)$ of the equation (4) on $[a,\infty)$ such that $\mathbb{P}[u(a)=x]=1$ and the equation (4) is (a,x)-unique in law. Then we say that *the equation* (4) *is* (a,x)*-well-posed.* Moreover, we denote by $P^{a,x}$ the law of the process $(\vec{u}(t\vee a))_t$ on $\overline{\mathcal{F}}$ and define $\mathscr{P}(a,x,t,V)=\mathbb{P}[u(t)\in V]$ for $t\geqslant a$ and $V\in\mathbb{B}(X)$.

The next theorem states that well-posedness is, in fact, sufficient for Borel measurability of the family of probability measures $\{P^{a,x}\}_{a,x}$ provided that the equation (4) is autonomous, and that the shift operator θ is closely related to this system of probability measures.

Theorem 19. *Let the coefficients F, G and the J-function J be time independent, i.e. $J(t,x) = J(x)$, $F(t,x) = F(x)$ and $G(t,x) = G(x)$ for every $t \geqslant 0$ and $x \in X$.*

- ▷ *If the equation (4) is (a,x)-well-posed for some $a \in [0,\infty)$ and $x \in X$ then the equation (4) is (b,x)-well-posed for every $b \in [0,\infty)$.*
- ▷ *If O is a Borel subset of X and (4) is $(0,x)$-well-posed for every $x \in O$ then the mapping $[0,\infty) \times O \to \mathbb{M}\colon\ (a,x) \mapsto P^{a,x}$ is Borel measurable.*

In both cases $P^{b,x} = \mathfrak{Law}_{P^{a,x}}(\theta_{a-b})$ on $\overline{\mathcal{F}}$ for every $a \geqslant 0$, $b \geqslant 0$ and $x \in X$.

Theorem 20 is an infinite dimensional version of 6.7.4, p. 167 in [17] and a generalization of the second part (Borel measurability of the family of probability measures) of Theorem 19 to non-autonomous equations.

Theorem 20. *Let E be a nonempty Borel subset of $[0,\infty) \times X$ and let the equation (4) be (a,x)-well-posed for every $(a,x) \in E$. Then the function $E \to \mathbb{M}\colon\ (a,x) \mapsto P^{a,x}$ is Borel measurable.*

Now we are coming to the essential theorem of this section where regular versions of conditional probabilities with respect to filtrations stopped at a random time are characterized via the family $\{P^{a,x}\}_{a,x}$. Its finite dimensional predecessor is due to Stroock and Varadhan Theorem 6.2.2, p. 146 in [17].

Theorem 21. *Let $a \geqslant 0$ and let O be a nonempty Borel subset of X. Let the equation (4) be (b,x)-well-posed for every $b \geqslant a$ and $x \in O$. Further, let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, u)$ be a solution of the equation (4) on $[a,\infty)$ and denote by P the law of the process $(\vec{u}(t \vee a))_t$ on $\overline{\mathcal{F}}$. Let $(\mathcal{G}_t)$ be a filtration of $\overline{\mathcal{F}}$ such that $\overline{\mathcal{F}}_t \subseteq \mathcal{G}_t \subseteq \overline{\mathcal{F}}_t^P$ for every $t \geqslant 0$, and let τ be an $[a,\infty)$-valued $(\mathcal{G}_t)$-stopping time. Assume that $P[\pi_\tau \in e[O]] = 1$ and denote by $r\colon \overline{\Omega} \times \overline{\mathcal{F}} \to \mathbb{R}$ a regular version of the conditional probability P with respect to $\mathcal{G}_\tau$. Then there exists a set N in $\mathcal{G}_\tau$, $P(N) = 1$, $N \subseteq [\pi_\tau \in e[O]]$ such that*

$$r(\omega, V) = P^{\tau(\omega), e^{-1}(\pi_{\tau(\omega)}(\omega))}(V)$$

for every $V \in \overline{\mathcal{F}}_{\tau(\omega),\infty}$ and every $\omega \in N$.

Theorem 21 has two immediate corollaries. Corollary 22 is about the weak existence and the uniqueness in law for general initial distribution and Corollary 23 deals with the measurability of the transition function $\{\mathscr{P}(a,x,t,V)\colon\ a,x,t\}$.

Corollary 22. *Let O be a Borel subset of X and μ a probability measure on $\mathbb{B}(X)$ such that $\mu(O) = 1$. Suppose that the equation* (4) *is (a, x)-well-posed for every $a \geqslant 0$ and $x \in O$. Then*

- *there exists a solution $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, u)$ of the equation* (4) *on $[a, \infty)$ such that $\mathfrak{Law}_{\mathbb{P}}\,(u(a)) = \mu$;*
- *whenever $(\Omega^k, \mathcal{F}^k, (\mathcal{F}^k_t), W^k, u^k)$, $k = 1, 2$, are two solutions of the equation* (4) *on $[0, \infty)$ such that $\mathfrak{Law}_{\mathbb{P}^1}\left(u^1(0)\right) = \mathfrak{Law}_{\mathbb{P}^2}\left(u^2(0)\right) = \mu$ then*

$$\mathfrak{Law}_{\mathbb{P}^1}\left(u^1(t_i)\colon\ i \leqslant m\right) = \mathfrak{Law}_{\mathbb{P}^2}\left(u^2(t_i)\colon\ i \leqslant m\right)$$

for every $t_1, \dots t_m$ in $[0, \infty)$.

Corollary 23. *Let O be a nonempty Borel subset of X and let the equation* (4) *be (a, x)-well-posed for every $a \geqslant 0$ and $x \in O$. Then the transition function*

$$(a, x, t) \mapsto \mathscr{P}(a, x, t, V)\colon\ \{(a, x, t)\colon\ 0 \leqslant a \leqslant t < \infty, x \in O\} \to \mathbb{R}$$

is jointly measurable for every $V \in \mathbb{B}(X)$.

5. Markov and strong Markov property

In this section we aim at proving that the famous Stroock-Varadhan theorem [17] holds for stochastic evolution equations in Banach spaces, too. More precisely, we prove that well-posedness of the equation (4) implies that the solutions of (4) define a (strong) Markov process. It is known that the equation (4) defines a (strong) Markov process if the coefficients have a particular form—for instance, equations with Lipschitz coefficients (Chapter 9.2 in [3]) or 2D stochastic Navier-Stokes equation (in [11]). In such cases, strong solutions exist and are pathwise unique, hence the equation is well-posed by the Yamada-Watanabe theorem (e.g. Theorem 2 in [14]) and so the results of this section are applicable.

In the following, we will consider two different settings of reference stochastic bases where a Markov and a strong Markov process, respectively, will be constructed. In the first case where Ω is a product space and no additional regularity of paths of solutions of the equation (4) is needed we prove that the laws of solutions of the equation (4) together with the canonical process define a Markov process and the Markov kernel defines a semigroup on the space of bounded Borel functions. In the other case, Ω is a space of continuous functions and we assume additionally that the paths of solutions of the equation (4) are continuous. Here we prove that the laws of solutions of the equation (4) together with the canonical process define a (strong) Markov process.

- Let O be a non empty Borel subset in X and denote

▷ $\mathcal{O} = \{C \in \mathbb{B}(X)\colon\ C \subseteq O\}$,

▷ $\Omega = O^{[0,\infty)} = \{\omega\colon\ [0,\infty) \to O\}$,

▷ $Y_t\colon\ \Omega \to O\colon\ \omega \mapsto \omega(t)$, $t \geqslant 0$,

▷ $\mathcal{F} = \mathcal{O}^{[0,\infty)} = \sigma(Y_t\colon\ t \geqslant 0)$,

▷ $\mathcal{F}_t = \sigma(Y_s\colon\ s \leqslant t)$, $t \geqslant 0$.

The following theorems are essentially applications of Theorem 21.

Theorem 24. *Let the equation (4) be (a,x)-well-posed for every $a \geqslant 0$ and $x \in O$. Further, suppose that, for every $a \geqslant 0$, $x \in O$, there exists a solution*

$$(\Omega_{a,x}, \mathcal{F}_{a,x}, (\mathcal{F}_{a,x;t}), \mathbb{P}_{a,x}, W_{a,x}, u_{a,x})$$

of the equation (4) on $[a,\infty)$ such that $\mathbb{P}_{a,x}[u_{a,x}(a) = x,\ u_{a,x}(t) \in O] = 1$ for every $t \geqslant a$. Denote by $\mathbb{P}^{a,x}$ the law of the process $(u_{a,x}(t \vee a))_t$ on $\mathcal{F}$. Then $(\Omega, \mathcal{F}, (\mathcal{F}_t), Y, (\mathbb{P}^{a,x})_{a \geqslant 0, x \in O})$ is a Markov family with the transition probability $\mathscr{P}$, i.e. the mapping

$$[0,\infty) \times (O,\mathcal{O}) \to \mathbb{R}\colon\ (a,x) \mapsto \mathbb{P}^{a,x}(V)$$

is jointly measurable for every $V \in \mathcal{F}$, and if we denote by $(\mathcal{H}_t)$ the augmentation of the filtration $(\mathcal{F}_t)$ with respect to $\mathbb{P}^{a,x}$ for $a \geqslant 0$, $x \in O$ fixed, and $b \leqslant c$ are fixed times in $[a,\infty)$ then

$$\mathbb{P}^{a,x}[Y_c \in V | \mathcal{H}_b] = \mathscr{P}(b, Y_b, c, V) \qquad \mathbb{P}^{a,x}\text{-almost surely}$$

for every $V \in \mathcal{O}$.

Let the equation (4) be (a,x)-well-posed for every $a \geqslant 0$ and $x \in O$ and define an operator

$$\mathscr{P}_{a,b}\,\varphi(x) = \int_{\Omega_{a,x}} \varphi(u_{a,x}(b))\,\mathrm{d}\mathbb{P}_{a,x} = \int_{\Omega} \varphi(Y_b)\,\mathrm{d}\mathbb{P}^{a,x} = \int_{O} \varphi\,\mathrm{d}\mathscr{P}(a,x,b,\cdot)$$

for $0 \leqslant a \leqslant b < \infty$, $x \in O$ and $\varphi\colon\ O \to \mathbb{R}$ bounded and Borel measurable. Corollary 23 and Theorem 24 have an immediate consequence:

Corollary 25. *Let the assumptions in Theorem 24 hold, let $0 \leqslant a \leqslant b \leqslant c < \infty$ and let $\varphi\colon O \to \mathbb{R}$ be a bounded and Borel measurable function. Then $\mathscr{P}_{a,b}\varphi\colon O \to \mathbb{R}$ is bounded and Borel measurable, and $\mathscr{P}_{a,c}\varphi = \mathscr{P}_{a,b}(\mathscr{P}_{b,c}\varphi)$.*

- Let O be a linear subspace of X, consider a norm $\|\cdot\|_O$ on O such that $(O, \|\cdot\|_O)$ is a separable Banach space continuously embedded in X and denote

▷ $Y_t\colon C([0,\infty),(O,\|\cdot\|_O)) \to O\colon \omega \mapsto \omega(t)$, $t \geqslant 0$,

▷ $\Omega = C([0,\infty),(O,\|\cdot\|_O))$,

▷ $\mathcal{F} = \mathbb{B}(C([0,\infty),(O,\|\cdot\|_O))) = \sigma(Y_t\colon\ t \geqslant 0)$,

▷ $\mathcal{F}_t = \sigma(Y_s\colon\ s \leqslant t)$, $t \geqslant 0$.

