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# ON HOMOMORPHISMS BETWEEN $C^*$-ALGEBRAS AND LINEAR DERIVATIONS ON $C^*$-ALGEBRAS

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*Abstract.* It is shown that every almost linear Pexider mappings $f$, $g$, $h$ from a unital $C^*$-algebra $\mathscr{A}$ into a unital $C^*$-algebra $\mathscr{B}$ are homomorphisms when $f(2^n uy) = f(2^n u)f(y)$, $g(2^n uy) = g(2^n u)g(y)$ and $h(2^n uy) = h(2^n u)h(y)$ hold for all unitaries $u \in \mathscr{A}$, all $y \in \mathscr{A}$, and all $n \in \mathbb{Z}$, and that every almost linear continuous Pexider mappings $f$, $g$, $h$ from a unital $C^*$-algebra $\mathscr{A}$ of real rank zero into a unital $C^*$-algebra $\mathscr{B}$ are homomorphisms when $f(2^n uy) = f(2^n u)f(y)$, $g(2^n uy) = g(2^n u)g(y)$ and $h(2^n uy) = h(2^n u)h(y)$ hold for all $u \in \{v \in \mathscr{A} \colon v = v^* \text{ and } v \text{ is invertible}\}$, all $y \in \mathscr{A}$ and all $n \in \mathbb{Z}$.

Furthermore, we prove the Cauchy-Rassias stability of $*$-homomorphisms between unital $C^*$-algebras, and $\mathbb{C}$-linear $*$-derivations on unital $C^*$-algebras.

*Keywords*: $C^*$-algebra homomorphism, $C^*$-algebra, real rank zero, $\mathbb{C}$-linear $*$-derivation, stability

*MSC 2000*: 39B52, 47B48, 46L05

## 1. Introduction

Let $X$ and $Y$ be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Consider $f \colon X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\theta \geqslant 0$ and $p \in [0,1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leqslant \theta(||x||^p + ||y||^p)$$

for all $x, y \in X$. Rassias [8] showed that there exists a unique $\mathbb{R}$-linear mapping $T \colon X \to Y$ such that $\|f(x) - T(x)\| \leqslant 2\theta/(2-2^p)||x||^p$ for all $x \in X$. Găvruta [2] generalized Rassias' result.

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Jun, Kim and Shin [4] proved the following: Let $X$ and $Y$ be Banach spaces. Denote by $\varphi\colon X\times X\to[0,\infty)$ a function such that

$$\text{(a)}\qquad \varepsilon_\varphi(x):=\sum_{j=1}^{\infty}2^{-j}(\varphi(2^{j-1}x,0)+\varphi(0,2^{j-1}x)+\varphi(2^{j-1}x,2^{j-1}x))<\infty$$

for all $x\in X$. Suppose that $f,g,h\colon X\to Y$ are mappings satisfying

$$\left\|2f\Bigl(\frac{x+y}{2}\Bigr)-g(x)-h(y)\right\|\leqslant\varphi(x,y)$$

for all $x,y\in X$. Then there exists a unique additive mapping $T\colon X\to Y$ such that

$$\begin{aligned}\left\|2f\Bigl(\frac{x}{2}\Bigr)-T(x)\right\|&\leqslant\|g(0)\|+\|h(0)\|+\varepsilon_\varphi(x),\\ \|g(x)-T(x)\|&\leqslant\|g(0)\|+2\|h(0)\|+\varphi(x,0)+\varepsilon_\varphi(x),\\ \|h(x)-T(x)\|&\leqslant 2\|g(0)\|+\|h(0)\|+\varphi(0,x)+\varepsilon_\varphi(x)\end{aligned}$$

for all $x\in X$.

