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REPRESENTATION OF CONNECTED MONOUNARY ALGEBRAS BY MEANS OF IRREDUCIBLES

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Abstract. The aim of the present paper is to describe all connected monounary algebras for which there exists a representation by means of connected monounary algebras which are retract irreducible in the class $\mathscr{U}_c$ (or in $\mathscr{U}$).

Keywords: monounary algebra, connectedness, retract, retract irreducibility, representation

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0. Introduction

The relations between retracts and direct product decompositions have been investigated by Duffus and Rival [1] (for ordered sets), by Imrich and Klavžar [2] and by Klavžar [11] (for graphs) and by Kuczmanov, Reich, Schmidt and Stachura [12] (for metric spaces).

The author [5]–[10] dealt with these relations for monounary algebras. Let us denote by $\mathscr{U}$ (and $\mathscr{U}_c$, respectively) the class of all (all connected) monounary algebras.

In [5] and [4] all connected monounary algebras which are retract irreducible in the class $\mathscr{U}_c$ were described. Next, in [6] all connected monounary algebras retract irreducible in the class $\mathscr{U}$ were found. In [10] it was proved that there exist connected monounary algebras which are not representable by means of connected monounary algebras which are retract irreducible in $\mathscr{U}_c$. An analogous result was shown for the representability in $\mathscr{U}$.

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The aim of the present paper is to describe all connected monounary algebras for which there exists a representation by means of connected monounary algebras which are retract irreducible in $\mathscr{U}_c$ (or in $\mathscr{U}$).

1. Preliminaries

Let $\underline{A} = (A, F)$ be an algebra. A subalgebra $\underline{B} = (B, F)$ of $\underline{A}$ is called a retract of $\underline{A}$ if there is an endomorphism φ of $\underline{A}$ such that $\varphi(b) = b$ for each $b \in B$. For an algebra $\underline{A}$ let $R(\underline{A})$ be the class of all algebras which are isomorphic to a retract of $\underline{A}$.

Let $\mathscr{K}$ be a class of algebras of the same type. If $\underline{A} \in R\Big(\prod_{i\in I} \underline{A_i}\Big)$, $\underline{A_i} \in \mathscr{K}$ for each $i \in I$, then the system $\{\underline{A_i}\}_{i\in I}$ is called a representation of $\underline{A}$ in the class $\mathscr{K}$; we say that $\underline{A}$ is representable by means of $\{\underline{A_i}\}_{i\in I}$.

The algebra $\underline{A}$ is said to be retract irreducible in a class of algebras $\mathscr{K}$ if the following condition is satisfied:

(1) $$\text{If } \underline{A} \in R\Big(\prod_{i\in I} \underline{A_i}\Big),\ \ \underline{A_i} \in \mathscr{K} \text{ for each } i \in I, \\ \text{then there is } j \in I \text{ such that } \underline{A} \in R(\underline{A_j}).$$

We will say that $\underline{A}$ is representable by means of irreducibles in $\mathscr{K}$ if there is a representation $\{\underline{A_i}\}_{i\in I}$ of $\underline{A}$ such that each $\underline{A_i}$ is retract irreducible in $\mathscr{K}$.

Let us remark that if $\{\underline{A_i}\}_{i\in I}$ is a representation of $\underline{A}$ in $\mathscr{K}$ and if for each $i \in I$ the system $\{\underline{A_{ij_i}}\}_{j_i\in J_i}$ is a representation of $\underline{A_i}$ in $\mathscr{K}$, then the system

$$\{\underline{A_{ij_i}}\}_{i\in I,\, j_i\in J_i}$$

is a representation of $\underline{A}$, too.

In what follows we will apply the previous notion to the case of monounary algebras.

For the sake of completeness we recall some basic notions concerning monounary algebras.

Let $\underline{A} = (A, f)$ be a monounary algebra. As usual, a nonempty subset M of A is said to be a retract of $\underline{A}$ if there is a mapping h of A onto M such that h is an endomorphism of $\underline{A}$ and $h(x) = x$ for each $x \in M$. The mapping h is then called a retraction endomorphism corresponding to the retract M. Let us remark that then $\underline{M} = (M, f)$ is a subalgebra of $\underline{A}$. (In fact, for each subalgebra $\underline{A_1}$ of $\underline{A} = (A, f)$, the corresponding operation in $\underline{A_1}$ is denoted by the same symbol f.)