Remark 26. The assumption of the continuous injection of O into X implies that $\mathbb{B}(O) \subseteq \mathbb{B}(X)$ by Chap. 3, Par. 39, Sect. IV, p. 487 in [9]. In particular, O is a Borel subset of X and $\mathbb{B}(O) = \mathcal{O} = \{C \in \mathbb{B}(X)\colon\ C \subseteq O\}$.

Theorem 27. *Let the equation (4) be (a,x)-well-posed for every $a \geqslant 0$ and $x \in O$. Further, suppose that, for every $a \geqslant 0$, $x \in O$, there exists a solution*

$$(\Omega_{a,x}, \mathcal{F}_{a,x}, (\mathcal{F}_{a,x;t}), \mathbb{P}_{a,x}, W_{a,x}, u_{a,x})$$

of the equation (4) on $[a,\infty)$ such that $\mathbb{P}_{a,x}[u_{a,x}(a) = x] = 1$ and the paths of $u_{a,x}$ are continuous in $(O,\|\cdot\|_O)$, and denote by $\mathbb{P}^{a,x}$ the law of the process $(u_{a,x}(t \vee a))_t$ on $\mathbb{B}(C([0,\infty),(O,\|\cdot\|_O)))$. Then $(\Omega, \mathcal{F}, (\mathcal{F}_t), Y, (\mathbb{P}^{a,x})_{a\geqslant 0, x\in O})$ is a strong Markov family with the transition probability $\mathscr{P}$, i.e. the function

$$[0,\infty) \times (O, \mathcal{O}) \to \mathbb{R}\colon\ (a,x) \mapsto \mathbb{P}^{a,x}(V)$$

is jointly measurable for every $V \in \mathcal{F}$, and if we denote by $(\mathcal{H}_t)$ the augmentation of the filtration $(\mathcal{F}_t)$ with respect to $\mathbb{P}^{a,x}$ for $a \geqslant 0$, $x \in O$ fixed, and τ is an $(\mathcal{H}_t)$-stopping time with values in $[a,\infty)$ then

$$\mathbb{P}^{a,x}[Y_{t+\tau} \in V | \mathcal{H}_\tau] = \mathscr{P}(\tau, Y_\tau, t+\tau, V) \qquad \mathbb{P}^{a,x}\text{-almost surely}$$

for every $t \geqslant 0$ and $V \in \mathcal{O}$.

Proof of Theorem 2.

First we will give a classical result on real local martingales. Then we recall the definition of the stochastic integral with respect to a cylindrical martingale and finally we conclude by the actual proof of Theorem 2.

The following proposition is an easy consequence of the Lenglart inequality (Lemma III.6.3, p. 106 in [8]):

Proposition 28. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space and $(M_n\colon\ n \in \mathbb{N})$ a sequence of real continuous local $(\mathcal{F}_t)$-martingales on $[0,\infty)$ starting from zero and, for every $\delta > 0$, $\Delta > \infty$ and $t > 0$. Let there exist $n_0 \in \mathbb{N}$ such that whenever $m \geqslant n_0$ and $n \geqslant n_0$ then $\mathbb{P}[\langle M_n - M_m\rangle(t) \geqslant \delta] \leqslant \Delta$. Then there exists a continuous local martingale M such that $|M_n - M| + \langle M_n - M\rangle + |\langle M_n\rangle - \langle M\rangle|$ converges to zero in $C([0,\infty))$ in probability.*

Now we will use Proposition 28 to show that the family of continuous local martingales $(M(x^*)\colon\ x^* \in D)$ in Theorem 2 can be extended in a unique way to a family of continuous local martingales $(M(x^*)\colon\ x^* \in X^*)$ with the same quadratic variation.

Corollary 29. *Let g and D be the same as in Theorem 2 and let $(Z(x^*)\colon\ x^* \in D)$ be a family of continuous local $(\mathcal{F}_t)$-martingales starting from zero such that (3) holds for Z. Then Z can be extended in a unique way for indices $x^* \in X^* \setminus D$ so that $(Z(x^*)\colon\ x^* \in X^*)$ is a family of continuous local $(\mathcal{F}_t)$-martingales starting from zero such that*

$$\langle Z(x^*)\rangle_t = \int_0^t \|g^* x^*\|_{U_0}^2 \,\mathrm{d}s \tag{9}$$

and

$$Z_t(ax^* + y^*) = aZ_t(x^*) + Z_t(y^*) \tag{10}$$

hold for every $t \geqslant 0$, $a \in \mathbb{R}$, $x^* \in X^*$ and $y^* \in X^*$.

Proof. Z may be clearly extended in a unique way to the linear span of D by algebraic operations so that $Z(x^*)$ is a continuous local $(\mathcal{F}_t)$-martingale and (9) and (10) hold just by bilinearity of the cross-variation operator $\langle\cdot,\cdot\rangle$ and $(x^*, y^*) \mapsto \int_0^\cdot \langle g_s^* x^*, g_s^* y^*\rangle_{U_0}\,\mathrm{d}s$, and by the fact that $\langle N\rangle = 0$ only for the null local martingale N. Let $\mathscr{M}$ be the set of all $(N(x^*)\colon\ x^* \in D_N)$ such that

- $\triangleright$ $D \subseteq D_N \subseteq X^*$ and D_N is a linear space,
- $\triangleright$ $(N_t(x^*)\colon\ t \geqslant 0)$ is a continuous local $(\mathcal{F}_t)$-martingale for each $x^* \in D_N$,
- $\triangleright$ (9) and (10) hold for N,
- $\triangleright$ $N(x^*) = Z(x^*)$ almost surely for every $x^* \in D$,

and let us write $N^1 \preccurlyeq N^2$ provided that $D_{N^1} \subseteq D_{N^2}$ and $N^1(x^*) = N^2(x^*)$ almost surely for every $x^* \in D_{N^1}$. Then $\preccurlyeq$ is a partial ordering on $\mathscr{M}$, every ordered chain in $(\mathscr{M}, \preccurlyeq)$ has an upper bound and so, by the principle of maximality, $(\mathscr{M}, \preccurlyeq)$ contains a maximal element N. The maximality of N and Proposition 28 imply that D_N is closed if X is reflexive, and weak star sequentially closed if g is almost surely compact. Therefore, if X is reflexive, D_N is weak star closed by Mazur's theorem as the weak topology coincides with the weak star topology on reflexive Banach spaces. In fact, D_N is weak star closed even if X is not reflexive since the weak star topology is metrizable on any closed ball B in X^* by Theorem V.5.1 in [6], thus $B \cap D_N$ is weak star closed and, by definition, D_N is bounded weak star closed in X^*. Now D_N is weak star closed by the Krein-Shmulyan Theorem V.5.7 in [6]. Further, D_N separates points of X since it contains D, and so D_N is weak star dense in X^*, whence $D_N = X^*$ no matter if X is reflexive or not.

To show uniqueness, let $(\tilde{N}(x^*)\colon\ x^* \in X^*)$ be another family of continuous local $(\mathcal{F}_t)$-martingales such that (9) and (10) hold for $\tilde{N}$ and $\tilde{N}(x^*) = Z(x^*)$ almost surely for every $x^* \in D$. The set $S = \{x^* \in X^*\colon\ N(x^*) = \tilde{N}(x^*)\}$ is a vector space and contains D. Moreover, S is closed if X is reflexive, or sequentially weak star closed if X is not reflexive. We get, by the same reasoning as in the previous part of this proof, that $S = X^*$. $\square$

Now we will construct an elementary stochastic integral with respect to a cylindrical continuous local martingale with quadratic variation in an integral form:

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space, H a separable Hilbert space, c a progressively measurable process in $L(U_0, H)$ such that

$$\int_0^T \|c(s)\|^2_{L(U_0,H)}\,\mathrm{d}s < \infty \qquad \text{almost surely for every } T > 0$$

and $(N(h)\colon\ h \in H)$ is a family of continuous local $(\mathcal{F}_t)$-martingales starting from zero such that

$$\langle N(h)\rangle_t = \int_0^t \|c^* h\|^2_{U_0}\,\mathrm{d}s, \qquad t \geqslant 0,\ h \in H.$$

We define

$$\int_0^{\cdot} \psi\,\mathrm{d}N = \lim_{n\to\infty} \sum_{k=1}^n \int_0^{\cdot} \langle \psi, h_k\rangle_H\,\mathrm{d}N(h_k) \tag{11}$$

for a progressively measurable H-valued uniformly bounded process $(\psi(t)\colon\ t \geqslant 0)$ where $(h_k\colon\ k \in \mathbb{N})$ is an orthonormal basis in H, the limit is taken in probability in $C([0,\infty))$ and due to Proposition 28, it is a continuous local $(\mathcal{F}_t)$-martingale. Moreover, the definition (11) is independent of the choice of the orthonormal basis.

If $(\psi(t)\colon\ t \geqslant 0)$ is a progressively measurable H-valued process such that

$$\int_0^t \|c_s^*(\psi_s)\|_{U_0}^2 \,\mathrm{d}s < \infty \qquad \text{almost surely for every } t \geqslant 0$$

then we define

$$\int_0^{\cdot} \psi \,\mathrm{d}N = \lim_{n\to\infty} \int_0^{\cdot} I_{[\|\psi\|_X \leqslant n]} \psi \,\mathrm{d}N$$

where the limit is, again, taken in probability in $C([0,\infty))$ and due to Proposition 28, is a continuous local $(\mathcal{F}_t)$-martingale. The following two remarks are elementary and can be easily proved using Proposition 28:

Remark 30. Let, in addition, φ be another progressively measurable H-valued process such that $\|c_s^*(\varphi_s)\|_{U_0} \in L^2(0,T)$ almost surely for every $T > 0$. Then the cross-variation process satisfies

$$\left\langle \int_0^{\cdot} \psi \,\mathrm{d}N, \int_0^{\cdot} \varphi \,\mathrm{d}N \right\rangle (t) = \int_0^t \langle c^*(\psi), c^*(\varphi)\rangle_{U_0} \,\mathrm{d}s$$

for every $t \geqslant 0$.