B. E. Johnson [3, Theorem 7.2] also investigated almost algebra $*$-homomorphisms between Banach $*$-algebras: Suppose that $\mathscr{U}$ and $\mathscr{B}$ are Banach $*$-algebras which satisfy the conditions of [3, Theorem 3.1]. Then for each positive $\varepsilon$ and $K$ there is a positive $\delta$ such that if $T\in L(\mathscr{U},\mathscr{B})$ with $\|T\|<K$, $\|T^\vee\|<\delta$ and $\|T(x^*)^*-T(x)\|\leqslant\delta\|x\|$ $(x\in\mathscr{U})$ then there is a $*$-homomorphism $T'\colon\mathscr{U}\to\mathscr{B}$ with $\|T-T'\|<\varepsilon$. Here $L(\mathscr{U},\mathscr{B})$ is the space of bounded linear mappings from $\mathscr{U}$ into $\mathscr{B}$, and $T^\vee(x,y)=T(xy)-T(x)T(y)$ $(x,y\in\mathscr{U})$. See [3] for details.

Throughout this paper, let $\mathscr{A}$ be a unital $C^*$-algebra with norm $||\cdot||$ and unit $e$, and $\mathscr{B}$ a unital $C^*$-algebra with norm $\|\cdot\|$. Let $\mathscr{U}(\mathscr{A})$ be the set of unitary elements in $\mathscr{A}$, $\mathscr{A}_{\rm sa}=\{x\in\mathscr{A}\colon x=x^*\}$ and $I_1(\mathscr{A}_{\rm sa})=\{v\in\mathscr{A}_{\rm sa}\colon ||v||=1,\ v\text{ is invertible}\}$.

In this paper, we prove that every almost linear Pexider mappings $f,g,h\colon\mathscr{A}\to\mathscr{B}$ are homomorphisms when $f(2^nuy)=f(2^nu)f(y)$, $g(2^nuy)=g(2^nu)g(y)$ and $h(2^nuy)=h(2^nu)h(y)$ hold for all $u\in\mathscr{U}(\mathscr{A})$, all $y\in\mathscr{A}$ and all $n\in\mathbb{Z}$, and that for a unital $C^*$-algebra $\mathscr{A}$ of real rank zero (see [1]), every almost linear continuous Pexider mappings $f,g,h\colon\mathscr{A}\to\mathscr{B}$ are homomorphisms when $f(2^nuy)=f(2^nu)f(y)$, $g(2^nuy)=g(2^nu)g(y)$ and $h(2^nuy)=h(2^nu)h(y)$ hold for all $u\in I_1(\mathscr{A}_{\rm sa})$, all $y\in\mathscr{A}$ and all $n\in\mathbb{Z}$.

Furthermore, we prove the Cauchy-Rassias stability of $*$-homomorphisms between unital $C^*$-algebras, and $\mathbb{C}$-linear $*$-derivations on unital $C^*$-algebras.

## 2. $*$-HOMOMORPHISMS BETWEEN UNITAL $C^*$-ALGEBRAS

In this section, let $f, g, h\colon \mathscr{A} \to \mathscr{B}$ be mappings satisfying $f(0) = g(0) = h(0) = 0$, and let $f(2^n uy) = f(2^n u)f(y)$, $g(2^n uy) = g(2^n u)g(y)$ and $h(2^n uy) = h(2^n u)h(y)$ for all $u \in \mathscr{U}(\mathscr{A})$, all $y \in \mathscr{A}$ and all $n \in \mathbb{Z}$, unless otherwise specified. We are going to investigate $*$-homomorphisms between unital $C^*$-algebras.

**Theorem 1.** *Assume that there exists a function* $\varphi\colon \mathscr{A} \times \mathscr{A} \to [0, \infty)$ *such that*

$$\text{(i)} \qquad \widetilde{\varphi}(x, y) := \sum_{j=0}^{\infty} 2^{-j}\varphi(2^{j-1}x, 2^{j-1}y) < \infty,$$

$$\text{(ii)} \qquad \Big\| 2f\Big(\frac{\mu x + \mu y}{2}\Big) - \mu g(x) - \mu h(y)\Big\| \leqslant \varphi(x, y),$$

$$\text{(iii)} \qquad \|f(2^n u^*) - f(2^n u)^*\| \leqslant \varphi(2^n u, 2^n u)$$

*for all* $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C}\colon\ |\lambda| = 1\}$, *all* $u \in \mathscr{U}(\mathscr{A})$, *all* $x, y \in \mathscr{A}$ *and all* $n \in \mathbb{Z}$. *If*

$$\text{(iv)} \qquad \lim_{n\to\infty} \frac{f(2^n e)}{2^n} \quad \textit{is invertible},$$

*then the mappings* $f$, $g$, $h$ *are* $*$*-homomorphisms and* $f = g = h$.