Let $\mathbb{Z}$ be the set of all integers, $\mathbb{N}$ the set of all positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A connected monounary algebra $\underline{A}$ is called unbounded if

$$(\forall\, x \in A)(\forall\, n \in \mathbb{N})(\exists\, m \in \mathbb{N})(f^{-(n+m)}(f^m(x)) \neq \emptyset).$$

The notion of the degree $s_f(x)$ of an element $x \in A$ was introduced in [13] (cf. also [3]) as follows. Let us denote by $A^{(\infty)}$ the set of all elements $x \in A$ such that there exists a sequence $\{x_n\}_{n \in \mathbb{N}\cup\{0\}}$ of elements belonging to A with the property $x_0 = x$ and $f(x_n) = x_{n-1}$ for each $n \in \mathbb{N}$. Further, we put $A^{(0)} = \{x \in A \colon\ f^{-1}(x) = \emptyset\}$. Now we define a set $A^{(\lambda)} \subseteq A$ for each ordinal λ by induction. Assume that we have defined $A^{(\alpha)}$ for each ordinal $\alpha < \lambda$. Then we put

$$A^{(\lambda)} = \Bigl\{x \in A - \bigcup_{\alpha<\lambda} A^{(\alpha)} \colon\ f^{-1}(x) \subseteq \bigcup_{\alpha<\lambda} A^{(\alpha)}\Bigr\}.$$

The sets $A^{(\lambda)}$ are pairwise disjoint. For each $x \in A$, either $x \in A^{(\infty)}$ or there is an ordinal λ with $x \in A^{(\lambda)}$. In the former case we put $s_f(x) = \infty$, in the latter we set $s_f(x) = \lambda$. We put $\lambda < \infty$ for each ordinal λ.

The following assertions are consequences of the definition of $s_f(x)$ (cf. also [14]) and we will sometimes use them without further reference.

(T1) If $s_f(x) \neq \infty$, then $s_f(f(x)) > s_f(x)$.

(T2) If h is a homomorphism of (A, f) into (B, g), then $s_g(h(x)) \geqslant s_f(x)$ for each $x \in A$.

(T3) Let $\{(A_i, f_i) \colon\ i \in I\}$ be a system of monounary algebras, $(A, f) = \prod\limits_{i\in I} (A_i, f_i)$. If $z \in \prod\limits_{i\in I} A_i$, then $s_f(z) \leqslant s_{f_i}(z(i))$ for each $i \in I$.

Further, by [5], Thm. 1.3, in which a characterization of retracts was given, we obtain

1.1. Lemma. *Let $\underline{A} = (A, f)$ be a monounary algebra and let $\underline{M} = (M, f)$ be a subalgebra of $\underline{A}$. Then M is a retract of $\underline{A}$ if and only if the following conditions are satisfied:*

(a) *if $y \in f^{-1}(M)$, then there is $z \in M$ such that $f(y) = f(z)$ and $s_f(y) \leqslant s_f(z)$;*

(b) *for any connected component K of $\underline{A}$ with $K \cap M = \emptyset$ there is a homomorphism of $\underline{K}$ into $\underline{M}$.*

1.2. Notation. For $k \in \mathbb{N}$ we denote by $\mathbb{Z}_k = \{0_k, 1_k, \ldots, (k-1)_k\}$ the set of all integers modulo k. Let us consider the following five types of monounary algebras:

a) $\underline{\mathbb{Z}} = (\mathbb{Z}, f)$, where $f(n) = n + 1$ for each $n \in \mathbb{Z}$;

b) $\underline{\mathbb{N}} = (\mathbb{N}, f)$, where $f(n) = n + 1$ for each $n \in \mathbb{N}$;

c) $\underline{\mathbb{N}_\infty} = (\mathbb{N}, f)$, where $f(n) = n - 1$ for each $n \in \mathbb{N} - \{1\}$, $f(1) = 1$;

d) $\underline{\mathbb{Z}_k} = (\mathbb{Z}_k, f)$, where $f(n_k) = (n+1)_k$ for each $n_k \in \mathbb{Z}_k$;

e) $\underline{\mathbb{N}_k}$ is a subalgebra of $\underline{\mathbb{N}_\infty}$, where $\mathbb{N}_k = \{1, 2, \ldots, k\}$.