Remark 31. Let, in addition, K be a separable Hilbert space and ψ a progressively measurable process in $L(H,K)$ such that $\|\psi c\|_{L(U_0,K)} \in L^2(0,T)$ almost surely for every $T > 0$. Then $L(k) = \int \psi^* k \,\mathrm{d}N$, $k \in K$ is a family of continuous local martingales, and if φ is a progressively measurable process in K such that $\|c^*\psi^*\varphi\|_{U_0} \,\mathrm{d}s \in L^2(0,T)$ almost surely for every $T > 0$ then

$$\int_0^{\cdot} \varphi \,\mathrm{d}L = \int_0^{\cdot} \psi^* \varphi \,\mathrm{d}N.$$

The Borel functional calculus will enter into the proof of Theorem 2. Here we point out one of its important features. To this end, let us denote by $L_s(U)$ the space of linear bounded symmetric operators on U equipped with the point σ-algebra (i.e. the σ-algebra generated by the mappings $B \mapsto Bu$ for every $u \in U$), and consider the Borel functional calculus on $L_s(U)$.

Proposition 32. *The mapping $L_s(U) \to L_s(U)\colon B \mapsto f(B)$ is measurable for every locally bounded Borel function $f\colon \mathbb{R} \to \mathbb{R}$.*

P r o o f. The Borel functional calculus is defined via finite signed measures $\mu^B_{x,y}$ with support in the spectrum of B, for $x \in X$, $y \in X$ and $B \in L_s(U)$ where

$$\langle f(B)x, y\rangle_U = \int_{-\infty}^{+\infty} f \,\mathrm{d}\mu^B_{x,y}$$

for every locally bounded Borel measurable function f.

The mapping $B \mapsto f(B)$ is measurable by definition for $f(t) = t^n$, $n \in \mathbb{N}$, since then $f(B) = B^n$. Indeed, the mapping $B \mapsto B^n$ is measurable for $n = 1$ by definition. Proceeding by induction, suppose that it is measurable for some $n \geqslant 1$. Then

$$B^{n+1}x = \sum_j \langle B^n x, e_j\rangle_U \, Be_j$$

for an orthonormal basis $(e_j)_j$ in U and every $x \in U$, so $B \mapsto B^{n+1}$ is measurable. Consequently, $B \mapsto f(B)$ is measurable for every polynomial f.

If f is continuous then there exist polynomials p_n converging to f uniformly on every compact in $\mathbb{R}$, hence $\langle p_n(B)x, y\rangle_U$ converges to $\langle f(B)x, y\rangle_U$ for every B and $x, y \in U$, so $B \mapsto f(B)$ is measurable.

If f is the indicator function of a closed set $F \subseteq \mathbb{R}$ then there exist continuous functions $p_n \in [0, 1]$ converging to f pointwise, hence $\langle p_n(B)x, y\rangle_U$ converges to $\langle f(B)x, y\rangle_U$ for every B and $x, y \in U$ by Lebesgue's theorem, and $B \mapsto f(B)$ is measurable.

The system of Borel subsets of $\mathbb{R}$ for which $B \mapsto I_F(B)$ is measurable, is a Dynkin system containing closed sets, and so $B \mapsto I_F(B)$ is measurable for every Borel set F. Consequently, $B \mapsto f(B)$ is measurable for every finite valued Borel function f.

Finally, if f is Borel measurable and locally bounded then it is a pointwise limit of simple functions p_n where p_n are uniformly bounded (with respect to n) on every compact in $\mathbb{R}$ and so $\langle p_n(B)x, y\rangle_U$ converges to $\langle f(B)x, y\rangle_U$ for every B and $x, y \in U$ by Lebesgue's theorem. □

Remark 33. Let $B \in L(U, H)$ where H is a Hilbert space and let f be a locally bounded Borel function. Then, mimicking the proof of Proposition 32 (i.e. f is a monomial, a polynomial, a continuous function, the indicator of a closed set, the indicator of a Borel set, a simple Borel function, a locally bounded Borel function), we get that $Bf(B^*B)B^* = f_0(BB^*)$ where $f_0(t) = tf(t)$. Consequently, choosing $f = I_{\{0\}}$, we get $BI_{\{0\}}(B^*B)B^* = 0$ and so $I_{\{0\}}(B^*B) = 0$ provided B is injective.

P r o o f of Theorem 2. We consider the extension of M from Corollary 29.

Step 1. Let us first suppose that X is a Hilbert space (we will identify the dual X^* with X), the covariance operator Q is the identity operator on U (therefore $U = U_0$) and there exists a cylindrical Wiener process $\overline{W}$ with identity covariance operator on U, independent of M. Further, decompose the interval $(0, \infty)$ into Borel sets $B_1, B_2, \ldots$ such that each of them has a positive distance from the origin, and define functions $\psi_i(t) = t^{-1} I_{B_i}(t)$, $i \geqslant 1$, and $\psi_0 = I_{\{0\}}$. Let us also denote $C_i = I_{B_i}$, $i \in \mathbb{N}$, and $C_0 = \psi_0$. The process $(t, \omega) \mapsto g^*(t, \omega) g(t, \omega)$ is progressively measurable in $L(U)$ and takes values in the subspace of the symmetric operators. So we have, using the Borel functional calculus (Proposition 32), the $L(U)$-valued progressively measurable processes

$$(t, \omega) \mapsto \psi_i(g^*(t, \omega) g(t, \omega)), \quad (t, \omega) \mapsto C_i(g^*(t, \omega) g(t, \omega)), \quad i \geqslant 0.$$

The processes

$$W_i(u) = \int_0^\cdot g \psi_i(g^* g)(u) \, \mathrm{d}M, \quad i = 1, 2, \ldots, \quad W_0(u) = \int_0^\cdot \psi_0(g^* g)(u) \, \overline{W}$$

are real local martingales for every $u \in U$ by Proposition 23 as

$$\| g^* g \psi_i(g^* g) \|_{L(U)} \leqslant 1, \quad i = 1, 2, \ldots, \qquad \| \psi_0(g^* g) \|_{L(U)} \leqslant 1,$$

and the cross-variation processes satisfy $\langle W_i(u_1), W_j(u_2) \rangle = 0$ for every $u_1 \in U$, $u_2 \in U$ and $0 \leqslant i < j$ since $g^* g \psi_j(g^* g) = C_j(g^* g)$ and $C_i(g^* g) C_j(g^* g) = 0$, and $\langle W_i(u) \rangle = \int_0^\cdot \| C_i(g^* g)(u) \|_U^2 \, \mathrm{d}s$ for every $u \in U$ and $i \geqslant 0$ by Remark 30. Now

$$\begin{aligned} \sum_{i=0}^\infty \langle W_i(u) \rangle(t) &= \sum_{i=0}^\infty \int_0^t \| C_i(g^* g)(u) \|_U^2 \, \mathrm{d}s = \sum_{i=0}^\infty \int_0^t \langle C_i(g^* g)(u), u \rangle_U \, \mathrm{d}s \\ &= \int_0^t \langle u, u \rangle_U \, \mathrm{d}s = t \| u \|_U^2, \end{aligned}$$

so $W(u) = \sum\limits_{i=0}^\infty W_i(u)$ is convergent in $C([0, \infty))$ in probability for every $u \in U$ by Proposition 28, and $W = (W(u) \colon u \in U)$ is a cylindrical Wiener process on U with the identity covariance operator. On the other hand, to show that $M(h) = \int_0^\cdot \langle h, g \rangle \, \mathrm{d}W$ for every $h \in X^*$, it is obviously sufficient to verify that $\int_0^\cdot \| g^* h \|_U^2 \, \mathrm{d}s$ is the cross-variation of $M(h)$ and $\int_0^\cdot \langle h, g \rangle \, \mathrm{d}W$ since

$$\left\langle M(h) - \int_0^\cdot \langle h, g \rangle \, \mathrm{d}W \right\rangle = \int_0^\cdot \| g^* h \|_U^2 \, \mathrm{d}s - 2 \left\langle M(h), \int_0^\cdot \langle h, g \rangle \, \mathrm{d}W \right\rangle + \int_0^\cdot \| g^* h \|_U^2 \, \mathrm{d}s.$$

To carry out this, recalling Remark 31, we get that

$$\left\langle \int_0^{\cdot} g^* h \, \mathrm{d}W_i, M(h) \right\rangle = \left\langle \int_0^{\cdot} g\psi_i(g^*g)g^* h \, \mathrm{d}M, \int_0^{\cdot} h \, \mathrm{d}M \right\rangle$$
$$= \int_0^{\cdot} \langle g^* g\psi_i(g^*g)g^* h, g^* h\rangle_U \, \mathrm{d}s$$

for $i = 1, 2, \dots$ by Remark 30, and

$$\left\langle \int_0^{\cdot} g^* h \, \mathrm{d}W_0, M(h) \right\rangle = \left\langle \int_0^{\cdot} \psi_0(g^*g)g^* h \, \mathrm{d}\overline{W}, M(h) \right\rangle = 0.$$

Since

$$\left\langle \sum_{i=0}^{N} \int_0^{\cdot} g^* h \, \mathrm{d}W_i - \int_0^{\cdot} g^* h \, \mathrm{d}W \right\rangle (t)$$

converges to zero for every $t \geqslant 0$ we arrive at

$$\left\langle \int_0^{\cdot} \langle h, g\cdot\rangle \, \mathrm{d}W, M(h) \right\rangle = \sum_{i=1}^{\infty} \int_0^{\cdot} \langle C_i(g^*g)g^* h, g^* h\rangle_U \, \mathrm{d}s$$
$$= \int_0^{\cdot} \|g^* h\|_U^2 \, \mathrm{d}s - \int_0^{\cdot} \langle C_0(g^*g)g^* h, g^* h\rangle_U \, \mathrm{d}s$$

but $\langle C_0(g^*g)g^*h, g^*h\rangle_U = \langle gC_0(g^*g)g^*h, h\rangle_X = \langle C(gg^*)h, h\rangle_X = 0$ where $C(t) = tC_0(t) = 0$ by Remark 33.

Step 2. If the process g is injective $dt \otimes \mathbb{P}$-almost surely then we do not need the process $\overline{W}$ since then $C_0(g^*g) = 0$ by Remark 33.

Step 3. If X is still a Hilbert space but the covariance operator Q is general then we consider $Q^{1/2}$, the square root of Q in $L(U)$, and K, the orthogonal complement of $\operatorname{Ker} Q^{1/2}$ in U considered with the norm of U. The operator $gQ^{1/2}$ belongs to $L(K, X)$ and $\|g^*h\|_{U_0} = \|(gQ^{1/2})^*h\|_K$ for every $h \in X$ since $Q^{1/2}\colon K \to U_0$ is an isometry. Moreover, $g\colon U_0 \to X$ is injective if and only if $gQ^{1/2}\colon K \to X$ is. Applying the above result we get that

$$M(h) = \int_0^{\cdot} \langle h, gQ^{1/2}\cdot\rangle \, \mathrm{d}\widetilde{W} = \int_0^{\cdot} \langle h, g\cdot\rangle \, \mathrm{d}W, \qquad h \in X$$

for some cylindrical Wiener process $\widetilde{W}$ with the identity covariance on K. Finally, $W(u) = \widetilde{W}(Q^{1/2}u)$, $u \in U$ is a Q-Wiener process on U.