P r o o f. Let $x \in \mathscr{A}$ be arbitrary. Put $\mu = 1 \in \mathbb{T}^1$ in (ii). It follows from [4, Corollary 2.5] that there exists a unique additive mapping $H\colon \mathscr{A} \to \mathscr{B}$ such that

$$(\dagger) \qquad \begin{aligned} \Big\| 2f\Big(\frac{x}{2}\Big) - H(x)\Big\| &\leqslant \varepsilon(x), \\ \|g(x) - H(x)\| &\leqslant \varphi(x, 0) + \varepsilon(x), \\ \|h(x) - H(x)\| &\leqslant \varphi(0, x) + \varepsilon(x), \end{aligned}$$

where $\varepsilon(x) := \varepsilon_\varphi(x)$ is given by (a). The additive mapping $H$ is given by

$$H(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} = \lim_{n\to\infty} \frac{g(2^n x)}{2^n} = \lim_{n\to\infty} \frac{h(2^n x)}{2^n}.$$

Let $\widetilde{f}(x) = 2f(\frac{1}{2}x)$, then $\lim\limits_{n\to\infty} 2^{-n}\widetilde{f}(2^n x) = \lim\limits_{n\to\infty} 2^{-n} f(2^n x)$.

Let $\mu \in \mathbb{T}^1$ and $x \in \mathscr{A}$ be arbitrary. By the assumption,

$$\begin{aligned} \|f(2^n \mu x) - \mu f(2^n x)\| &= \Big\| f(2^n \mu x) - \frac{1}{2}\mu g(2^n x) - \frac{1}{2}\mu h(2^n x) \\ &\qquad + \frac{1}{2}\mu g(2^n x) + \frac{1}{2}\mu h(2^n x) - \mu f(2^n x)\Big\| \\ &\leqslant \frac{1}{2}\varphi(2^n x, 2^n x) + \frac{1}{2}|\mu|\varphi(2^n x, 2^n x) \\ &= \varphi(2^n x, 2^n x). \end{aligned}$$

Thus $2^{-n}\|f(2^n\mu x)-\mu f(2^n x)\|\to 0$ as $n\to\infty$. Hence

$$H(\mu x)=\mu H(x). \tag{1}$$

Now let $\lambda\in\mathbb{C}$ and $M$ an integer greater than $2|\lambda|$. Since $|\lambda/M|<\frac{1}{2}$, there is $t\in(\frac{\pi}{3},\frac{\pi}{2}]$ such that $|\lambda/M|=\cos t=\frac{1}{2}(\mathrm{e}^{\mathrm{i}t}+\mathrm{e}^{-\mathrm{i}t})$. Now $\lambda/M=|\lambda/M|\mu$ for some $\mu\in\mathbb{T}^1$. And $H(x)=2H(\frac{1}{2}x)$ for all $x\in\mathscr{A}$. So $H(\frac{1}{2}x)=\frac{1}{2}H(x)$ for all $x\in\mathscr{A}$. Thus, by (1), $H(\lambda x)=H(M(\lambda/M)x)=\lambda H(x)$ for all $x\in\mathscr{A}$. So the unique additive mapping $H\colon\mathscr{A}\to\mathscr{B}$ is $\mathbb{C}$-linear.

By (i) and (iii), we get $H(u^*)=H(u)^*$ for all $u\in\mathscr{U}(\mathscr{A})$. Since $H$ is $\mathbb{C}$-linear and each $x\in\mathscr{A}$ is a finite linear combination of unitary elements (see [6, Theorem 4.1.7]), say, $x=\sum_{j=1}^{m}\lambda_j u_j$ ($\lambda_j\in\mathbb{C}$, $u_j\in\mathscr{U}(\mathscr{A})$), $H(x^*)=\sum_{j=1}^{m}\overline{\lambda_j}H(u_j)^*=H(x)^*$ for all $x\in\mathscr{A}$.