Let $\mathscr{U}$ and $\mathscr{U}_c$ be as above. Connected monounary algebras which are retract irreducible in the above classes were characterized in [4], [5] and [6] as follows:

1.3. Theorem (cf. [5], R1 and [4], R). *Let $\underline{A}$ be a connected monounary algebra. Then $\underline{A}$ is retract irreducible in $\mathscr{U}_c$ if and only if $\underline{A}$ is isomorphic to one of the algebras $\underline{\mathbb{N}}$, $\underline{\mathbb{N}_\infty}$, $\underline{\mathbb{N}_k}$ or $\underline{\mathbb{Z}_{p^k}}$ for some $k \in \mathbb{N}$, p prime.*

1.4. Theorem (cf. [6], Thm.). *Let $\underline{A}$ be a connected monounary algebra. Then $\underline{A}$ is retract irreducible in $\mathscr{U}$ if and only if $\underline{A}$ is isomorphic to one of the algebras $\underline{\mathbb{N}}$, $\underline{\mathbb{N}_\infty}$ or $\underline{\mathbb{N}_k}$ for some $k \in \mathbb{N}$.*

In the remaining part of this section we assume that $\underline{A} = (A, f)$ is a connected monounary algebra. We start by proving some auxiliary results.

1.5. Lemma. *Let $\underline{A}$ be representable by means of irreducibles in $\mathscr{U}_c$ or in $\mathscr{U}$. Then $\{s_f(x) \colon x \in A\} \subseteq \mathbb{N}_0 \cup \{\infty\}$.*

P r o o f. The assumption yields that $\underline{A} \in R\Big(\prod\limits_{i \in I} \underline{A_i}\Big)$, where $\underline{A_i}$ are irreducibles (in $\mathscr{U}_c$, in $\mathscr{U}$). There is a retract T of $\prod\limits_{i\in I} \underline{A_i}$ such that $\underline{A} \cong \underline{T}$. By way of contradiction, let $a \in A$ be such that $s_f(a) = \lambda \in \text{Ord-}\mathbb{N}_0$. Let $\iota \colon \underline{A} \to \underline{T}$ be an isomorphism, $\iota(a) = b$. Then, for each $i \in I$, $s_f(b(i)) \geqslant s_f(b) = s(a) = \lambda$ in view of T3). By 1.3 or 1.4, $\{s_f(x) \colon x \in A_i\} \subseteq \mathbb{N}_0 \cup \{\infty\}$, which implies that $s_f(b(i)) = \infty$ for each $i \in I$, hence $s_f(b) = \infty \neq \lambda = s_f(a)$, which is a contradiction. $\square$

Now suppose that

$$(*) \qquad \{s_f(x) \colon x \in A\} \subseteq \mathbb{N}_0 \cup \{\infty\}$$

is valid.

1.6. Lemma. *Let A_0 be a retract of $\underline{A}$, $A_0 \neq A$. Then $\underline{A_0}$ is a subalgebra of $\underline{A}$ and there exists a representation $\{\underline{A_i}\}_{i \in I \cup \{0\}}$ of $\underline{A}$ such that if $i \in I$, then $\underline{A_i} = (A_i, f_i)$, $A_i \subseteq A$ and*

(i) *$A_i \cap A_0$ is a one-element cycle of A_i,*
(ii) *if $x \in A_i$ is noncyclic, then $f_i(x) = f(x)$,*
(iii) *if $x \in A_i$ is cyclic, then $|f_i^{-1}(x)| = 2$.*

First, we introduce some notation and then we prove Lemmas 1.6.1–1.6.3; as a consequence we then obtain that 1.6 is valid.

We denote $I = f^{-1}(A_0) - A_0$. If $I = \emptyset$, then $\{\underline{A_0}\}$ is a required representation. Suppose that $I \neq \emptyset$. For $i \in I$ let us put

$$A_i = \bigcup_{l \in \mathbb{N} \cup \{0\}} f^{-l}(i) \cup \{f(i)\}$$

and let the corresponding unary operation f_i on A_i be defined by the formula

$$f_i(x) = \begin{cases} f(x) & \text{if } x \in A_i - \{f(i)\}, \\ f(i) & \text{if } x = f(i). \end{cases}$$

