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ON HOMOMORPHISMS BETWEEN $C^{*}$-ALGEBRAS AND LINEAR DERIVATIONS ON $C^{*}$-ALGEBRAS

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Abstract. It is shown that every almost linear Pexider mappings $f, g, h$ from a unital $C^{*}$-algebra $A$ into a unital $C^{*}$-algebra $B$ are homomorphisms when $f(2^{n}uy) = f(2^{n}u)f(y)$, $g(2^{n}uy) = g(2^{n}u)g(y)$ and $h(2^{n}uy) = h(2^{n}u)h(y)$ hold for all unitaries $u \in A$, all $y \in A$, and all $n \in \mathbb{Z}$, and that every almost linear continuous Pexider mappings $f, g, h$ from a unital $C^{*}$-algebra $A$ of real rank zero into a unital $C^{*}$-algebra $B$ are homomorphisms when $f(2^{n}uy) = f(2^{n}u)f(y)$, $g(2^{n}uy) = g(2^{n}u)g(y)$ and $h(2^{n}uy) = h(2^{n}u)h(y)$ hold for all $u \in \{v \in A : v = v^{*}$ and $v$ is invertible$\}$, all $y \in A$ and all $n \in \mathbb{Z}$.

Furthermore, we prove the Cauchy-Rassias stability of $*$-homomorphisms between unital $C^{*}$-algebras, and $C$-linear $*$-derivations on unital $C^{*}$-algebras.

Keywords: $C^{*}$-algebra homomorphism, $C^{*}$-algebra, real rank zero, $C$-linear $*$-derivation, stability

MSC 2000: 39B52, 47B48, 46L05

1. Introduction

Let $X$ and $Y$ be Banach spaces with norms $\| \cdot \|$ and $\| \cdot \|$, respectively. Consider $f : X \rightarrow Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \theta(\|x\|^{p} + \|y\|^{p})$$

for all $x, y \in X$. Rassias [8] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \rightarrow Y$ such that $\|f(x) - T(x)\| \leq 2\theta/(2 - 2^{p})\|x\|^{p}$ for all $x \in X$. Găvruta [2] generalized Rassias’ result.

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Jun, Kim and Shin [4] proved the following: Let $X$ and $Y$ be Banach spaces. Denote by $\varphi: X \times X \to [0, \infty)$ a function such that

(a) $\varepsilon \varphi(x) := \sum_{j=1}^{\infty} 2^{-j}(\varphi(2^{j-1}x, 0) + \varphi(0, 2^{j-1}x) + \varphi(2^{j-1}x, 2^{j-1}x)) < \infty$

for all $x \in X$. Suppose that $f, g, h: X \to Y$ are mappings satisfying

$$\|2f\left(\frac{x+y}{2}\right) - g(x) - h(y)\| \leq \varphi(x, y)$$

for all $x, y \in X$. Then there exists a unique additive mapping $T: X \to Y$ such that

$$\|2f\left(\frac{x}{2}\right) - T(x)\| \leq \|g(0)\| + \|h(0)\| + \varepsilon \varphi(x),$$

$$\|g(x) - T(x)\| \leq \|g(0)\| + 2\|h(0)\| + \varphi(x, 0) + \varepsilon \varphi(x),$$

$$\|h(x) - T(x)\| \leq 2\|g(0)\| + \|h(0)\| + \varphi(0, x) + \varepsilon \varphi(x)$$

for all $x \in X$.

B.E. Johnson [3, Theorem 7.2] also investigated almost algebra $*$-homomorphisms between Banach $*$-algebras: Suppose that $\mathcal{A}$ and $\mathcal{B}$ are Banach $*$-algebras which satisfy the conditions of [3, Theorem 3.1]. Then for each positive $\varepsilon$ and $K$ there is a positive $\delta$ such that if $T \in L(\mathcal{A}, \mathcal{B})$ with $\|T\| < K$, $\|T^\vee\| < \delta$ and $\|T(x^*)^* - T(x)\| \leq \delta \|x\|$ ($x \in \mathcal{A}$) then there is a $*$-homomorphism $T': \mathcal{A} \to \mathcal{B}$ with $\|T - T'\| < \varepsilon$. Here $L(\mathcal{A}, \mathcal{B})$ is the space of bounded linear mappings from $\mathcal{A}$ into $\mathcal{B}$, and $T^\vee(x, y) = T(xy) - T(x)T(y)$ ($x, y \in \mathcal{A}$). See [3] for details.