Step 4. Let X be a separable Banach space. Choose a subset $\{x_n^*\colon n \in \mathbb{N}\}$ of the unit sphere of X^* which separates points of X and define H as the completion of X in the norm $\|x\|_H^2 = \sum_{n=1}^{\infty} 2^{-n} |\langle x_n^*, x\rangle|^2$. Then the inclusion $j\colon X \to H$ is continuous and we apply the previous result to the processes $N(h) = M(j^*h)$, $h \in H$. The quadratic variation of $N(h)$ is then $\int_0^{\cdot} \|(jg)^*h\|_{U_0}^2 \, \mathrm{d}s$ and so there exists a Q-Wiener process W such that

$$M(j^*h) = N(h) = \int_0^{\cdot} \langle h, jg\cdot \rangle \, \mathrm{d}W = \int_0^{\cdot} (j^*h) \circ g \, \mathrm{d}W, \qquad h \in H.$$

Consequently,

$$M(x^*) = \int_0^{\cdot} x^* \circ g \, \mathrm{d}W, \qquad x^* \in \operatorname{Rng} j^*. \tag{12}$$

In fact, (12) holds for every $x^* \in X^*$ by the uniqueness part of Corollary 29. $\square$

Proof of Theorem 14.

We will need the following Lemma which is a modification of Proposition 4.6, Chapter 5, p. 315 in [7]:

Lemma 34. *Let $0 \leqslant a < \infty$ and let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space with a real continuous adapted process π and real progressively measurable processes b_1 and b_2 with locally integrable paths. Moreover, suppose that the processes*

$$M(t) = \pi(t) - \pi(a) - \int_a^t b_1(s) \, \mathrm{d}s$$

and

$$Z(t) = \pi^2(t) - \pi^2(a) - 2\int_a^t \pi(s) b_1(s) \, \mathrm{d}s - \int_a^t b_2(s) \, \mathrm{d}s$$

are local martingales. Then

$$t \mapsto \int_a^t b_2(s) \, \mathrm{d}s$$

is the quadratic variation of M.

Proof. The process π is a semimartingale, so

$$\pi^2(t) = \pi^2(a) + 2\int_a^t \pi(s) b_1(s) \, \mathrm{d}s + 2\int_a^t \pi(s) \, \mathrm{d}M(s) + \langle M \rangle(t)$$

by Ito's formula. But then

$$Z(t) - 2\int_a^t \pi(s) \, \mathrm{d}M(s) = \langle M \rangle(t) - \int_a^t b_2(s) \, \mathrm{d}s.$$

The left hand side is a local martingale while the right hand side is a process of bounded variation. Hence it is null everywhere. $\square$

We will need the following proposition on the preservation of laws of Bochner integrals (see e.g. Corollary 8.2 and Theorem 8.3 in [14] for a proof) for proving Theorem 14.

Proposition 35. *Suppose that* $(f^i(t)\colon\ t \leqslant T)$ *is a* $[0,\infty]$*-valued measurable process on* $(\Omega^i, \mathcal{F}^i, P^i)$, $i = 1, 2$ *such that*

$$\mathfrak{Law}_{P^1}\big(f^1(r_l)\colon\ l \leqslant m\big) = \mathfrak{Law}_{P^2}\big(f^2(r_l)\colon\ l \leqslant m\big)$$

for every partition $0 = r_0 < \ldots < r_m \leqslant T$. *Then*

$$P^1\bigg[\int_0^T f^1(s)\,\mathrm{d}s < \infty\bigg] = P^2\bigg[\int_0^T f^2(s)\,\mathrm{d}s < \infty\bigg].$$

Moreover, if $(Y, \mathcal{Y})$ *is a measurable space,* ξ^i *a* Y*-valued random variable and* $(f_j^i(t)\colon$ $t \leqslant T)$, $j \leqslant N$, *a family of measurable processes on* $(\Omega^i, \mathcal{F}^i, P^i)$, $i = 1, 2$, *satisfying*

$$P^1\bigg[\int_0^T |f_j^1(s)|\,\mathrm{d}s < \infty\bigg] = P^2\bigg[\int_0^T |f_j^2(s)|\,\mathrm{d}s < \infty\bigg] = 1, \qquad j \leqslant N,$$

and

$$\mathfrak{Law}_{P^1}\big(f_j^1(r_l), \xi^1\colon\ j \leqslant N, l \leqslant m\big) = \mathfrak{Law}_{P^2}\big(f_j^2(r_l), \xi^2\colon\ j \leqslant N, l \leqslant m\big)$$

for every partition $0 = r_0 < \ldots < r_m \leqslant T$, *then*

$$\mathfrak{Law}_{P^1}\bigg(\int_0^{t_k} f_j^1(s)\,\mathrm{d}s, \xi^1\colon\ k, j\bigg) = \mathfrak{Law}_{P^2}\bigg(\int_0^{t_k} f_j^2(s)\,\mathrm{d}s, \xi^2\colon\ k, j\bigg)$$

for every partition $0 = t_0 < \ldots < t_n \leqslant T$.

P r o o f of Theorem 14, 1°. First we see that

$$\mathfrak{Law}_{\overline{\mathbb{P}}}\big(e^{-1}\pi_{r_i}\colon\ i \leqslant k\big) = \mathfrak{Law}_{\mathbb{P}}\,(u(r_i \vee a)\colon\ i \leqslant k) \tag{13}$$

for every $r_1, \ldots, r_k$ in $[0, \infty)$, so $\overline{\mathbb{P}}(M_a) = 1$ by (5) and Proposition 35, and

$$\overline{\mathbb{P}}[\pi_t \in \operatorname{Rng} e] = \mathbb{P}[\vec{u}(t) \in \operatorname{Rng} e] = 1$$

for every $t \geqslant a$. Fix $n \in \mathbb{N}$, $f \in C^2(\mathbb{R})$ and define an auxiliary process

$$\begin{aligned} Y(t) = {} & f(\langle h_n, u(t)\rangle) - f(\langle h_n, u(a)\rangle) \\ & - \int_a^t \dot{f}(\langle h_n, u(s)\rangle)\,(\langle A^* h_n, u(s)\rangle + \langle h_n, F(s, u(s))\rangle)\,\mathrm{d}s \\ & - \frac{1}{2}\int_a^t \ddot{f}(\langle h_n, u(s)\rangle)\,\|G^*(s, u(s))h_n\|_{U_0}^2, \qquad t \geqslant a. \end{aligned}$$

Then

$$Y(t)=\int_a^t \dot f(\langle h_n,u(s)\rangle)\,\langle h_n,G(s,u(s))\cdot\rangle\,\mathrm{d}W,\qquad t\geqslant a$$

is a local martingale by Ito's formula and

$$\mathfrak{Law}_{\overline{\mathbb{P}}}\left(e^{-1}\pi_{r_i},L_n(f)(t_j)\colon\ i\leqslant k,j\leqslant k\right)=\mathfrak{Law}_{\mathbb{P}}\left(u(r_i\vee a),Y(t_j)\colon\ i\leqslant k,j\leqslant k\right)$$

for every $r_1,\ldots,r_k$ in $[0,\infty)$ and every $t_1,\ldots,t_k$ in $[a,\infty)$ by (13) and Proposition 35. The mapping τ_m in (8) is lower-semicontinuous from $C([a,\infty))$ to $[a,\infty]$ as sets $[\tau_m\leqslant\Delta]$ are closed for every $\Delta\in\mathbb{R}$, and so

$$C([a,\infty))\to\mathbb{R}\colon\ f\mapsto f(t\wedge\tau_m(f))$$

is Borel measurable for every $t\geqslant a$. Having in mind that the Borel σ-algebra over $C([a,\infty))$ coincides with the σ-algebra generated by the projection mappings $C([a,\infty))\to\mathbb{R}\colon\ f\mapsto f(t)$, $t\geqslant a$, we conclude that

$$\mathrm{stop}\colon\ C([a,\infty))\to C([a,\infty))\colon\ f\mapsto (f(t\wedge\tau_m(f))\colon\ t\geqslant a)$$

is Borel measurable. Consequently,

$$\begin{aligned}&\mathfrak{Law}_{\overline{\mathbb{P}}}\left(e^{-1}\pi_{r_i},L_{nm}(f)(t_j)\colon\ i\leqslant k,j\leqslant k\right)\\ &=\mathfrak{Law}_{\mathbb{P}}\left(u(r_i\vee a),\mathrm{stop}\,Y(t_j)\colon\ i\leqslant k,j\leqslant k\right)\end{aligned}$$

for every $r_1,\ldots,r_k$ in $[0,\infty)$ and every $t_1,\ldots,t_k$ in $[a,\infty)$, and so

$$\mathfrak{Law}_{\overline{\mathbb{P}}}(\pi_{r_i},L_{nm}(f)(t_j)\colon\ i\leqslant k,j=1,2)=$$

$$=\mathfrak{Law}_{\mathbb{P}}\left(\vec u(r_i\vee a),\mathrm{stop}\,Y(t_j)\colon\ i\leqslant k,j=1,2\right)$$

for every $a\leqslant t_1<t_2$ and $0\leqslant r_1\leqslant\ldots\leqslant r_k\leqslant t_1$, whence $L_{nm}(f)$ is a martingale on $[a,\infty)$ with respect to the $\overline{\mathbb{P}}$-augmentation of the filtration $(\mathcal{F}_t)$. □

P r o o f of Theorem 14, 2°. Let us denote $f^1(t)=t$, $f^2(t)=t^2$, $t\in\mathbb{R}$ and let f_k^1, f_k^2 be arbitrary $C^\infty(\mathbb{R})$-functions with compact supports such that $f_k^1=f^1$ and $f_k^2=f^2$ on $[-k,k]$ for every $k\in\mathbb{N}$. Let $B=\{f_k^1,f_k^2\colon\ k\in\mathbb{N}\}$ and let $\overline{\mathbb{P}}$ be the presumed probability measure on $\overline{\mathcal{F}}$ in 2° for the set B that we have just specified. Now fix $n\in\mathbb{N}$ and $m\in\mathbb{N}$. Then $L_{nm}(f_k^i)$ are bounded martingales on $[a,\infty)$ for $\overline{\mathbb{P}}$ with respect to the $\overline{\mathbb{P}}$-augmentation of the filtration $(\overline{\mathcal{F}}_{a,t})$ for $k\in\mathbb{N}$, $i=1,2$, and so are the stopped processes

$$(t,\overline{\omega})\mapsto L_{nm}(f_k^i)(t\wedge\tau_k(\pi^n(\cdot,\overline{\omega})),\overline{\omega})$$

where τ_k was defined in (8). However,

$$L_{nm}(f_k^i)(t \wedge \tau_k(\pi^n(\cdot,\overline{\omega})),\overline{\omega}) = L_{nm}(f^i)(t \wedge \tau_k(\pi^n(\cdot,\overline{\omega})),\overline{\omega})$$

and $\tau_k(\pi^n(\cdot,\overline{\omega})) \to \infty$ as $k \to \infty$, so the processes $L_{nm}(f^i)$, $i = 1,2$, are bounded martingales on $[a,\infty)$ as well, therefore $L_n(f^i)$, $i = 1,2$, are local martingales on $[a,\infty)$. Define $M(h_n) = L_n(f^1)$ and $Z_n(t) = L_n(f^2)$. Then

$$\langle M(h_n)\rangle(t) = \int_a^t \left\| G^*(s,e^{-1}\pi(s))h_n \right\|_{U_0}^2 \,\mathrm{d}s$$

by Lemma 34. We are ready to extend the objects onto a product space.