Let $u\in\mathscr{U}(\mathscr{A})$ and $y\in\mathscr{A}$ be arbitrary. Since $f(2^n uy)=f(2^n u)f(y)$ for all $n\in\mathbb{Z}$,

$$H(uy)=H(u)f(y). \tag{2}$$

So

$$H(uy)=H(u)\frac{1}{2^n}f(2^n y) \tag{3}$$

for all $n\in\mathbb{Z}$. Taking the limit in (3) as $n\to\infty$, we obtain

$$H(uy)=H(u)H(y). \tag{4}$$

Since $H$ is $\mathbb{C}$-linear and each $x\in\mathscr{A}$ is a finite linear combination of unitary elements, it follows from (4) that $H(xy)=H(x)H(y)$ for all $x\in\mathscr{A}$.

By (2) and (4), $H(e)H(y)=H(e)f(y)$ for all $y\in\mathscr{A}$. Since $\lim_{n\to\infty}f(2^n e)2^{-n}=H(e)$ is invertible, $H(y)=f(y)$ for all $y\in\mathscr{A}$. Similarly, $H(y)=g(y)=h(y)$ for all $y\in\mathscr{A}$.

Therefore, the mappings $f$, $g$, $h$ are $*$-homomorphisms and $f=g=h$. □

**Corollary 2.** *Assume that there exist constants $\theta\geqslant 0$ and $p\in[0,1)$ such that*

$$\Big\|2f\Big(\frac{\mu x+\mu y}{2}\Big)-\mu g(x)-\mu h(y)\Big\|\leqslant\theta(||x||^p+||y||^p),$$
$$\|f(2^n u^*)-f(2^n u)^*\|\leqslant 2^{np+1}\theta$$

*for all $\mu\in\mathbb{T}^1$, all $u\in\mathscr{U}(\mathscr{A})$, all $x,y\in\mathscr{A}$ and all $n\in\mathbb{Z}$. If $f$ satisfies* (iv), *the mappings $f$, $g$, $h$ are $*$-homomorphisms and $f=g=h$.*

P r o o f. Define $\varphi(x,y)=\theta(||x||^p+||y||^p)$ and apply Theorem 1. □

**Theorem 3.** *Assume that there exists a function* $\varphi\colon \mathscr{A}\times\mathscr{A}\to[0,\infty)$ *satisfying* (i) *and* (iii) *such that*

$$\text{(v)}\qquad \Big\|2f\Big(\frac{\mu x+\mu y}{2}\Big)-\mu g(x)-\mu h(y)\Big\|\leqslant\varphi(x,y)$$

*for* $\mu=1,\mathrm{i}$, *and all* $x,y\in\mathscr{A}$. *If* $f$ *satisfies* (iv) *and* $f(tx)$ *is continuous in* $t\in\mathbb{R}$ *for each fixed* $x\in\mathscr{A}$, *then the mappings* $f$, $g$, $h$ *are* $*$*-homomorphisms and* $f=g=h$.

Proof. Put $\mu=1$ in (v). By the same reasoning as in the proof of Theorem 1, there exists a unique additive mapping $H\colon \mathscr{A}\to\mathscr{B}$ satisfying (†). By the same reasoning as in the proof of [8, Theorem], the additive mapping $H$ is $\mathbb{R}$-linear.

Put $\mu=\mathrm{i}$ in (v). By the same method as in the proof of Theorem 1, one can obtain that $H(\mathrm{i}x)=\mathrm{i}H(x)$ for all $x\in\mathscr{A}$. For each $\lambda\in\mathbb{C}$, $\lambda=s+\mathrm{i}t$, where $s,t\in\mathbb{R}$. So $H(\lambda x)=sH(x)+\mathrm{i}tH(x)=\lambda H(x)$ for all $\lambda\in\mathbb{C}$ and all $x\in\mathscr{A}$. Hence the additive mapping $H$ is $\mathbb{C}$-linear.