Obviously, (i)–(iii) are valid. We denote

$$\underline{B} = (B, f) = \prod_{i \in I \cup \{0\}} \underline{A_i}.$$

Since A_0 is a retract of $\underline{A}$, there exists a retraction homomorphism φ of A onto A_0. Let us define a mapping $\tau\colon\ \underline{A} \to \underline{B}$ as follows:

a) If $a \in A_0$, then $\tau(a) \in B$ is such that

$$(\tau(a))(j) = \begin{cases} a & \text{if } j = 0, \\ f(j) & \text{if } j \in I. \end{cases}$$

b) If $a \in A - A_0$, then there exists a uniquely determined $i \in I$ such that $a \in A_i - A_0$; let us denote by $\tau(a) \in B$ such an element that

$$(\tau(a))(j) = \begin{cases} \varphi(a) & \text{if } j = 0, \\ a & \text{if } j = i, \\ f(j) & \text{if } j \in I - \{i\}. \end{cases}$$

Let $T = \{\tau(a)\colon\ a \in A\}$.

1.6.1. Lemma. *The mapping $\tau\colon\ A \to B$ is injective.*

Proof. Assume that $a, b \in A$, $\tau(a) = \tau(b)$. If $a, b \in A_0$, then

$$a = (\tau(a))(0) = (\tau(b))(0) = b.$$

Let $a \notin A_0$, thus $a \in A_i - A_0$ for some $a \in I$. Then

$$(\tau(b))(i) = (\tau(a))(i) = a \in A_i - A_0$$

and the definition of $\tau(b)$ implies that $b = a$. □

1.6.2. Lemma. *The mapping τ is a homomorphism of $\underline{A}$ into $\underline{B}$.*

Proof. First let $a \in A$ be such that $\{a, f(a)\} \subseteq A_i - A_0$, $i \in I$. Then

$$(\tau(f(a))(j) = \begin{cases} \varphi(f(a)) & \text{if } j = 0, \\ f(a) & \text{if } j = i, \\ f(j) & \text{otherwise.} \end{cases}$$

Therefore $\tau(f(a)) = f(\tau(a))$.

Now let $a \in A_i - A_0$, $f(a) \in A_0$, $i \in I$. Then $a = i$ and for $j \in I - \{i\}$ we obtain

$$\begin{aligned} f_0(\tau(a)(0)) &= f_0(\varphi(a)) = f(\varphi(a)) = \varphi(f(a)) = f(a) = (\tau(f(a)))(0); \\ f_i(\tau(a)(i)) &= f_i(a) = f(a) = f(i) = (\tau(f(i)))(i) = (\tau(f(a)))(a); \\ f_j(\tau(a)(j)) &= f_j(f(j)) = f(j) = (\tau(f(a)))(j); \end{aligned}$$

i.e., $f(\tau(a)) = \tau(f(a))$.

Finally, if $\{a, f(a)\} \subseteq A_0$, then obviously $\tau(f(a)) = f(\tau(a))$. $\square$

1.6.3. Lemma. *T is a retract of $\underline{B}$.*

Proof. a) First let us prove the validity of a) of 1.1. Assume that $y \in f^{-1}(T)$, $f(y) = \tau(a)$, $a \in A$. First let $a \in A_0$. We obtain

$$f(y(0)) = (f(y))(0) = (\tau(a))(0) = a.$$

Put $z = \tau(y(0))$. Then $z \in T$ and $f(z) = f(\tau(y(0))) = \tau(f(y(0))) = \tau(a) = f(y)$. Further, by T3) and T2),

$$s_f(y) \leqslant s_f(y(0)) \leqslant s_f(\tau(y(0))) = s_f(z).$$

Now suppose that $a \in A_i - A_0$, $i \in I$. Then

$$f(y(i)) = f_i(y(i)) = (f(y))(i) = a.$$

Put $z = \tau(y(i))$. Then $z \in T$ and $f(z) = f(\tau(y(i))) = \tau(f(y(i))) = \tau(a) = f(y)$. Similarly as above,

$$s_f(y) \leqslant s_{f_i}(y(i)) \leqslant s_f(\tau(y(i))) = s_f(z).$$

To prove b) of 1.1 let K be a connected component of B with $K \cap T = \emptyset$. There is a projection p_i of K onto A_i for any $i \in I$. Then $p_i \circ \tau\colon \underline{K} \to \underline{T}$ is a homomorphism. $\square$

Proof of 1.6. According to 1.6.1–1.6.3, $\underline{A} \in R\Big(\prod_{i \in I \cup \{0\}} \underline{A_i}\Big)$ and we have constructed the required representation. $\square$

2. $\underline{A}$ WITH A ONE-ELEMENT CYCLE

As above, in 2.1–2.6 we suppose that $\underline{A} = (A, f)$ is a connected monounary algebra such that

$$(*) \qquad \{s_f(x)\colon\ x \in A\} \subseteq \mathbb{N}_0 \cup \{\infty\}$$

is valid. Further, we will speak shortly about a representation by means of irreducibles and we will mean a representation of $\underline{A}$ by means of connected monounary algebras which are retract irreducible in the class $\mathscr{U}_c$.