To this end, let $(\widetilde{\Omega},\widetilde{\mathcal{F}},(\widetilde{\mathcal{F}}_t),\widetilde{\mathbb{P}})$ be a filtered probability space with countably many independent real Brownian motions $(\beta^k\colon\ k \in \mathbb{N})$ and define $\Omega = \overline{\Omega} \times \widetilde{\Omega}$, $\mathcal{F} = \overline{\mathcal{F}} \otimes \widetilde{\mathcal{F}}$, $\mathbb{P} = \overline{\mathbb{P}} \otimes \widetilde{\mathbb{P}}$ and let $\mathcal{F}_t$ be the $\mathbb{P}$-augmentation of the filtration $(\overline{\mathcal{F}}_{a,t\vee a} \otimes \widetilde{\mathcal{F}}_t)$ in $\mathcal{F}$. Define processes $N(h)(t,\overline{\omega},\widetilde{\omega}) = M(h)(t\vee a,\overline{\omega})$, $\Pi(t,\overline{\omega},\widetilde{\omega}) = \pi(t,\overline{\omega})$ and $w_k(t,\overline{\omega},\widetilde{\omega}) = \beta_k(t,\widetilde{\omega})$ for $t \geqslant 0$, $k \in \mathbb{N}$, $h \in D$ and $(\overline{\omega},\widetilde{\omega}) \in \Omega$. Then $N(h)$ is a local martingale on $[0,\infty)$ for the filtration $(\mathcal{F}_t)$ and

$$\langle N(h)\rangle(t) = \int_0^t \left\| I_{[a,\infty)}(s)G^*(s,e^{-1}\Pi(s))h \right\|_{U_0}^2 \,\mathrm{d}s$$

holds for every $h \in D$. The processes $(w_k\colon\ k \in \mathbb{N})$ are independent $(\mathcal{F}_t)$-Wiener processes independent of $\sigma(N(h)\colon\ h \in D)$, so we can apply Theorem 2 obtaining a Q-$(\mathcal{F}_t)$-Wiener process W such that

$$N_t(h) = \int_0^t \left\langle h, I_{[a,\infty)}G(s,e^{-1}\Pi(s))\cdot \right\rangle \,\mathrm{d}W, \qquad h \in D,$$

or equivalently

$$\begin{aligned}\Pi^n(t) = \Pi^n(a) + \int_a^t \left(\left\langle A^*h, e^{-1}\Pi(s)\right\rangle + \left\langle h, F(s,e^{-1}\Pi(s))\right\rangle\right) \mathrm{d}s \\ + \int_a^t \left\langle h, G(s,e^{-1}\Pi(s))\cdot \right\rangle \,\mathrm{d}W \qquad \mathbb{P}\text{-almost surely}\end{aligned}$$

for every $h \in D$ and $t \in [a,\infty)$, or equivalently

$$\begin{aligned}\left\langle h, e^{-1}\Pi(t)\right\rangle = \left\langle h, e^{-1}\Pi(a)\right\rangle + \int_a^t \left(\left\langle A^*h, e^{-1}\Pi(s)\right\rangle + \left\langle h, F(s,e^{-1}\Pi(s))\right\rangle\right) \mathrm{d}s \\ + \int_a^t \left\langle h, G(s,e^{-1}\Pi(s))\cdot \right\rangle \,\mathrm{d}W \qquad \mathbb{P}\text{-almost surely}\end{aligned}$$

for every $h \in D$, $t = a$ and almost every $t \in [a, \infty)$ as

$$\mathbb{P}[\Pi_t^n = \left\langle h_n, e^{-1}\Pi(t)\right\rangle, n \in \mathbb{N}] = \mathbb{P}[\Pi_t = e(e^{-1}\Pi(t))] \geqslant \mathbb{P}[\Pi(t) \in \operatorname{Rng} e] = 1$$

for $t = a$ and almost every $t \in [a, \infty)$. If we define a process

$$u(t) = S_{t-a}e^{-1}\Pi(a) + \int_a^t S_{t-s}F(s, e^{-1}\Pi(s))\,\mathrm{d}s + \int_a^t S_{t-s}G(s, e^{-1}\Pi(s))\,\mathrm{d}W$$

for $t \geqslant a$ then u is $(\mathcal{F}_t)$-adapted and continuous in probability, hence we may assume that u is $(\mathcal{F}_t)$-predictable by Proposition I.3.2 in [3], and

$$\begin{aligned}\langle h, u(t)\rangle = {} & \left\langle h, e^{-1}\Pi(a)\right\rangle + \int_a^t \left(\left\langle A^*h, e^{-1}\Pi(s)\right\rangle + \left\langle h, F(s, e^{-1}\Pi(s))\right\rangle\right)\,\mathrm{d}s \\ & + \int_a^t \left\langle h, G(s, e^{-1}\Pi(s))\cdot\right\rangle\,\mathrm{d}W\end{aligned}$$

holds $\mathbb{P}$-almost surely for every $t \geqslant a$ and every $h \in D$ by the Chojnowska-Michalik theorem ([14] or [2]), so $\mathbb{P}[e^{-1}\Pi(t) = u(t)] = 1$ for $t = a$ and almost every $t \in [a, \infty)$. Consequently, $e^{-1}\Pi(\cdot, \omega) = u(\cdot, \omega)$ almost surely with respect to the Lebesgue measure on $[a, \infty)$ for $\mathbb{P}$-almost every $\omega \in \Omega$, and so

$$u(t) = S_{t-a}u(a) + \int_a^t S_{t-s}F(s, u(s))\,\mathrm{d}s + \int_a^t S_{t-s}G(s, u(s))\,\mathrm{d}W$$

and $\mathbb{P}[\Pi(t) = \vec{u}(t)] = 1$ for every $t \geqslant a$ by continuity of Π and $\vec{u}$. $\square$

Proofs of Theorem 19 and Theorem 20.

Lemma 36. *Let O be a Borel subset of X and let the equation* (4) *be* $(0, x)$*-well-posed for every $x \in O$. Then the mapping $O \to \mathbb{M}\colon\ x \mapsto P^{0,x}$ is Borel measurable.*

P r o o f. The mapping

$$\mathbb{M} \to \mathbb{R}\colon\ \mu \mapsto \int_{\overline{\Omega}} \varphi\,\mathrm{d}\mu$$

is Borel measurable for every bounded or $[0, \infty]$-valued measurable function $\varphi\colon\ \overline{\Omega} \to \mathbb{R}$,

$$K = M_0 \cap [\pi_0 \in \operatorname{Rng} e] \cap \left\{\omega \in \overline{\Omega}\colon\ \mathrm{d}t\,\{s \in [0, \infty)\colon\ \pi_s(\omega) \notin \operatorname{Rng} e\} = 0\right\}$$

is a Borel set since the third set is measurable by Fubini's theorem. Further, let V_1 be the set of all measures μ for which

$$\int_C L_{nm}(f)(t)\,\mathrm{d}\mu = \int_C L_{nm}(f)(s)\,\mathrm{d}\mu$$

for every C in some countable subset generating $\overline{\mathcal{F}}_s$ for every $0 \leqslant s < t$ in $\mathbb{Q}$, every $n \in \mathbb{N}$, $m \in \mathbb{N}$ and $f \in B$ where B is the countable subset of $C^2(\mathbb{R})$ introduced in Theorem 14, 2°. Then V_1 is a Borel set and $V_2 = \{\mu\colon\ \mu(K) = 1\} \cap V_1$ is the Borel set of all probability measures μ which satisfy

$$\mu(M_0) = \mu[\pi_0 \in \operatorname{Rng} e] = \mu[\pi_t \in \operatorname{Rng} e] = 1$$

for almost every $t \geqslant 0$, and render the bounded processes $L_{nm}(f)$ martingales with respect to the μ-augmentation of the filtration $(\overline{\mathcal{F}}_t)$ for every $f \in B$. Now

$$X \xrightarrow{e} \mathbb{R}^{\mathbb{N}} \xrightarrow{j} \mathbb{M}(\mathbb{R}^{\mathbb{N}}) \xleftarrow{i} \mathbb{M}$$

where e was defined in Definition 7, $j\colon\ y \mapsto \delta_y$ is a homeomorphism, and $i\colon\ \mu \mapsto \mathfrak{Law}_\mu(\pi_0)$ is continuous. By this scheme we get that

$$V_3 = i^{-1}[j[e[O]]] = \{\mu\colon \text{ there exists } x \in O \text{ such that } \mu[\pi_0 = \vec{x}] = 1\}$$

is a Borel set by Proposition 8. Consequently,

$$V = V_2 \cap V_3 = \{\mu \in V_2\colon \text{ there exists } x \in O \text{ such that } \mu[e^{-1}\pi_0 = x] = 1\}$$

is a Borel set and the mapping i is injective on V. Indeed, let μ_1 and μ_2 be two elements of V such that $i(\mu_1) = i(\mu_2) = \delta_{\vec{x}}$ for some $x \in O$. Then there exist two solutions $(\Omega^k, \mathcal{F}^k, (\mathcal{F}^k_t), \mathbb{P}^k, W^k, u^k)$, $k = 1, 2$, of the equation (4) on $[0, \infty)$ such that $\mathfrak{Law}_{\mathbb{P}^k}\big(e(u^k)\big) = \mu_k$ for $k = 1, 2$ on $\overline{\mathcal{F}}$ by Theorem 14, 2°. But then $\mathbb{P}^k[u^k(0) = x] = 1$ for $k = 1, 2$ and so, by $(0, x)$-uniqueness, we get that $\mu_1 = \mu_2$. Therefore $i\colon V \to \mathbb{M}(\mathbb{R}^{\mathbb{N}})$ is a continuous injection and so is the mapping $j^{-1}i\colon\ V \to e[O]$. In fact, $j^{-1}i$ is a bijection from V to $e[O]$ by Theorem 14, 1° due to the well-posedness of the equation (4) on O. In particular, $j^{-1}i[C]$ is a Borel set in $\mathbb{R}^{\mathbb{N}}$ for every Borel subset C of V by Chap. 3, Par 39, Sect. IV, p. 487 in [9] and so $i^{-1}j\colon\ e[O] \to \mathbb{M}$ is Borel measurable. Eventually, $i^{-1}je\colon\ O \to \mathbb{M}$ is Borel measurable. □

Proof of Theorem 19. Fix $y \geqslant -a$ and consider the shift operator θ_y from Definition 15.