The rest of the proof is the same as in the proof of Theorem 1. □

From now on, assume that $\mathscr{A}$ is a unital $C^*$-algebra of real rank zero, where "real rank zero" means that the set of invertible self-adjoint elements is dense in the set of self-adjoint elements (see [1]). Let $f$, $g$, $h$ be continuous and $f(0)=g(0)=h(0)=0$ and let $f(2^nuy)=f(2^nu)f(y)$, $g(2^nuy)=g(2^nu)g(y)$ and $h(2^nuy)=h(2^nu)h(y)$ for all $u\in I_1(\mathscr{A}_{\mathrm{sa}})$, all $y\in\mathscr{A}$ and all $n\in\mathbb{Z}$.

Now we are going to investigate continuous $*$-homomorphisms between unital $C^*$-algebras.

**Theorem 4.** *Assume that there exists a function* $\varphi\colon \mathscr{A}\times\mathscr{A}\to[0,\infty)$ *satisfying* (i), (ii) *and* (iii). *If f satisfies* (iv), *then the mappings* $f$, $g$, $h$ *are* $*$*-homomorphisms and* $f=g=h$.

Proof. By the same reasoning as in the proof of Theorem 1, there exists a unique $\mathbb{C}$-linear involutive mapping $H\colon \mathscr{A}\to\mathscr{B}$ satisfying the system of the inequalities (†).

Let $u\in I_1(\mathscr{A}_{\mathrm{sa}})$ and $y\in\mathscr{A}$ be arbitrary. Since $f(2^nuy)=f(2^nu)f(y)$ for all $n\in\mathbb{Z}$,

$$\text{(5)}\qquad H(uy)=H(u)f(y).$$

So

$$\text{(6)}\qquad H(uy)=H(u)\frac{1}{2^n}f(2^ny)$$

for all $n \in \mathbb{Z}$. Taking the limit in (6) as $n \to \infty$, we obtain

$$H(uy) = H(u)H(y). \tag{7}$$

Let $y \in \mathscr{A}$ be arbitrary. By (5) and (7),

$$H(e)H(y) = H(e)f(y).$$

Since $\lim_{n\to\infty} f(2^n e)/2^n = H(e)$ is invertible, $H(y) = f(y)$. Similarly, $H(y) = g(y) = h(y)$. So $H\colon \mathscr{A} \to \mathscr{B}$ is continuous. But by the assumption that $\mathscr{A}$ has real rank zero, it is easy to show that the set of linear combinations of elements of $I_1(\mathscr{A}_{\mathrm{sa}})$ is dense in $\mathscr{A}$. So for each $x \in \mathscr{A}$, there is a sequence $\{\kappa_j\}$ such that $\kappa_j \to x$ as $j \to \infty$ and $\kappa_j$ is a linear combination of elements of $I_1(\mathscr{A}_{\mathrm{sa}})$. Since $H$ is continuous, it follows from (7) and the $\mathbb{C}$-linearity of $H$ that

$$H(xy) = \lim_{j\to\infty} H(\kappa_j)H(y) = H(x)H(y) \tag{8}$$

for all $x \in \mathscr{A}$.

Therefore, the mappings $f$, $g$, $h$ are $*$-homomorphisms and $f = g = h$. □

**Corollary 5.** *Assume that there exist constants $\theta \geqslant 0$ and $p \in [0, 1)$ such that*

$$\Big\|2f\Big(\frac{\mu x + \mu y}{2}\Big) - \mu g(x) - \mu h(y)\Big\| \leqslant \theta(||x||^p + ||y||^p),$$
$$\|f(2^n u^*) - f(2^n u)^*\| \leqslant 2^{np+1}\theta$$

*for all $\mu \in \mathbb{T}^1$, all $u \in I_1(\mathscr{A}_{\mathrm{sa}})$, all $x, y \in \mathscr{A} \setminus \{0\}$ and all $n \in \mathbb{Z}$. If $f$ satisfies* (iv), *the mappings $f$, $g$, $h$ are $*$-homomorphisms.*