Throughout this paper, let $\mathcal{A}$ be a unital $C^*$-algebra with norm $\|\cdot\|$ and unit $e$, and $\mathcal{B}$ a unital $C^*$-algebra with norm $\|\cdot\|$. Let $\mathcal{U}(\mathcal{A})$ be the set of unitary elements in $\mathcal{A}$, $\mathcal{A}_{sa} = \{x \in \mathcal{A}: x = x^*\}$ and $I_1(\mathcal{A}_{sa}) = \{v \in \mathcal{A}_{sa}: \|v\| = 1, \ v \text{ is invertible}\}$.

In this paper, we prove that every almost linear Pexider mappings $f, g, h: \mathcal{A} \to \mathcal{B}$ are homomorphisms when $f(2^nuy) = f(2^nuf(y))$, $g(2^nuy) = g(2^nu)g(y)$ and $h(2^nuy) = h(2^nu)h(y)$ hold for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$, and that for a unital $C^*$-algebra $\mathcal{A}$ of real rank zero (see [1]), every almost linear continuous Pexider mappings $f, g, h: \mathcal{A} \to \mathcal{B}$ are homomorphisms when $f(2^nuy) = f(2^nuf(y))$, $g(2^nuy) = g(2^nu)g(y)$ and $h(2^nuy) = h(2^nu)h(y)$ hold for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$.

Furthermore, we prove the Cauchy-Rassias stability of $*$-homomorphisms between unital $C^*$-algebras, and $C$-linear $*$-derivations on unital $C^*$-algebras.
In this section, let \( f, g, h : \mathcal{A} \to \mathcal{B} \) be mappings satisfying \( f(0) = g(0) = h(0) = 0 \), and let \( f(2^nuy) = f(2^nu)f(y), g(2^nuy) = g(2^nu)g(y) \) and \( h(2^nuy) = h(2^nu)h(y) \) for all \( u \in \mathcal{U}(\mathcal{A}) \), all \( y \in \mathcal{A} \) and all \( n \in \mathbb{Z} \), unless otherwise specified. We are going to investigate \(*\)-homomorphisms between unital \( \mathcal{C}^* \)-algebras.

**Theorem 1.** Assume that there exists a function \( \varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty) \) such that

(i) \( \bar{\varphi}(x, y) := \sum_{j=0}^{\infty} 2^{-j} \varphi(2^{j-1}x, 2^{j-1}y) < \infty \),

(ii) \( \|2f\left(\frac{\mu x + \mu y}{2}\right) - \mu g(x) - \mu h(y)\| \leq \varphi(x, y) \),

(iii) \( \|f(2^nu^*) - f(2^nu^)\| \leq \varphi(2^nu, 2^nu) \)

for all \( \mu \in \mathbb{T}^1 := \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \), all \( u \in \mathcal{U}(\mathcal{A}) \), all \( x, y \in \mathcal{A} \) and all \( n \in \mathbb{Z} \). If

(iv) \( \lim_{n \to \infty} f(2^ne) = 2^ne \) is invertible,

then the mappings \( f, g, h \) are \(*\)-homomorphisms and \( f = g = h \).

**Proof.** Let \( x \in \mathcal{A} \) be arbitrary. Put \( \mu = 1 \in \mathbb{T}^1 \) in (ii). It follows from [4, Corollary 2.5] that there exists a unique additive mapping \( H : \mathcal{A} \to \mathcal{B} \) such that

\[
\|2f\left(\frac{x}{2}\right) - H(x)\| \leq \varepsilon(x),
\]

(\dagger) \( \|g(x) - H(x)\| \leq \varphi(x, 0) + \varepsilon(x) \),

\( \|h(x) - H(x)\| \leq \varphi(0, x) + \varepsilon(x) \),

where \( \varepsilon(x) := \varepsilon_{\varphi}(x) \) is given by (a). The additive mapping \( H \) is given by

\[
H(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} = \lim_{n \to \infty} \frac{g(2^n x)}{2^n} = \lim_{n \to \infty} \frac{h(2^n x)}{2^n}.
\]

Let \( \tilde{f}(x) = 2f\left(\frac{x}{2}\right) \), then \( \lim_{n \to \infty} 2^{-n} \tilde{f}(2^n x) = \lim_{n \to \infty} 2^{-n} f(2^n x) \).