Step 1: Let μ be a probability measure on $\overline{\mathcal{F}}$ such that $\mu(M_{a+y}) = \mu[\pi_s \in \operatorname{Rng} e] = 1$ for every $s \in [a+y, \infty)$ and let the processes $L^{a+y}_{nm}(f)$ be martingales on $[a+y, \infty)$ under μ with respect to the μ-augmentation of the filtration $(\overline{\mathcal{F}}_{a+y,t})$ for every $f \in C^2(\mathbb{R})$. Then $\theta_y^{-1}[M_a] = M_{a+y}$ and $L^{a+y}_{nm}(t+y, \omega) = L^a_{nm}(f)(t, \theta_y(\omega))$ for every $\omega \in M_{a+y}$ and $t \geqslant a$ since the coefficients J, F and G do not depend on time. If we define $\nu = \mathfrak{Law}_\mu(\theta_y)$ then $\nu(M_a) = \mu(\theta_y^{-1}[M_a]) = \mu(M_{a+y}) = 1$, $\nu[\pi_s \in \operatorname{Rng} e] = \mu[\pi_{s+y} \in \operatorname{Rng} e] = 1$ for every $s \in [a,\infty)$. Further,

$$\begin{aligned}
&\int_{\overline{\Omega}} I_{\Gamma_1}(\pi_{s_1}(\omega)) \dots I_{\Gamma_k}(\pi_{s_1}(\omega)) L^a_{nm}(f)(t,\omega)\,\mathrm{d}\nu \\
&= \int_{\overline{\Omega}} I_{\Gamma_1}(\pi_{s_1+y}(\omega)) \dots I_{\Gamma_k}(\pi_{s_1+y}(\omega)) L^a_{nm}(f)(t,\theta_y(\omega))\,\mathrm{d}\mu \\
&= \int_{\overline{\Omega}} I_{\Gamma_1}(\pi_{s_1+y}(\omega)) \dots I_{\Gamma_k}(\pi_{s_1+y}(\omega)) L^{a+y}_{nm}(f)(t+y,\omega)\,\mathrm{d}\mu
\end{aligned}$$

for every $s_1, \dots, s_k$ and t in $[a,\infty)$, and $\Gamma_1, \dots, \Gamma_k$ in $(\mathbb{R}^{\mathbb{N}})$. Hence the processes $L^a_{nm}(f)$ are martingales on $[a,\infty)$ under ν with respect to the ν-augmentation of the filtration $(\overline{\mathcal{F}}_{a,t})$ for every $f \in C^2(\mathbb{R})$.

Step 2: Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), W, u)$ be a solution of the equation (4) on $[a+y,\infty)$ with $\mathbb{P}[u(a+y) = x] = 1$ and denote by μ the law of the process $(\vec{u}(t \vee (a+y)))_t$. Then μ satisfies the assumptions of Step 1 by Theorem 14, 1°. Applying Theorem 14, 2° to ν we get a solution $(\Omega^1, \mathcal{F}^1, (\mathcal{F}^1_t), W^1, u^1)$ of the equation (4) on $[a,\infty)$ with $\mathbb{P}^1[u^1(a) = x] = 1$.

Step 3: Let the equation (4) be (a,x)-unique in law. Let $(\Omega^k, \mathcal{F}^k, (\mathcal{F}^k_t), W^k, u^k)$, $k = 1,2$ be two solutions of the equation (4) on $[a+y,\infty)$ with $\mathbb{P}^k[u^k(a+y) = x] = 1$, $k = 1,2$ and denote by μ^k the law of the process $(\vec{u}^k(t \vee (a+y)))_t$, $k = 1,2$. Then μ^k satisfy the assumptions of Step 1 by Theorem 14, 1°. Applying Theorem 14, 2° to ν^k we get two solution $(\Omega_k, \mathcal{F}_k, (\mathcal{F}_{kt}), W_k, u_k)$, $k = 1,2$ of the equation (4) on $[a,\infty)$ such that ν^k coincides with the law of the process $(\vec{u}_k(t \vee a))_t$ on $\overline{\mathcal{F}}_{a,\infty}$ for $k = 1,2$. In particular, $\mathbb{P}_k[u_k(a) = x] = 1$ for $k = 1,2$, and so, by (a,x)-uniqueness in law, $\nu^1 = \nu^2$ on $\overline{\mathcal{F}}_{a,\infty}$. Consequently, $\mu^1 = \mu^2$ on $\overline{\mathcal{F}}_{a+y,\infty}$. Hence $(a+y,x)$-uniqueness in law.

Step 4: Let the equation (4) be $(0,x)$-well-posed for every $x \in O$. Then $O \to \mathbb{M}\colon x \mapsto P^{0,x}$ is Borel measurable by Lemma 36. Moreover, $P^{a,x} = \mathfrak{Law}_{P^{0,x}}(\theta_{-a})$ by Step 2. Hence $[0,\infty) \times O \to \mathbb{M}\colon x \mapsto P^{a,x}$ is Borel measurable. Indeed, the mapping $\theta\colon \mathbb{R} \times \overline{\Omega} \to \overline{\Omega}$ is jointly measurable. Consider a continuous bounded function φ on $\overline{\Omega}$. Then the mapping

$$(a,x) \mapsto \int_{\overline{\Omega}} \varphi \,\mathrm{d}P^{a,x} = \int_{\overline{\Omega}} \varphi \circ \theta_{-a} \,\mathrm{d}P^{0,x}$$

is continuous in a for fixed x, and measurable in x for fixed a. The joint measurability of such a function then follows, for instance, from Carathéodory's conditions. □

Let us consider an auxiliary equation

$$d\widetilde{u} = \{\widetilde{A}\widetilde{u}(t) + \widetilde{F}(\widetilde{u}(t))\}\,\mathrm{d}t + \widetilde{G}(\widetilde{u}(t))\,\mathrm{d}W \tag{14}$$

on the separable 2-smoothable Banach space $\widetilde{X} = \mathbb{R} \times X$, the C_0-semigroup

$$\widetilde{S}_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ S_t u_2 \end{pmatrix}, \qquad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \widetilde{X}, \quad t \geqslant 0$$

with the infinitesimal generator

$$\widetilde{A} = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}, \qquad \mathrm{Dom}(\widetilde{A}) = \mathbb{R} \times D(A),$$

the nonlinearities

$$\widetilde{F} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ F(u_1^+, u_2) \end{pmatrix}, \quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \widetilde{X},$$
$$\widetilde{G} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \xi = \begin{pmatrix} 0 \\ G(u_1^+, u_2)\xi \end{pmatrix}, \quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \widetilde{X}, \quad \xi \in U_0,$$

and the function

$$\widetilde{J} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = |u_1| + J(u_1^+, u_2), \qquad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \widetilde{X}.$$

Solutions of the equation (14) are characterized by the following lemma where we write (u_1, u_2) instead of the vertical vector notation:

Lemma 37. *Let $a \geqslant 0$, $x \in X$ and let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W)$ be a filtered probability space with a Q-$(\mathcal{F}_t)$-Wiener process W. Let also (u_1, u_2) be a progressively measurable $\widetilde{X}$-valued process on $[a, \infty)$ with $\mathbb{P}[u_1(a) = a] = 1$ and $\mathbb{P}[u_2(a) = x] = 1$. Then $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, (u_1, u_2))$ is a solution of (14) on $[a, \infty)$ if and only if*
- ▷ *$\mathbb{P}[u_1(t) = t] = 1$ for every $t \geqslant a$,*
- ▷ *$(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, u_2)$ is a solution of (4) on $[a, \infty)$.*

Moreover, the equation (14) is $(a, (a, x))$-unique in law if and only if the equation (4) is (a, x)-unique in law.

Corollary 38. *Let $a \geqslant 0$ and $x \in X$. Then the equation (14) is $(a, (a, x))$-well-posed if and only if the equation (4) is (a, x)-well-posed.*

P r o o f of Theorem 20. The equation (14) is $(r, (a, x))$-well-posed for every $r \geqslant 0$ and $(a, x) \in E$ by Corollary 38. Denote by $\tilde{e}\colon \widetilde{X} \to \mathbb{R}^{\mathbb{N}}$ an embedding for the setting of the equation (14) as in Definition 7 and, provided $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, \widetilde{u} = (u_1, u_2))$ is a solution of the equation (14) on $[a, \infty)$ with $\mathbb{P}[\widetilde{u}(a) = (a, x)] = 1$, denote by

$P^{a,(a,x)}$ the law of the process $(\tilde{e}(\widetilde{u}(t \vee r)))_t$ on $\overline{\Omega}$ as in Definition 18. The mapping $E \to \mathbb{M}\colon (a,x) \mapsto P^{a,(a,x)}$ is Borel measurable by Theorem 19. But

$$\begin{aligned}
P^{a,x}[\pi_{t_1} \in \Gamma_1, \dots, \pi_{t_k} \in \Gamma_k] &= \mathbb{P}\left[u_2(t_1 \vee a) \in e^{-1}[\Gamma_1], \dots, u_2(t_k \vee a) \in e^{-1}[\Gamma_k]\right] \\
&= \mathbb{P}\left[\widetilde{u}(t_1 \vee a) \in \mathbb{R} \times e^{-1}[\Gamma_1], \dots, \widetilde{u}_2(t_k \vee a) \in \mathbb{R} \times e^{-1}[\Gamma_k]\right] \\
&= \mathbb{P}\left[\tilde{e}(\widetilde{u}(t_1 \vee a)) \in \tilde{e}\left[\mathbb{R} \times e^{-1}[\Gamma_1]\right], \dots, \tilde{e}(\widetilde{u}_2(t_k \vee a)) \in \tilde{e}\left[\mathbb{R} \times e^{-1}[\Gamma_k]\right]\right] \\
&= P^{a,(a,x)}\left[\pi_{t_1} \in \tilde{e}\left[\mathbb{R} \times e^{-1}[\Gamma_1]\right], \dots, \pi_{t_k} \in \tilde{e}\left[\mathbb{R} \times e^{-1}[\Gamma_k]\right]\right]
\end{aligned}$$

holds for every $t_1, \dots, t_k$ in $[0,\infty)$ and $\Gamma_1, \dots \Gamma_k$ in $\mathbb{B}(\mathbb{R}^{\mathbb{N}})$ by Lemma 37. Hence $E \to \mathbb{M}\colon (a,x) \mapsto P^{a,x}$ is Borel measurable. □

Proof of Theorem 21.