P r o o f. Define $\varphi(x, y) = \theta(||x||^p + ||y||^p)$ and apply Theorem 4. □

**Theorem 6.** *Assume that there exists a function $\varphi\colon \mathscr{A} \times \mathscr{A} \to [0, \infty)$ satisfying* (i), (iii) *and* (v). *If $f$ satisfies* (iv), *the mappings $f$, $g$, $h$ are $*$-homomorphisms and $f = g = h$.*

P r o o f. By the same reasoning as in the proof of Theorem 3, there exists a unique $\mathbb{C}$-linear mapping $H\colon \mathscr{A} \to \mathscr{B}$ satisfying the system of the inequalities (†).

The rest of the proof is the same as in the proofs of Theorems 1 and 4. □

## 3. Stability of $*$-homomorphisms between unital $C^*$-algebras

In this section, let $f, g, h\colon \mathscr{A} \to \mathscr{B}$ be mappings with $f(0) = g(0) = h(0) = 0$. We are going to show the Cauchy-Rassias stability of $*$-homomorphisms between unital $C^*$-algebras.

**Theorem 7.** *Assume that there exists a function* $\varphi\colon (\mathscr{A}\setminus\{0\})^4 \to [0,\infty)$ *such that*

$$\text{(vi)}\qquad \widetilde{\varphi}(x,y,z,w) = \sum_{j=0}^{\infty} 2^{-j}\varphi(2^j x, 2^j y, 2^j z, 2^j w) < \infty,$$

$$\text{(vii)}\qquad \Big\|2f\Big(\frac{\mu x+\mu y+zw}{2}\Big) - \mu g(x) - \mu h(y) - f(z)f(w)\Big\| \leqslant \varphi(x,y,z,w),$$

$$\text{(viii)}\qquad \|f(2^n u^*) - f(2^n u)^*\| \leqslant \varphi(2^n u, 2^n u, 0, 0)$$

*for all* $\mu\in\mathbb{T}^1$, *all* $u\in\mathscr{U}(\mathscr{A})$, *all* $x,y,z,w\in\mathscr{A}$ *and all* $n\in\mathbb{Z}$. *Then there exists a unique* $*$*-homomorphism* $H\colon \mathscr{A}\to\mathscr{B}$ *such that*

$$\text{(ix)}\qquad \begin{aligned} \Big\|2f\Big(\frac{x}{2}\Big) - H(x)\Big\| &\leqslant \varepsilon(x),\\ \|g(x)-H(x)\| &\leqslant \varphi(x,0,0,0)+\varepsilon(x),\\ \|h(x)-H(x)\| &\leqslant \varphi(0,x,0,0)+\varepsilon(x) \end{aligned}$$

*for all* $x\in\mathscr{A}$, *where*

$$\text{(b)}\qquad \varepsilon(x) := \varepsilon_{\psi}(x)$$

*is given by* (a) *and* $\psi(x,y) := \varphi(x,y,0,0)$ *for all* $x,y\in\mathscr{A}$.

Proof. Put $z=w=0$ and $\mu=1\in\mathbb{T}^1$ in (vii). By the same reasoning as in the proof of Theorem 1, there exists a unique $\mathbb{C}$-linear involutive mapping $H\colon\mathscr{A}\to\mathscr{B}$ satisfying (ix). The $\mathbb{C}$-linear mapping $H$ is given by

$$\text{(9)}\qquad H(x) = \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x\in\mathscr{A}$.