Let \( \mu \in \mathbb{T}^1 \) and \( x \in \mathcal{A} \) be arbitrary. By the assumption,

\[
\|f(2^nu^*) - f(2^nu^)\| = \left\|f\left(2^n \mu x\right) - \frac{1}{2} \mu g(2^n x) - \frac{1}{2} \mu h(2^n x)
+ \frac{1}{2} \mu g(2^n x) + \frac{1}{2} \mu h(2^n x) - \mu f(2^n x)\right\|
\leq \frac{1}{2} \varphi(2^n x, 2^n x) + \frac{1}{2} |\mu| \varphi(2^n x, 2^n x)
= \varphi(2^n x, 2^n x).
\]
Thus $2^{-n}\|f(2^n\mu x) - \mu f(2^n x)\| \to 0$ as $n \to \infty$. Hence

(1) $H(\mu x) = \mu H(x)$.

Now let $\lambda \in \mathbb{C}$ and $M$ an integer greater than $2|\lambda|$. Since $|\lambda/M| < \frac{1}{2}$, there is $t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right]$ such that $|\lambda/M| = \cos t = \frac{1}{2}(e^{it} + e^{-it})$. Now $\lambda/M = |\lambda/M|\mu$ for some $\mu \in \mathbb{T}$. And $H(x) = 2H(\frac{1}{2}x)$ for all $x \in \mathcal{A}$. So $H(\frac{1}{2}x) = \frac{1}{2}H(x)$ for all $x \in \mathcal{A}$.

Thus, by (1), $H(\lambda x) = H(M(\lambda/M)x) = \lambda H(x)$ for all $x \in \mathcal{A}$. So the unique additive mapping $H: \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear.

By (i) and (iii), we get $H(u^*) = H(u)^*$ for all $u \in \mathcal{U}(\mathcal{A})$. Since $H$ is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements (see [6, Theorem 4.1.7]), say, $x = \sum_{j=1}^{m} \lambda_j u_j$ ($\lambda_j \in \mathbb{C}$, $u_j \in \mathcal{U}(\mathcal{A})$), $H(x^*) = \sum_{j=1}^{m} \lambda_j H(u_j)^* = H(x)^*$ for all $x \in \mathcal{A}$.

Let $u \in \mathcal{U}(\mathcal{A})$ and $y \in \mathcal{A}$ be arbitrary. Since $f(2^n uy) = f(2^n u)f(y)$ for all $n \in \mathbb{Z}$,

(2) $H(u y) = H(u)f(y)$.

So

(3) $H(u y) = H(u)\frac{1}{2^n}f(2^n y)$

for all $n \in \mathbb{Z}$. Taking the limit in (3) as $n \to \infty$, we obtain

(4) $H(u y) = H(u)H(y)$.

Since $H$ is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements, it follows from (4) that $H(xy) = H(x)H(y)$ for all $x \in \mathcal{A}$.

By (2) and (4), $H(e)H(y) = H(e)f(y)$ for all $y \in \mathcal{A}$. Since $\lim_{n \to \infty} f(2^n e)2^{-n} = H(e)$ is invertible, $H(y) = f(y)$ for all $y \in \mathcal{A}$. Similarly, $H(y) = g(y) = h(y)$ for all $y \in \mathcal{A}$.

Therefore, the mappings $f$, $g$, $h$ are $*$-homomorphisms and $f = g = h$. □

**Corollary 2.** Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$
\|2f\left(\frac{\mu x + \mu y}{2}\right) - \mu g(x) - \mu h(y)\| \leq \theta(\|x\|^p + \|y\|^p),
$$

$$
\|f(2^n u^*) - f(2^n u)^*\| \leq 2^{np+1}\theta
$$

for all $\mu \in \mathbb{T}$, all $u \in \mathcal{U}(\mathcal{A})$, all $x, y \in \mathcal{A}$ and all $n \in \mathbb{Z}$. If $f$ satisfies (iv), the mappings $f$, $g$, $h$ are $*$-homomorphisms and $f = g = h$.