Step 1: First we realize that $P(M_a) = P[\pi_s \in \operatorname{Rng} e] = 1$ for every $s \geqslant a$ and the processes $L^a_{nm}(f)$, $f \in C^2(\mathbb{R})$ are martingales on $[a,\infty)$ under P with respect to the P-augmentation of the filtration $(\overline{\mathcal{F}}_t)$ by Theorem 14, 1°. Now let $a \leqslant s \leqslant t$ and $C \in \overline{\mathcal{F}}_s$ be arbitrary. We have

$$\begin{aligned}
&\int_{C_1 \cap [\tau \leqslant s]} \left(\int_C L^a_{nm}(f)(t)\, \mathrm{d}r(\omega) \right) \mathrm{d}P(\omega) = \int_{C \cap C_1 \cap [\tau \leqslant s]} L^a_{nm}(f)(t)\, \mathrm{d}P(\omega) \\
&= \int_{C \cap C_1 \cap [\tau \leqslant s]} L^a_{nm}(f)(s)\, \mathrm{d}P(\omega) = \int_{C_1 \cap [\tau \leqslant s]} \left(\int_C L^a_{nm}(f)(s)\, \mathrm{d}r(\omega) \right) \mathrm{d}P(\omega)
\end{aligned}$$

for every $C_1 \in \mathcal{G}_\tau$ since $C_1 \cap [\tau \leqslant s] \in \mathcal{G}_s \cap \mathcal{G}_\tau$, $\mathcal{G}_s \subseteq \overline{\mathcal{F}}^P_s$ and due to the martingale property of $L^a_{nm}(f)$. In particular,

$$I_{[\tau \leqslant s]}(\omega) \int_C L^a_{nm}(f)(t)\, \mathrm{d}r(\omega) = I_{[\tau \leqslant s]}(\omega) \int_C L^a_{nm}(f)(s)\, \mathrm{d}r(\omega) \tag{15}$$

P-almost everywhere.

Step 2: Considering the countable set B from Theorem 14, 2° and the fact that every $\overline{\mathcal{F}}_s$ is generated by some countable algebra $\mathcal{A}_s$, we can find a set $N_1 \in \mathcal{G}_\tau$, $P(N_1) = 1$ such that (15) holds for every $C \in \mathcal{A}_s$, $f \in B$, $n \in \mathbb{N}$, $m \in \mathbb{N}$, $\omega \in N_1$ and $s \leqslant t$ in $[a,\infty) \cap \mathbb{Q}$. We can suppose as well that $r(\omega, M_a) = 1$ for every $\omega \in N_1$ since $P(M_a) = 1$. Now we see that, given $\omega \in N_1$,

$$\int_C L^a_{nm}(f)(t)\, \mathrm{d}r(\omega) = \int_C L^a_{nm}(f)(s)\, \mathrm{d}r(\omega)$$

holds for every $\tau(\omega) \leqslant s < t$, $C \in \overline{\mathcal{F}}_s$, $n \in \mathbb{N}$, $m \in \mathbb{N}$ and $f \in B$ by the continuity of the processes $L^a_{nm}(f)$. Moreover, $L^a_{nm}(f)$ are adapted to the $r(\omega)$-augmentation of the filtration $(\overline{\mathcal{F}}_{a,t})$ since $r(\omega, M_a) = 1$. Altogether, we have just got that

$$(L^a_{nm}(f)(t) - L^a_{nm}(f)(\tau(\omega))\colon\ t \geqslant \tau(\omega))$$

are uniformly bounded martingales under $r(\omega)$ on $[\tau(\omega),\infty)$ with respect to the $r(\omega)$-augmentation of the filtration $(\overline{\mathcal{F}}_{a,t})$ for every $\omega\in N_1$, $n\in\mathbb{N}$, $m\in\mathbb{N}$ and $f\in B$. The same is, of course, true for the stopped process

$$(L^a_{nm}(f)(t\wedge\sigma_k)-L^a_{nm}(f)(\tau(\omega))\colon\ t\geqslant\tau(\omega))$$

by the optional stopping theorem for the $(\overline{\mathcal{F}}^{r(\omega)}_{a,t})$-stopping time

$$\sigma_k=\inf\{s\geqslant\tau(\omega)\colon\ |L^{\tau(\omega)}_n(f)(s)|\geqslant k\}=\tau^{\tau(\omega)}_k(L^{\tau(\omega)}_n(f))$$

by definition in (8). But, still with an $\omega\in N_1$ fixed,

$$L^a_{nm}(f)(t\wedge\sigma_k)-L^a_{nm}(f)(\tau(\omega))=I_{[\tau^a_m(L^a_n(f))\geqslant\tau(\omega)]}L^{\tau(\omega)}_{nk}(t\wedge\tau^a_m(L^a_n(f)))$$

on the set M_a by definition. Hence, letting m tend to infinity, we get that the processes

$$(L^{\tau(\omega)}_{nk}(t)\colon\ t\geqslant\tau(\omega))$$

are uniformly bounded martingales under $r(\omega)$ on $[\tau(\omega),\infty)$ with respect to the $r(\omega)$-augmentation of the filtration $(\overline{\mathcal{F}}_{a,t})$ for every $\omega\in N_1$, $n\in\mathbb{N}$, $k\in\mathbb{N}$ and $f\in B$.

Step 3: We already know that $r(\omega,M_a)=1$ for every $\omega\in N_1$. Define the set

$$\begin{aligned}V_a&=\left\{\omega\in\overline{\Omega}\colon\ \int_a^\infty I_{\mathbb{R}^{\mathbb{N}}\setminus\operatorname{Rng}e}(\pi_s(\omega))\,\mathrm{d}s=0\right\}\\&=\left\{\omega\in\overline{\Omega}\colon\ \pi_s(\omega)\in\operatorname{Rng}e\text{ for a.e. }s\geqslant a\right\}.\end{aligned}$$

Then $P(V_a)=1$ by Theorem 14, 1°. Hence, there exists a set $N_2\in\mathcal{G}_\tau$, $P(N_2)=1$ such that $r(\omega,V_a)=1$ for every $\omega\in N_2$.

Step 4: There exists a set $N_3\in\mathcal{G}_\tau$, $P(N_3)=1$, such that $r(\omega,[\pi_{\tau(\omega)}(\omega)=\pi_{\tau(\omega)}])=1$ for every $\omega\in N_3$ by the elementary proprieties of r and the fact that $\overline{\mathcal{F}}_t\subseteq\mathcal{G}_t$, $t\geqslant 0$ (hence $(\omega,\omega_0)\mapsto\pi_{\tau(\omega)}(\omega)-\pi_{\tau(\omega)}(\omega_0)$ is $\mathcal{G}_\tau\otimes\overline{\mathcal{F}}$-measurable), and $\pi_{\tau(\omega)}(\omega)\in e[O]$ for every $\omega\in N_3$ by the assumptions of Theorem 21. So, given $\omega\in N=N_1\cap N_2\cap N_3$, we have

$$r\left(\omega,[\pi_{\tau(\omega)}\in\operatorname{Rng}e]\right)\geqslant r\left(\omega,[\pi_{\tau(\omega)}=\pi_{\tau(\omega)}(\omega)]\right)=1$$

and so there exists a solution $(\widetilde{\Omega},\widetilde{\mathcal{F}},(\widetilde{\mathcal{F}}_t),\widetilde{\mathbb{P}},\widetilde{W},v)$ of the equation (4) on $[\tau(\omega),\infty)$ such that the law of the process $(\vec{v}(t\vee\tau(\omega)))_t$ coincides with $r(\omega)$ on $\overline{\mathcal{F}}_{\tau(\omega),\infty}$ by Theorem 14, 2°. In particular, $\widetilde{\mathbb{P}}[v(\tau(\omega)=e^{-1}\pi_{\tau(\omega)}(\omega))]=1$ and so $r(\omega)=P^{\tau(\omega),e^{-1}(\pi_{\tau(\omega)}(\omega))}$ on $\overline{\mathcal{F}}_{\tau(\omega),\infty}$ since the equation (4) is $(\tau(\omega),e^{-1}\pi_{\tau(\omega)}(\omega))$-unique in law by assumption. □

P r o o f of Corollary 22. Define

$$P(V) = \int_O P^{a,x}(V)\,\mathrm{d}\mu(x), \qquad V \in \overline{\mathcal{F}}$$

in view of Theorem 20. Then, apparently, $P(M_a) = P[\pi_s \in \operatorname{Rng} e] = 1$ for every $s \geqslant a$ and

$$\int_C L^a_{nm}(t)\,\mathrm{d}P = \int_C L^a_{nm}(s)\,\mathrm{d}P$$

for every $C \in \overline{\mathcal{F}}_s$, $a \leqslant s \leqslant t$, $n \in \mathbb{N}$, $m \in \mathbb{N}$ and $f \in C^2(\mathbb{R})$ by Theorem 14, 1°. Hence there exists a solution $(\Omega, \mathcal{F}, (\mathcal{F}_t), W, u)$ of the equation (4) on $[a, \infty)$ such that $\mathfrak{Law}_{\mathbb{P}}\,(\vec{u}(\cdot \vee a)) = P$ by Theorem 14, 2°. In particular, $\mathfrak{Law}_{\mathbb{P}}\,(u(a)) = \mu$. For the second part of the corollary denote by P_k the laws of the processes $t \mapsto e(u^k(t))$ on $\overline{\mathcal{F}}$ for $k = 1, 2$ by Remark 11. Then $P_1 = P_2$ on $\overline{\mathcal{F}_0}$ and $P_1|\overline{\mathcal{F}}_0 = P_2|\overline{\mathcal{F}}_0$ by Theorem 21. Consequently, $P_1 = P_2$. □

P r o o f of Corollary 23. The mapping

$$(a, x) \mapsto \int_{\Omega} \varphi(\pi_t)\,\mathrm{d}P^{a,x} \quad \text{or} \quad t \mapsto \int_{\Omega} \varphi(\pi_t)\,\mathrm{d}P^{a,x}$$

is measurable by Theorem 20 or continuous for every $\varphi\colon \mathbb{R}^{\mathbb{N}} \to [0, 1]$ continuous, respectively. Hence it is jointly measurable since it satisfies the Carathéodory condition. The joint measurability then holds for every φ bounded, e.g. $\varphi = I_{e[V]}$ for $V \in \mathbb{B}(X)$. □