Let us consider the following conditions:

(c1) $\underline{A}$ possesses a one-element cycle $\{c\}$ and $|f^{-1}(c)| = 2$;
(c2) (c1) is valid and $s_f(x) = \infty$ for each $x \in A$;
(c3) (c1) is valid and $s_f(x) \neq \infty$ for each $x \in A - \{c\}$.

2.1. Lemma. *If* (c2) *holds in* $\underline{A}$, *then* $\underline{A}$ *is representable by means of irreducibles.*

P r o o f. Assume that (c2) is valid and that $\underline{A}$ is not irreducible. By 1.3, there are $a, b \in A$ with $a \neq b$ such that $f(a) = f(b) \neq c$. Then the condition (C4) from [4] is satisfied and in view of the construction 4.1 and the corresponding lemmas in [4] we obtain that $\underline{A}$ is representable by means of irreducibles (these irreducibles are isomorphic to $\underline{\mathbb{N}_\infty}$). □

2.2. Notation. Let $\underline{A}$ satisfy (c3). There exists a unique element $a \in f^{-1}(c) - \{c\}$. Put

$$d(\underline{A}) = s_f(a) + 1;$$

we will call this positive integer the depth of $\underline{A}$.

2.3. Lemma. *Let* $\underline{A}$ *satisfy* (c3). *Then there exists a representation* $\{\underline{A_j}\}_{j\in J}$ *of* $\underline{A}$ *such that, for* $j \in J$, $\underline{A_j}$ *satisfies* (c3) *and either* $\underline{A_j}$ *is irreducible or* $d(\underline{A_j}) < d(\underline{A})$.

P r o o f. In view of the notation introduced in 2.2, there exist distinct elements $a = a_1, a_2, \ldots, a_{d(A)} \in A$ such that

$$f(a_k) = \begin{cases} a_{k-1} & \text{if } k > 1,\\ c & \text{if } k = 1,\end{cases}$$
$$f^{-1}(a_{d(\underline{A})}) = \emptyset.$$

It is easy to verify that $A_0 = \{c, a_1, a_2, \ldots, a_{d(A)}\}$ is a retract of $\underline{A}$. If $A_0 = A$, then the assertion holds. Let $A_0 \neq A$. Thus there is a representation of $\underline{A}$ satisfying the assertion of 1.6. Put $J = I \cup \{0\}$. From 1.3 it follows that $\underline{A_0}$ is irreducible. Let $i \in I$. By 1.6 (ii), if $x \in A_i$ and x fails to be cyclic, then $s_{f_i}(x) \leqslant s_f(x)$. Further, let $\{c_i\}$ be a cycle of A_i and $\{a_i\} = f_i^{-1}(c_i) - \{c_i\}$. Then $c_i \in A_0$ by virtue of 1.6 (i). If $c_i = c$, then $c \neq a_i \in f^{-1}(c) = \{c, a\}$, hence $\{a_i, c_i\} = \{a, c\} \subseteq A_i \cap A_0$ and $|A_i \cap A_0| \geqslant 2$, which is a contradiction to (i) of 1.6. Thus $c_i = a_k$ for some $k \in \{1, \ldots, d(\underline{A})\}$. Hence

$$d(\underline{A_i}) = s_{f_i}(a_i) + 1 \leqslant s_f(a_i) + 1 \leqslant s_f(c_i) = s_f(a_k) < s_f(a) + 1 = d(\underline{A}).$$

□

2.4. Lemma. *Let $\underline{A}$ satisfy* (c3). *Then $\underline{A}$ is representable by means of irreducibles.*

Proof. If $\underline{A}$ is irreducible, then $\underline{A}$ is representable. Assume that $\underline{A}$ is not irreducible. In view of 2.3, $\underline{A}$ has a representation in which the factors are either irreducible or have smaller depth than $\underline{A}$. The factors $\underline{A_i}$ which are not irreducible have representations in which the factors have depths smaller than $\underline{A_i}$ has. After finitely many steps we obtain a representation in which all factors are irreducible. □