**Proof.** Define $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$ and apply Theorem 1. □
**Theorem 3.** Assume that there exists a function \( \varphi: A \times A \to [0, \infty) \) satisfying (i) and (iii) such that

\[
(\text{v}) \quad \left\| 2f\left( \frac{\mu x + \mu y}{2} \right) - \mu g(x) - \mu h(y) \right\| \leq \varphi(x, y)
\]

for \( \mu = 1, i, \) and all \( x, y \in A \). If \( f \) satisfies (iv) and \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in A \), then the mappings \( f, g, h \) are \( * \)-homomorphisms and \( f = g = h \).

**Proof.** Put \( \mu = 1 \) in (v). By the same reasoning as in the proof of Theorem 1, there exists a unique additive mapping \( H: A \to B \) satisfying \((\dagger)\). By the same reasoning as in the proof of [8, Theorem], the additive mapping \( H \) is \( \mathbb{R} \)-linear.

Put \( \mu = i \) in (v). By the same method as in the proof of Theorem 1, one can obtain that \( H(ix) = iH(x) \) for all \( x \in A \). For each \( \lambda \in \mathbb{C}, \lambda = s + it, \) where \( s, t \in \mathbb{R} \). So \( H(\lambda x) = sH(x) + itH(x) = \lambda H(x) \) for all \( \lambda \in \mathbb{C} \) and all \( x \in A \). Hence the additive mapping \( H \) is \( \mathbb{C} \)-linear.

The rest of the proof is the same as in the proof of Theorem 1. \( \square \)

From now on, assume that \( A \) is a unital \( C^* \)-algebra of real rank zero, where “real rank zero” means that the set of invertible self-adjoint elements is dense in the set of self-adjoint elements (see [1]). Let \( f, g, h \) be continuous and \( f(0) = g(0) = h(0) = 0 \) and let \( f(2^nu) = f(2^nu)f(y), g(2^nu) = g(2^nu)g(y) \) and \( h(2^nu) = h(2^nu)h(y) \) for all \( u \in I_1(A_{sa}), \) all \( y \in A \) and all \( n \in \mathbb{Z} \).

Now we are going to investigate continuous \( * \)-homomorphisms between unital \( C^* \)-algebras.

**Theorem 4.** Assume that there exists a function \( \varphi: A \times A \to [0, \infty) \) satisfying (i), (ii) and (iii). If \( f \) satisfies (iv), then the mappings \( f, g, h \) are \( * \)-homomorphisms and \( f = g = h \).

**Proof.** By the same reasoning as in the proof of Theorem 1, there exists a unique \( \mathbb{C} \)-linear involutive mapping \( H: A \to B \) satisfying the system of the inequalities \((\dagger)\).

Let \( u \in I_1(A_{sa}) \) and \( y \in A \) be arbitrary. Since \( f(2^nu) = f(2^nu)f(y) \) for all \( n \in \mathbb{Z}, \)

\[
(5) \quad H(uy) = H(u)f(y).
\]

So

\[
(6) \quad H(uy) = H(u)\frac{1}{2^n}f(2^ny)
\]
for all \( n \in \mathbb{Z} \). Taking the limit in (6) as \( n \to \infty \), we obtain

\[
H(uy) = H(u)H(y).
\]

Let \( y \in \mathcal{A} \) be arbitrary. By (5) and (7),

\[
H(e)H(y) = H(e)f(y).
\]

Since \( \lim_{n \to \infty} f(2^n e)/2^n = H(e) \) is invertible, \( H(y) = f(y) \). Similarly, \( H(y) = g(y) = h(y) \). So \( H: \mathcal{A} \to \mathcal{B} \) is continuous. But by the assumption that \( \mathcal{A} \) has real rank zero, it is easy to show that the set of linear combinations of elements of \( I_1(\mathcal{A}_{sa}) \) is dense in \( \mathcal{A} \). So for each \( x \in \mathcal{A} \), there is a sequence \( \{\kappa_j\} \) such that \( \kappa_j \to x \) as \( j \to \infty \) and \( \kappa_j \) is a linear combination of elements of \( I_1(\mathcal{A}_{sa}) \). Since \( H \) is continuous, it follows from (7) and the \( \mathbb{C} \)-linearity of \( H \) that

\[
H(xy) = \lim_{j \to \infty} H(\kappa_j)H(y) = H(x)H(y)
\]

for all \( x \in \mathcal{A} \).