P r o o f of measurability of the family of probability measures in Theorem 24 and Theorem 27. We have

$$\begin{aligned}\mathbb{P}^{a,x}\left[Y_{t_1} \in \Gamma_1, \ldots, Y_{t_m} \in \Gamma_m\right] &= \mathbb{P}_{a,x}\left[u_{a,x}(t_1) \in \Gamma_1, \ldots, u_{a,x}(t_m) \in \Gamma_m\right] \\ &= P^{a,x}\left[\pi_{t_1 \vee a} \in e\left[\Gamma_1\right], \ldots, \pi_{t_m \vee a} \in e\left[\Gamma_m\right]\right]\end{aligned}$$

for every $a \geqslant 0$, $x \in O$, $t_1, \ldots, t_m$ in $[0, \infty)$ and every $\Gamma_1, \ldots, \Gamma_m$ in $\mathcal{O} \subseteq \mathbb{B}(X)$. Hence the measurability of $(a, x) \to \mathbb{P}^{a,x}$ follows from Theorem 20. □

P r o o f of the Markov property in Theorem 24. Fixing $0 \leqslant a \leqslant b \leqslant c$ and $x \in O$, denote by r a regular version of the conditional probability $P^{a,x}$ with respect to $\overline{\mathcal{F}}_b$. Then

$$r(\omega, [\pi_c \in e[\Gamma]]) = P^{b, e^{-1}(\pi_b(\omega))}[\pi_c \in e[\Gamma]] \qquad P^{a,x}\text{-a.e.}$$

for every $\Gamma \in \mathcal{O}$ by Theorem 21. So, if $0 \leqslant s_1 \leqslant \ldots \leqslant s_m \leqslant b$ and $\Gamma_1, \ldots, \Gamma_m, \Gamma$ belong to $\mathcal{O}$ then

$$
\begin{aligned}
&\mathbb{P}^{a,x}[Y_{s_1} \in \Gamma_1, \ldots, Y_{s_m} \in \Gamma_m, Y_c \in \Gamma] \\
&\quad = \mathbb{P}_{a,x}[u_{a,x}(s_1 \vee a) \in \Gamma_1, \ldots, u_{a,x}(s_m \vee a) \in \Gamma_m, u_{a,x}(c) \in \Gamma] \\
&\quad = P^{a,x}[\pi_{s_1} \in e[\Gamma_1], \ldots, \pi_{s_m} \in e[\Gamma_m], \pi_c \in e[\Gamma]] \\
&\quad = \int_{\overline{\Omega}} I_{e[\Gamma_1]}(\pi_{s_1}(\omega)) \ldots I_{e[\Gamma_m]}(\pi_{s_m}(\omega)) P^{b,e^{-1}(\pi_b(\omega))}[\pi_c \in e[\Gamma]] \, \mathrm{d}P^{a,x}(\omega) \\
&\quad = \int_{\Omega_{a,x}} I_{\Gamma_1}(u_{a,x}(s_1 \vee a)) \ldots I_{\Gamma_m}(u_{a,x}(s_m \vee a)) P^{b,u_{a,x}(b)}[\pi_c \in e[\Gamma]] \, \mathrm{d}\mathbb{P}_{a,x} \\
&\quad = \int_{\Omega} I_{\Gamma_1}(Y_{s_1}) \ldots I_{\Gamma_m}(Y_{s_m}) P^{b,Y_b}[\pi_c \in e[\Gamma]] \, \mathrm{d}\mathbb{P}^{a,x}
\end{aligned}
$$

by definition. Hence

$$
\mathbb{P}^{a,x}[Y_c \in \Gamma | \mathcal{H}_b] = P^{b,Y_b}[\pi_c \in e[\Gamma]] = \mathbb{P}^{b,Y_b}[Y_c \in \Gamma].
$$

□

Proof of the strong Markov property in Theorem 27.

(a) Fixing an $a \geqslant 0$ and $x \in O$, denote by $\mathcal{G}_t$ the $P^{a,x}$-augmentation of $\overline{\mathcal{F}}_t$, $t \geqslant 0$, by

$$
i \colon C([0,\infty),(O,\|\cdot\|_O)) \to C\big([0,\infty),\mathbb{R}^{\mathbb{N}}\big) : \omega \mapsto (e(\omega(t)) \colon t \geqslant 0)
$$

the continuous injection and by

$$
j \colon C\big([0,\infty),\mathbb{R}^{\mathbb{N}}\big) \to C([0,\infty),(O,\|\cdot\|_O)) : \overline{\omega} \mapsto \begin{cases} i^{-1}(\overline{\omega}), & \overline{\omega} \in \operatorname{Rng} i, \\ 0, & \overline{\omega} \in (\operatorname{Rng} i)^c \end{cases}
$$

its Borel pseudoinverse. Actually, j is Borel measurable by Chap. 3, Par. 39, Sect. IV, p. 487 in [9].

(b) Apparently, $\mathfrak{Law}_{\mathbb{P}^{a,x}}(i) = P^{a,x}$ on $\overline{\mathcal{F}}$ and $\mathfrak{Law}_{P^{a,x}}(j) = \mathbb{P}^{a,x}$ on $\mathcal{F}$. In particular, $P^{a,x}(\operatorname{Rng} i) = 1$.

(c) $j^{-1}[Y_t \in \Gamma] \cap \operatorname{Rng} i = [\pi_t \in e[\Gamma]] \cap \operatorname{Rng} i$ for every $\Gamma \in \mathcal{O}$ and $t \geqslant 0$, so $j^{-1}[C] \in \mathcal{G}_t$ for every $C \in \mathcal{F}_t$ and, consequently, for every $C \in \mathcal{H}_t$ by (b), $t \geqslant 0$.

(d) $i^{-1}[\pi_t \in \Gamma] = [e(Y_t) \in \Gamma]$ for every $\Gamma \in \mathbb{B}(\mathbb{R}^{\mathbb{N}})$ and $t \geqslant 0$, so $i^{-1}[C] \in \mathcal{H}_t$ for every $C \in \overline{\mathcal{F}}_t$ and, consequently, for every $C \in \mathcal{G}_t$, $t \geqslant 0$.

(e) Defining $\overline{\tau}(\overline{\omega}) = \tau(j(\overline{\omega}))$, $\overline{\omega} \in \overline{\Omega}$ then $\overline{\tau}$ is a $(\mathcal{G}_t)$-stopping time by (c).

(f) The mapping $i \colon (\Omega, \mathcal{H}_\tau) \to (\overline{\Omega}, \mathcal{G}_{\overline{\tau}})$ is measurable by (d) and $j \colon (\overline{\Omega}, \mathcal{G}_{\overline{\tau}}) \to (\Omega, \mathcal{H}_\tau)$ is measurable by (c).

(g) If we denote by $r\colon\ \overline{\Omega}\times\overline{\mathcal{F}}\to\mathbb{R}$ a regular version of the conditional probability $P^{a,x}$ with respect to $\mathcal{G}_{\overline{\mathcal{T}}}$ then there exists a set $N\in\mathcal{G}_{\overline{\mathcal{T}}}$, $P^{a,x}(N)=1$ such that $\pi_{\overline{\mathcal{T}}(\overline{\omega})}(\overline{\omega})\in e[O]$ and $r(\overline{\omega})=P^{\overline{\mathcal{T}}(\overline{\omega}),e^{-1}(\pi_{\overline{\mathcal{T}}(\overline{\omega})}(\overline{\omega}))}$ on $\overline{\mathcal{F}}_{\overline{\mathcal{T}}(\overline{\omega}),\infty}$ for every $\overline{\omega}\in N$ by Theorem 21.

(i) The function $R\colon\ \Omega\times\mathcal{F}\to\mathbb{R}\colon\ (\omega,V)\mapsto r(i(\omega),j^{-1}[V])$ is $\mathcal{H}_\tau$-measurable in the first variable by (f). Moreover, R is a regular version of the conditional probability $\mathbb{P}^{a,x}$ with respect to $\mathcal{H}_\tau$. Indeed, let $C\in\mathcal{H}_\tau$ and $V\in\mathcal{F}$. Then

$$\begin{aligned}\int_C R(\cdot,V)\,\mathrm{d}\mathbb{P}^{a,x}&=\int_{[j\in C]}r(i(j(\cdot)),j^{-1}[V])\,\mathrm{d}P^{a,x}=\int_{[j\in C]}r(\cdot,j^{-1}[V])\,\mathrm{d}P^{a,x}\\&=P^{a,x}\left[j^{-1}[C\cap V]\right]=\mathbb{P}^{a,x}(V\cap C)\end{aligned}$$

by (b) and (f).

(j) Since $P^{a,x}(\operatorname{Rng} i)=1$ the set $N_1=\{\overline{\omega}\colon\ r(\overline{\omega},\operatorname{Rng} i)=1\}\in\mathcal{G}_{\overline{\mathcal{T}}}$ satisfies $P^{a,x}(N_1)=1$ and the set $N_2=\{\omega\colon\ R(\omega,[\tau(\omega)=\tau])=1\}\in\mathcal{H}_\tau$ satisfies $\mathbb{P}^{a,x}(N_2)=1$ by the elementary properties of conditional probabilities. So,

$$\begin{aligned}R(\omega,[Y_{t+\tau}\in\Gamma])&=R(\omega,[Y_{t+\tau(\omega)}\in\Gamma])=r(i(\omega),j^{-1}[Y_{t+\tau(\omega)}\in\Gamma])\\&=r(i(\omega),j^{-1}[Y_{t+\tau(\omega)}\in\Gamma]\cap\operatorname{Rng} i)=r(i(\omega),[\pi_{t+\tau(\omega)}\in e[\Gamma]]\cap\operatorname{Rng} i)\\&=r(i(\omega),[\pi_{t+\tau(\omega)}\in e[\Gamma]])=P^{\tau(\omega),e^{-1}(\pi_{\tau(\omega)}(i(\omega)))}[\pi_{t+\tau(\omega)}\in e[\Gamma]]\\&=P^{\tau(\omega),Y_{\tau(\omega)}(\omega)}[\pi_{t+\tau(\omega)}\in e[\Gamma]]=\mathbb{P}^{\tau(\omega),Y_{\tau(\omega)}(\omega)}[Y_{t+\tau(\omega)}\in\Gamma]\end{aligned}$$

by (c), (g) and Theorem 19 for every $\Gamma\in\mathbb{B}(O)$, $\omega\in i^{-1}[N\cap N_1]\cap N_2\in\mathcal{H}_\tau$ and $\mathbb{P}^{a,x}[i^{-1}[N\cap N_1]\cap N_2]=1$ by (b) and (f). $\square$

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