2.5. Lemma. *Let $\underline{A}$ be a monounary algebra with a cycle $\{c\}$, $|A| \geqslant 2$. Then there exists a representation $\{\underline{A_i}\}_{i\in I}$ of $\underline{A}$ such that $\underline{A_i}$ satisfies* (c1) *for each $i \in I$.*

Proof. The assertion follows from 1.6, if we take $A_0 = \{c\}$. □

2.6. Lemma. *Let* (c1) *hold in $\underline{A}$. There exists a representation $\{\underline{A_i}\}_{i\in I}$ of $\underline{A}$ such that if $i \in I$, then $\underline{A_i}$ satisfies either* (c2) *or* (c3).

Proof. Let $A_0 = \{x \in A\colon s_f(x) = \infty\}$. If $A_0 = \{c\}$, then $\underline{A}$ satisfies (c3) and $\{\underline{A}\}$ is a one-element representation of $\underline{A}$. Suppose that $A_0 \neq \{c\}$. It is easy to verify that A_0 is a retract of $\underline{A}$ and we can apply 1.6, i.e., there is a representation $\{\underline{A_i}\}_{i\in I\cup\{0\}}$ satisfying the assertion of 1.6. Then 1.6 (i) and (ii) imply that $s_{f_i}(x) \neq \infty$ for each noncyclic element x of A_i, $i \in I$. Therefore $\underline{A_0}$ satisfies (c2) and $\underline{A_i}$, for $i \in I$, satisfies (c3). □

2.7. Proposition. *Suppose that $\underline{A}$ is a connected monounary algebra with a one-element cycle. Then $\underline{A}$ is representable by means of irreducibles in $\mathscr{U}_c$; moreover, each factor of the representation contains a one-element cycle.*

Proof. According to 2.5, there exists a representation $\{\underline{A_i}\}_{i\in I}$ of $\underline{A}$ such that $\underline{A_i}$ satisfies (c1) for each $i \in I$. Further, 2.6 yields that for each $\underline{A_i}$ for $i \in I$ there exists a representation $\{\underline{A_{ij_i}}\}_{j_i} \in J_i$ such that $\underline{A_{ij_i}}$ satisfies either (c2) or (c3) for each $j_i \in J_i$. All factors are then representable by means of irreducibles in view of 2.1 or 2.4, respectively, therefore $\underline{A}$ is representable by means of irreducibles. All factors which were used contain a one-element cycle. $\square$

3. Representation in $\mathscr{U}_c$

In 3.1–3.5 we will suppose that $\underline{A}$ is a connected monounary algebra and that $(*)$ is valid.

3.1. Lemma. *If A is a cycle, then $\underline{A}$ is representable by means of irreducibles.*

Proof. Suppose that $\underline{A}$ is an n-element cycle, where the canonical form of n is $p_1^{\alpha_1}\dots p_k^{\alpha_k}$ ($p_1,\dots,p_k$ being distinct primes, $\alpha_1,\dots,\alpha_k \in \mathbb{N}$). For $i \in \{1,\dots,k\}$ let $\underline{A_i}$ be a cycle with $p_i^{\alpha_i}$ elements. Denote $\underline{B} = \prod_{i=1}^{k} \underline{A_i}$. Then $\underline{B}$ is a cycle with $p_1^{\alpha_1} p_i^{\alpha_i} \dots p_k^{\alpha_k} = n$ elements, i.e., $\underline{B} \cong \underline{A}$. The algebras $\underline{A_i}$ are irreducible for $i \in 1,\dots,k$ in view of 1.3, therefore $\underline{A}$ is representable by means of irreducibles. $\square$

3.2. Lemma. *The algebra $\underline{\mathbb{Z}}$ is representable by means of cycles.*

Proof. For $i \in \mathbb{N}$ let $\underline{A_i}$ be a 2^i-element cycle. Put $\underline{B} = \prod_{i\in\mathbb{N}} \underline{A_i}$.

Then $\underline{B}$ consists of $\aleph_0$ connected components, each component being isomorphic to $\underline{\mathbb{Z}}$. Obviously, $\underline{\mathbb{Z}} \in R(\underline{B})$, thus $\underline{\mathbb{Z}}$ is representable by the cycles $\underline{A_i}$, $i \in \mathbb{N}$. $\square$

3.3. Lemma. *Assume that there is $x \in A$ with $s_f(x) = \infty$. Then $\underline{A}$ is representable by means of irreducibles.*

Proof. The assumption implies that there is a subalgebra $\underline{A_0}$ of $\underline{A}$ such that either

1) $\underline{A_0} \cong \underline{\mathbb{Z}}$, or
2) $\underline{A_0}$ is a cycle.