Let $z,w\in\mathscr{A}$ be arbitrary. Taking $x=y=0$ in (vii), $\|2f(\frac{1}{2}zw) - f(z)f(w)\| \leqslant \varphi(0,0,z,w)$. So

$$\text{(10)}\qquad \frac{1}{2^{2n}}\Big\|2f\Big(\frac{1}{2}\,2^n z\cdot 2^n w\Big) - f(2^n z)f(2^n w)\Big\| \leqslant \frac{1}{2^n}\varphi(0,0,2^n z, 2^n w).$$

By (vi), (9) and (10), $2H\big(\frac{1}{2}zw\big) = H(z)H(w)$. But since $H$ is $\mathbb{C}$-linear, $H(zw) = H(z)H(w)$. Hence the $\mathbb{C}$-linear mapping $H$ is a $*$-homomorphism satisfying (ix). $\square$

**Corollary 8.** *Assume that there exist constants $\theta \geqslant 0$ and $p \in [0,1)$ such that*

$$\begin{aligned}\Big\|2f\Big(\frac{\mu x+\mu y+zw}{2}\Big)-\mu g(x)-\mu h(y)-f(z)f(w)\Big\| \\ \leqslant \theta(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p),\end{aligned}$$
$$\|f(2^n u^*)-f(2^n u)^*\| \leqslant 2^{np+1}\theta$$

*for all $\mu \in \mathbb{T}^1$, all $u \in \mathscr{U}(\mathscr{A})$, all $x,y,z,w \in \mathscr{A}$ and all $n \in \mathbb{Z}$. Then there exists a unique $*$-homomorphism $H\colon \mathscr{A} \to \mathscr{B}$ such that*

$$\begin{aligned}\Big\|2f\Big(\frac{x}{2}\Big)-H(x)\Big\| &\leqslant \frac{1}{2-2^p}\theta\|x\|^p,\\ \|g(x)-H(x)\| &\leqslant \frac{3-2^p}{2-2^p}\theta\|x\|^p,\\ \|h(x)-H(x)\| &\leqslant \frac{3-2^p}{2-2^p}\theta\|x\|^p\end{aligned}$$

*for all $x \in \mathscr{A}$.*

Proof. Define $\varphi(x,y,z,w)=\theta(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)$ and apply Theorem 7. □

**Theorem 9.** *Assume that there exists a function $\varphi\colon \mathscr{A}^4 \to [0,\infty)$ satisfying* (vi) *and* (viii) *such that*

$$\Big\|2f\Big(\frac{\mu x+\mu y+zw}{2}\Big)-\mu g(x)-\mu h(y)-f(z)f(w)\Big\| \leqslant \varphi(x,y,z,w)$$

*for $\mu = 1, \mathrm{i}$, and all $x,y,z,w \in \mathscr{A}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathscr{A}$, then there exists a unique $*$-homomorphism $H\colon \mathscr{A} \to \mathscr{B}$ satisfying* (ix).

Proof. By the same reasoning as in the proof of Theorem 3, there exists a unique $\mathbb{C}$-linear mapping $H\colon \mathscr{A} \to \mathscr{B}$ satisfying (ix).

The rest of the proof is the same as in the proofs of Theorems 1 and 7. □

## 4. Stability of linear $*$-derivations on unital $C^*$-algebras

From now on, let $\mathscr{A} = \mathscr{B}$. We are going to show the Cauchy-Rassias stability of linear $*$-derivations on unital $C^*$-algebras.

**Theorem 10.** *Assume that there exists a function* $\varphi\colon \mathscr{A}^4 \to [0,\infty)$ *satisfying* (vi) *and* (viii) *such that*

$$\text{(x)}\qquad \Bigl\|2f\Bigl(\frac{\mu x+\mu y+zw}{2}\Bigr)-\mu g(x)-\mu h(y)-zf(w)-wf(z)\Bigr\| \leqslant \varphi(x,y,z,w)$$

*for all* $\mu\in\mathbb{T}^1$ *and all* $x,y,z,w\in\mathscr{A}$. *Then there exists a unique* $\mathbb{C}$*-linear* $*$*-derivation* $D\colon \mathscr{A}\to\mathscr{A}$ *such that*

$$\text{(xi)}\qquad \begin{aligned} \Bigl\|2f\Bigl(\frac{x}{2}\Bigr)-D(x)\Bigr\| &\leqslant \varepsilon(x),\\ \|g(x)-D(x)\| &\leqslant \varphi(x,0,0,0)+\varepsilon(x),\\ \|h(x)-D(x)\| &\leqslant \varphi(0,x,0,0)+\varepsilon(x) \end{aligned}$$

*for all* $x\in\mathscr{A}$, *where* $\varepsilon(x)$ *is given by* (b).