Therefore, the mappings \( f, g, h \) are \(*\)-homomorphisms and \( f = g = h \).

**Corollary 5.** Assume that there exist constants \( \theta \geq 0 \) and \( p \in [0, 1) \) such that

\[
\|2f\left(\frac{\mu x + \mu y}{2}\right) - \mu g(x) - \mu h(y)\| \leq \theta(\|x\|^p + \|y\|^p),
\]

\[
\|f(2^n u^*) - f(2^n u)^*\| \leq 2^{np+1}\theta
\]

for all \( \mu \in \mathbb{T} \), all \( u \in I_1(\mathcal{A}_{sa}) \), all \( x, y \in \mathcal{A} \setminus \{0\} \) and all \( n \in \mathbb{Z} \). If \( f \) satisfies (iv), the mappings \( f, g, h \) are \(*\)-homomorphisms.

**Proof.** Define \( \varphi(x,y) = \theta(\|x\|^p + \|y\|^p) \) and apply Theorem 4.

**Theorem 6.** Assume that there exists a function \( \varphi: \mathcal{A} \times \mathcal{A} \to [0, \infty) \) satisfying (i), (iii) and (v). If \( f \) satisfies (iv), the mappings \( f, g, h \) are \(*\)-homomorphisms and \( f = g = h \).

**Proof.** By the same reasoning as in the proof of Theorem 3, there exists a unique \( \mathbb{C} \)-linear mapping \( H: \mathcal{A} \to \mathcal{B} \) satisfying the system of the inequalities (\( \dagger \)).

The rest of the proof is the same as in the proofs of Theorems 1 and 4.

3. Stability of \(*\)-homomorphisms between unital \( C^* \)-algebras

In this section, let \( f, g, h: \mathcal{A} \to \mathcal{B} \) be mappings with \( f(0) = g(0) = h(0) = 0 \). We are going to show the Cauchy-Rassias stability of \(*\)-homomorphisms between unital \( C^* \)-algebras.
**Theorem 7.** Assume that there exists a function \( \varphi : (\mathcal{A} \setminus \{0\})^4 \to [0, \infty) \) such that

\[
(\text{vi}) \quad \tilde{\varphi}(x, y, z, w) = \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x, 2^j y, 2^j z, 2^j w) < \infty,
\]

\[
(\text{vii}) \quad \|2f\left(\frac{\mu x + \mu y + zw}{2}\right) - \mu g(x) - \mu h(y) - f(z)f(w)\| \leq \varphi(x, y, z, w),
\]

\[
(\text{viii}) \quad \|f(2^n u^\ast) - f(2^n u)^\ast\| \leq \varphi(2^n u, 2^n u, 0, 0)
\]

for all \( \mu \in \mathbb{T}^1 \), all \( u \in \mathcal{H}(\mathcal{A}) \), all \( x, y, z, w \in \mathcal{A} \) and all \( n \in \mathbb{Z} \). Then there exists a unique \(*\)-homomorphism \( H : \mathcal{A} \to \mathcal{B} \) such that

\[
(\text{ix}) \quad \left\| 2f\left(\frac{x}{2}\right) - H(x) \right\| \leq \varepsilon(x),
\]

\[
\|g(x) - H(x)\| \leq \varphi(x, 0, 0, 0) + \varepsilon(x),
\]

\[
\|h(x) - H(x)\| \leq \varphi(0, x, 0, 0) + \varepsilon(x)
\]

for all \( x \in \mathcal{A} \), where

\[
(\text{b}) \quad \varepsilon(x) := \varepsilon\varphi(x)
\]

is given by (a) and \( \psi(x, y) := \varphi(x, y, 0, 0) \) for all \( x, y \in \mathcal{A} \).