It is easy to see that A_0 is a retract of $\underline{A}$. We get by 1.6 that there is a representation $\{\underline{A_i}\}_{i\in I\cup\{0\}}$ such that if $i \in I$, then $\underline{A_i}$ contains a one-element cycle. According to 2.7, each $\underline{A_i}$, $i \in I$, is representable by means of irreducibles. Further, $\underline{A_0}$ is representable by means of cycles in view of 3.2, thus it is representable by means of irreducibles in view of 3.1. Therefore $\underline{A}$ is representable by means of irreducibles. $\square$

3.4. Lemma. *Assume that $s_f(x) \neq \infty$ for each $x \in A$ and that $\underline{A}$ is bounded. Then $\underline{A}$ is representable by means of irreducibles, which are not cycles.*

Proof. From [7], 1.6 it follows that there is a subalgebra $\underline{A_0}$ of $\underline{A}$ such that $\underline{\mathbb{N}} \cong \underline{A_0}$ and that $\underline{A_0}$ is a retract of $\underline{A}$. If $A_0 = A$, then $\underline{A}$ is irreducible. Let $A \neq A_0$. Then according to 1.6, $\underline{A}$ is representable by $\underline{A_0}$ and by nontrivial algebras containing one-element cycles. Then $\underline{A}$ is representable by means of irreducibles in view of 2.7; one of the factors is $\underline{A_0} \cong \underline{\mathbb{N}}$ and the other factors contain a one-element cycle and are nontrivial, thus none of the factors is a cycle. $\square$

3.5. Lemma. *Assume that $s_f(x) \neq \infty$ for each $x \in A$ and that $\underline{A}$ is unbounded. Then there is no representation by means of irreducibles for the algebra $\underline{A}$.*

Proof. By way of contradiction, suppose that there is a representation $\{\underline{A_i}\}_{i\in I}$ of $\underline{A}$ by irreducibles. Denote $\underline{B} = \prod_{i\in I} \underline{A_i}$ and assume that T is a retract of $\underline{B}$ with $\underline{T} \cong \underline{A}$. Then $s_f(x) \neq \infty$ for each $x \in T$. If for each $i \in I$ there is $a_i \in A_i$ with $s_f(a_i) = \infty$, then the element $b \in B$ such that $b(i) = a_i$ for each $i \in I$ satisfies the relation $s_f(b) = \infty$; by T2) this element cannot be mapped by any endomorphism of $\underline{B}$ into T, which yields a contradiction. Thus there is $i \in I$ such that $s_f(x) \neq \infty$ for each $x \in A_i$. Since $\underline{A_i}$ is irreducible, $\underline{A_i} \cong \underline{\mathbb{N}}$.

Further, we have supposed that $\underline{A}$ is unbounded, thus $\underline{T}$ is unbounded and

(2) $$(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(f^{-(n+m)}(f^m(x)) \neq \emptyset) \text{ is valid for each } x \in T.$$

Then

(3) $$(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(f^{-(n+m)}(f^m(x(i)) \neq \emptyset),$$

which is a contradiction to the relation $\underline{A_i} \cong \underline{\mathbb{N}}$. $\square$

3.6. Theorem. *Let $\underline{A}$ be a connected monounary algebra. Then $\underline{A}$ is representable by means of connected monounary algebras which are retract irreducible in the class $\mathscr{U}_c$ if and only if*

1) $\{s_f(x)\colon x \in A\} \subseteq \mathbb{N}_0 \cup \{\infty\}$,
2) *if* $\{s_f(x)\colon x \in A\} \subseteq \mathbb{N}_0$, *then A is bounded.*

Proof. Suppose that $\underline{A}$ is representable by means of irreducibles in $\mathscr{U}_c$. By 1.5, 1) is valid. Let $\{s_f(x)\colon x \subseteq A\} \subseteq \mathbb{N}_0$. If $\underline{A}$ is unbounded, then 3.5 yields a contradiction; therefore the condition 2) is satisfied, too.