P r o o f. Put $z=w=0$ and $\mu=1\in\mathbb{T}^1$ in (x). By the same reasoning as in the proof of Theorem 1, there exists a unique $\mathbb{C}$-linear involutive mapping $D\colon\mathscr{A}\to\mathscr{A}$ satisfying (xi). The $\mathbb{C}$-linear mapping $D\colon\mathscr{A}\to\mathscr{A}$ is given by

$$\text{(11)}\qquad D(x)=\lim_{n\to\infty}\frac{1}{2^n}f(2^nx)$$

for all $x\in\mathscr{A}$.

Let $z,w\in\mathscr{A}$ be arbitrary. Taking $x=y=0$ in (x),

$$\Bigl\|2f\Bigl(\frac{zw}{2}\Bigr)-zf(w)-wf(z)\Bigr\|\leqslant\varphi(0,0,z,w).$$

So

$$\text{(12)}\qquad \frac{1}{2^{2n}}\Bigl\|2f\Bigl(\frac{1}{2}2^nz\cdot2^nw\Bigr)-2^nzf(2^nw)-2^nwf(2^nz)\Bigr\|\leqslant\frac{1}{2^n}\varphi(0,0,2^nz,2^nw).$$

By (x), (11) and (12), $2D(\frac{1}{2}zw)=zD(w)+wD(z)$. But since $D$ is $\mathbb{C}$-linear,

$$D(zw)=zD(w)+wD(z).$$

Hence the $\mathbb{C}$-linear mapping $D$ is a $*$-derivation satisfying (xi). □

**Corollary 11.** *Assume that there exist constants $\theta \geqslant 0$ and $p \in [0,1)$ such that*

$$\begin{aligned}\Big\|2f\Big(\frac{\mu x+\mu y+zw}{2}\Big)-\mu g(x)-\mu h(y)-zf(w)-wf(z)\Big\| \\ \leqslant \theta(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p),\end{aligned}$$
$$\|f(2^n u^*)-f(2^n u)^*\| \leqslant 2^{np+1}\theta$$

*for all $\mu \in \mathbb{T}^1$, all $u \in \mathscr{U}(\mathscr{A})$, all $x,y,z,w \in \mathscr{A}$ and all $n \in \mathbb{Z}$. Then there exists a unique $\mathbb{C}$-linear $*$-derivation $D\colon \mathscr{A} \to \mathscr{A}$ such that*

$$\begin{aligned}\Big\|2f\Big(\frac{x}{2}\Big)-D(x)\Big\| &\leqslant \frac{1}{2-2^p}\theta\|x\|^p,\\ \|g(x)-D(x)\| &\leqslant \frac{3-2^p}{2-2^p}\theta\|x\|^p,\\ \|h(x)-D(x)\| &\leqslant \frac{3-2^p}{2-2^p}\theta\|x\|^p\end{aligned}$$

*for all $x \in \mathscr{A}$.*

Proof. Define $\varphi(x,y,z,w)=\theta(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)$ and apply Theorem 10. □

**Theorem 12.** *Assume that there exists a function $\varphi\colon \mathscr{A}^4 \to [0,\infty)$ satisfying* (vi) *and* (viii) *such that*

$$\Big\|2f\Big(\frac{\mu x+\mu y+zw}{2}\Big)-\mu g(x)-\mu h(y)-zf(w)-wf(z)\Big\| \leqslant \varphi(x,y,z,w)$$

*for $\mu = 1, \mathrm{i}$, and all $x,y,z,w \in \mathscr{A}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathscr{A}$, then there exists a unique $\mathbb{C}$-linear $*$-derivation $D\colon \mathscr{A} \to \mathscr{A}$ satisfying* (xi).

Proof. By the same reasoning as in the proof of Theorem 3, there exists a unique $\mathbb{C}$-linear mapping $D\colon \mathscr{A} \to \mathscr{A}$ satisfying (xi).

The rest of the proof is the same as in the proofs of Theorems 1 and 10. □

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