**Proof.** Put \( z = w = 0 \) and \( \mu = 1 \in \mathbb{T}^1 \) in (vii). By the same reasoning as in the proof of Theorem 1, there exists a unique \( \mathbb{C}\)-linear involutive mapping \( H : \mathcal{A} \to \mathcal{B} \) satisfying (ix). The \( \mathbb{C}\)-linear mapping \( H \) is given by

\[
(9) \quad H(x) = \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)
\]

for all \( x \in \mathcal{A} \).

Let \( z, w \in \mathcal{A} \) be arbitrary. Taking \( x = y = 0 \) in (vii), \( \|2f\left(\frac{1}{2}zw\right) - f(z)f(w)\| \leq \varphi(0, 0, z, w) \). So

\[
(10) \quad \frac{1}{2^n} \left\| 2f\left(\frac{1}{2} 2^n z \cdot 2^n w\right) - f(2^n z)f(2^n w) \right\| \leq \frac{1}{2^n} \varphi(0, 0, 2^n z, 2^n w).
\]

By (vi), (9) and (10), \( 2H\left(\frac{1}{2}zw\right) = H(z)H(w) \). But since \( H \) is \( \mathbb{C}\)-linear, \( H(zw) = H(z)H(w) \). Hence the \( \mathbb{C}\)-linear mapping \( H \) is a \(*\)-homomorphism satisfying (ix). \( \square \)
Corollary 8. Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that
\[
\left\| 2f\left(\frac{\mu x + \mu y + zw}{2}\right) - \mu g(x) - \mu h(y) - f(z)f(w) \right\| \\
\leq \theta \left( \|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p \right),
\]
\[
\left\| f(2^n u^*) - f(2^n u)^* \right\| \leq 2^{np+1}\theta
\]
for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U} (\mathcal{A})$, all $x, y, z, w \in \mathcal{A}$ and all $n \in \mathbb{Z}$. Then there exists a unique $*$-homomorphism $H: \mathcal{A} \to \mathcal{B}$ such that
\[
\left\| 2f\left(\frac{x}{2}\right) - H(x) \right\| \leq \frac{1}{2 - 2^p}\theta\|x\|^p,
\]
\[
\left\| g(x) - H(x) \right\| \leq \frac{3 - 2^p}{2 - 2^p}\theta\|x\|^p,
\]
\[
\left\| h(x) - H(x) \right\| \leq \frac{3 - 2^p}{2 - 2^p}\theta\|x\|^p
\]
for all $x \in \mathcal{A}$.

Proof. Define $\varphi(x, y, z, w) = \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$ and apply Theorem 7.
\[\square\]

Theorem 9. Assume that there exists a function $\varphi: \mathcal{A}^4 \to [0, \infty)$ satisfying (vi) and (viii) such that
\[
\left\| 2f\left(\frac{\mu x + \mu y + zw}{2}\right) - \mu g(x) - \mu h(y) - f(z)f(w) \right\| \leq \varphi(x, y, z, w)
\]
for $\mu = 1, i$, and all $x, y, z, w \in \mathcal{A}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then there exists a unique $*$-homomorphism $H: \mathcal{A} \to \mathcal{B}$ satisfying (ix).

Proof. By the same reasoning as in the proof of Theorem 3, there exists a unique $\mathbb{C}$-linear mapping $H: \mathcal{A} \to \mathcal{B}$ satisfying (ix).

The rest of the proof is the same as in the proofs of Theorems 1 and 7. \[\square\]

4. Stability of linear $*$-derivations on unital $C^*$-algebras

From now on, let $\mathcal{A} = \mathcal{B}$. We are going to show the Cauchy-Rassias stability of linear $*$-derivations on unital $C^*$-algebras.
Theorem 10. Assume that there exists a function \( \varphi: \mathcal{A}^4 \to [0, \infty) \) satisfying (vi) and (viii) such that

\[
(x) \quad \left\| 2f\left(\frac{\mu x + \mu y + zw}{2}\right) - \mu g(x) - \mu h(y) - zf(w) - wf(z) \right\| \leq \varphi(x, y, z, w)
\]

for all \( \mu \in \mathbb{T}^1 \) and all \( x, y, z, w \in \mathcal{A} \). Then there exists a unique \( \mathbb{C} \)-linear \(*\)-derivation \( D: \mathcal{A} \to \mathcal{A} \) such that

\[
(xi) \quad \left\| 2f\left(\frac{x}{2}\right) - D(x) \right\| \leq \varepsilon(x), \\
\left\| g(x) - D(x) \right\| \leq \varphi(x, 0, 0, 0) + \varepsilon(x), \\
\left\| h(x) - D(x) \right\| \leq \varphi(0, x, 0, 0) + \varepsilon(x)
\]

for all \( x \in \mathcal{A} \), where \( \varepsilon(x) \) is given by (b).