Conversely, let 1) and 2) hold. If there is $x \in A$ with $s_f(x) = \infty$, then $\underline{A}$ is representable by means of irreducibles according to 3.3. If $\{s_f(x)\colon x \in A\} \subseteq \mathbb{N}_0$, then $\underline{A}$ is bounded by 2), and then 3.4 implies that $\underline{A}$ is representable by means of irreducibles in $\mathscr{U}_c$. $\square$

4. Representation in $\mathscr{U}$

The aim of this section is to characterize the connected monounary algebras which are representable by means of connected monounary algebras which are retract irreducible in the class $\mathscr{U}$.

Let $\underline{A}$ be a connected monounary algebra. It follows from 1.3 and 1.4 that if $\underline{A}$ is representable by means of connected monounary algebras which are retract irreducible in $\mathscr{U}$, then $\underline{A}$ is representable by means of irreducibles in $\mathscr{U}_c$.

4.1. Lemma. *Suppose there is a subalgebra $\underline{C}$ of $\underline{A}$ such that either $\underline{C} \cong \underline{\mathbb{Z}}$ or $\underline{C} \cong \underline{\mathbb{Z}_n}$ for some $n \in A$, $n > 1$. Then $\underline{A}$ is not representable by means of connected monounary algebras which are retract irreducible in $\mathscr{U}$.*

Proof. Assume that $\{\underline{A_i}\}_{i\in I}$ is a representation by means of connected monounary algebras which are retract irreducible in $\mathscr{U}$, corresponding to the algebra $\underline{A}$. By assumption, there is $a \in A$ with $s_f(a) = \infty$. Put $\underline{B} = \prod_{i\in I} \underline{A_i}$ and let $\iota\colon A \to T$ be an isomorphism of $\underline{A}$ onto a retract T of $\underline{B}$. Then $s_f(\iota(a)) = \infty$, hence T3) yields that $s_f((\iota(a))(i)) = \infty$ for each $i \in I$. Thus if $i \in I$, then there is $a_i \in A_i$ with $s_f(a_i) = \infty$. Since $\underline{A_i}$ is irreducible in $\mathscr{U}$, 1.4 implies that a_i forms a one-element cycle. This yields that $\underline{B}$ is a connected monounary algebra possessing a one-element cycle. Therefore neither $\underline{B}$ nor the retract T of $\underline{B}$ contains a subalgebra isomorphic to $\underline{\mathbb{Z}}$ ($\underline{\mathbb{Z}_n}$ for $n > 1$, respectively), which contradicts the relation $\underline{T} \cong \underline{A}$. □

4.2. Theorem. *Let $\underline{A}$ be a connected monounary algebra. Then $\underline{A}$ is representable by means of connected monounary algebras which are retract irreducible in $\mathscr{U}$ if and only if the following conditions are satisfied:*

(1) $\{s_f(x)\colon\ x \in A\} \subseteq \mathbb{N}_0 \cup \{\infty\}$,

(2) *if* $\{s_f(x)\colon\ x \in A\} \subseteq \mathbb{N}_0$, *then* $\underline{A}$ *is bounded,*

(3) *if* $x \in A, s_f(x) = \infty$, *then* $f(x) = x$.

Proof. As was already remarked, if $\underline{A}$ is representable by means of connected monounary algebras which are retract irreducible in $\mathscr{U}$, then $\underline{A}$ is representable by means of irreducibles in $\mathscr{U}_c$, thus then (1) and (2) hold. If in this case (3) fails to hold, then 4.1 yields a contradiction.

Conversely, suppose that (1)–(3) are valid. Therefore one of the following possibilities occurs:

(a) $\underline{A}$ contains a one-element cycle;

(b) $\underline{A}$ contains no cycle, $\{s_f(x)\colon\ x \in A\} \subseteq \mathbb{N}_0$, $\underline{A}$ is bounded.

Let (a) hold. By 2.7, $\underline{A}$ is representable by means of irreducibles in $\mathscr{U}_c$, each factor of the representation containing a one-element cycle, i.e., it is irreducible in $\mathscr{U}$ as well. Thus $\underline{A}$ is representable by means of irreducibles in $\mathscr{U}$.

Now let (b) be valid. According to 3.4 there exists a representation of $\underline{A}$ by means of irreducibles in $\mathscr{U}_c$, where none of the factors in the representation is a cycle. Thus this is a representation of $\underline{A}$ by means of irreducibles in $\mathscr{U}$, which concludes the proof. □

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