Proof. Put \( z = w = 0 \) and \( \mu = 1 \in \mathbb{T}^1 \) in (x). By the same reasoning as in the proof of Theorem 1, there exists a unique \( \mathbb{C} \)-linear involutive mapping \( D: \mathcal{A} \to \mathcal{A} \) satisfying (xi). The \( \mathbb{C} \)-linear mapping \( D: \mathcal{A} \to \mathcal{A} \) is given by

\[
(11) \quad D(x) = \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)
\]

for all \( x \in \mathcal{A} \).

Let \( z, w \in \mathcal{A} \) be arbitrary. Taking \( x = y = 0 \) in (x),

\[
\left\| 2f\left(\frac{zw}{2}\right) - zf(w) - w f(z) \right\| \leq \varphi(0, 0, z, w).
\]

So

\[
(12) \quad \frac{1}{2^{2n}} \left\| 2f\left(\frac{1}{2} 2^n z \cdot 2^n w\right) - 2^n z f(2^n w) - 2^n w f(2^n z) \right\| \leq \frac{1}{2^n} \varphi(0, 0, 2^n z, 2^n w).
\]

By (x), (11) and (12), \( 2D(\frac{1}{2} zw) = zD(w) + wD(z) \). But since \( D \) is \( \mathbb{C} \)-linear,

\[
D(zw) = zD(w) + wD(z).
\]

Hence the \( \mathbb{C} \)-linear mapping \( D \) is a \(*\)-derivation satisfying (xi). \( \square \)
Corollary 11. Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$
\left\| 2f\left(\frac{\mu x + \mu y + zw}{2}\right) - \mu g(x) - \mu h(y) - \mu f(w) - \mu f(z) \right\| \\
\leq \theta (\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p),
$$

$$
\|f(2^n u^*) - f(2^n u^*)\| \leq 2^{np+1}\theta
$$

for all $\mu \in \mathbb{T}$, all $u \in \mathcal{U}(\mathcal{A})$, all $x, y, z, w \in \mathcal{A}$ and all $n \in \mathbb{Z}$. Then there exists a unique $\mathbb{C}$-linear $*$-derivation $D : \mathcal{A} \to \mathcal{A}$ such that

$$
\left\| 2f\left(\frac{x}{2}\right) - D(x) \right\| \leq \frac{1}{2 - 2p}\theta \|x\|^p,
$$

$$
\|g(x) - D(x)\| \leq \frac{3 - 2p}{2 - 2p}\theta \|x\|^p,
$$

$$
\|h(x) - D(x)\| \leq \frac{3 - 2p}{2 - 2p}\theta \|x\|^p
$$

for all $x \in \mathcal{A}$.

Proof. Define $\varphi(x, y, z, w) = \theta (\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$ and apply Theorem 10.

Theorem 12. Assume that there exists a function $\varphi : \mathcal{A}^4 \to [0, \infty)$ satisfying (vi) and (viii) such that

$$
\left\| 2f\left(\frac{\mu x + \mu y + zw}{2}\right) - \mu g(x) - \mu h(y) - \mu f(w) - \mu f(z) \right\| \leq \varphi(x, y, z, w)
$$

for $\mu = 1, i$, and all $x, y, z, w \in \mathcal{A}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then there exists a unique $\mathbb{C}$-linear $*$-derivation $D : \mathcal{A} \to \mathcal{A}$ satisfying (xi).

Proof. By the same reasoning as in the proof of Theorem 3, there exists a unique $\mathbb{C}$-linear mapping $D : \mathcal{A} \to \mathcal{A}$ satisfying (xi).

The rest of the proof is the same as in the proofs of Theorems 1 and 10.

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