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Krzysztof Ciepliński

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GENERAL CONSTRUCTION OF NON-DENSE DISJOINT ITERATION GROUPS ON THE CIRCLE

Krzysztof Ciepliński, Kraków

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Abstract. Let $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \to \mathbb{S}^1, v \in V\}$ be a disjoint iteration group on the unit circle $\mathbb{S}^1$, that is a family of homeomorphisms such that $F^{v_1} \circ F^{v_2} = F^{v_1+v_2}$ for $v_1, v_2 \in V$ and each F^v either is the identity mapping or has no fixed point ($(V,+)$ is a 2-divisible nontrivial Abelian group). Denote by $L_{\mathscr{F}}$ the set of all cluster points of $\{F^v(z), v \in V\}$ for $z \in \mathbb{S}^1$. In this paper we give a general construction of disjoint iteration groups for which $\emptyset \neq L_{\mathscr{F}} \neq \mathbb{S}^1$.

Keywords: (disjoint, non-singular, singular, non-dense) iteration group, (strictly) increasing mapping

MSC 2000: 37E10, 20F38, 39B12

1. Introduction

Let X be a topological space and $(V,+)$ be a 2-divisible nontrivial (i.e., $\operatorname{card} V > 1$) Abelian group.

Recall that a family $\{F^v\colon X \to X, v \in V\}$ of homeomorphisms with $F^{v_1} \circ F^{v_2} = F^{v_1+v_2}$ for $v_1, v_2 \in V$ is called an *iteration group* or a *flow* (on X). An iteration group $\{F^v\colon X \to X, v \in V\}$ is said to be *disjoint* if each of its elements either is the identity mapping or has no fixed point. The structure of such iteration groups on open real intervals in the case where $V = \mathbb{R}$ has been studied in [8]. Some special cases of disjoint iteration groups on the unit circle $\mathbb{S}^1$ under the assumption that $V = \mathbb{R}$ have been investigated in [1] and [2].

By the *limit set* of a disjoint iteration group $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \to \mathbb{S}^1, v \in V\}$ we mean the set $L_{\mathscr{F}} := \{F^v(z), v \in V\}^{\mathrm{d}}$, where z is an arbitrary element of $\mathbb{S}^1$ and A^{d} stands for the set of all cluster points of A. An iteration group $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \to \mathbb{S}^1, v \in V\}$ is said to be *non-singular* if at least one its element has no periodic point, otherwise

$\mathscr{F}$ is called a *singular* iteration group. By the *limit set* of a non-singular iteration group $\mathscr{F}$ we mean the set $L_{\mathscr{F}} := L_{F^v}$, where $F^v \in \mathscr{F}$ is an arbitrary homeomorphism with irrational rotation number $\alpha(F^v)$ and L_{F^v} is the limit set of F^v. A non-singular or disjoint iteration group $\mathscr{F} = \{F^v \colon \mathbb{S}^1 \to \mathbb{S}^1, v \in V\}$ is called: *dense*, if $L_{\mathscr{F}} = \mathbb{S}^1$; *non-dense*, if $\emptyset \neq L_{\mathscr{F}} \neq \mathbb{S}^1$; *discrete*, if $L_{\mathscr{F}} = \emptyset$. It is worth pointing out that every discrete iteration group is both disjoint and singular, and every dense iteration group is disjoint (see [5]).

The aim of this paper is to present a general construction of non-dense disjoint iteration groups $\mathscr{F} = \{F^v \colon \mathbb{S}^1 \to \mathbb{S}^1, v \in V\}$. This together with [5] gives a complete description of disjoint iteration groups on the circle.

2. Preliminaries

We begin by recalling the basic definitions and introducing some notation.

For any $v, w, z \in \mathbb{S}^1$ there exist unique $t_1, t_2 \in [0, 1)$ such that $we^{2\pi i t_1} = z$ and $we^{2\pi i t_2} = v$, so we can put

$$v \prec w \prec z \quad \text{if and only if} \quad 0 < t_1 < t_2,$$
$$v \preceq w \preceq z \quad \text{if and only if} \quad t_1 \leqslant t_2 \text{ or } t_2 = 0$$

(see [2]). Some properties of these relations can be found in [3] and [4]. It is easily seen that we also have

Lemma 1 (see also [6]). *For any $v, u, w, z \in \mathbb{S}^1$:*

(i) *$v \prec w \prec z$ implies $u \cdot v \prec u \cdot w \prec u \cdot z$,*

(ii) *$u \prec v \prec w$ and $u \prec w \prec z$ imply $v \prec w \prec z$.*

For any $v, w, z \in \mathbb{S}^1$ set

$$v \preceq w \prec z \quad \text{if and only if} \quad v \prec w \prec z \text{ or } v = w,$$
$$v \prec w \preceq z \quad \text{if and only if} \quad v \prec w \prec z \text{ or } w = z.$$

A set $A \subset \mathbb{S}^1$ is said to be an *open arc* if there are distinct $v, z \in \mathbb{S}^1$ with

$$A = \overrightarrow{(v, z)} := \{w \in \mathbb{S}^1 \colon v \prec w \prec z\} = \{e^{2\pi i t}, \ t \in (t_v, t_z)\},$$

where $t_v, t_z \in \mathbb{R}$ are such that $e^{2\pi i t_v} = v$, $e^{2\pi i t_z} = z$ and $0 < t_z - t_v < 1$. A mapping $F \colon A \to \mathbb{S}^1$ is said to be *linear* if there are $a, b \in \mathbb{R}$, $a > 0$ with $F(e^{2\pi i x}) = e^{2\pi i (ax+b)}$ for $x \in (t_v, t_z)$.

Given a subset A of $\mathbb{S}^1$ with $\operatorname{card} A \geqslant 3$ and a function F mapping A into $\mathbb{S}^1$ we say that F is *increasing* (respectively, *strictly increasing*) if for any $v, w, z \in A$ such that $v \prec w \prec z$ we have $F(v) \preceq F(w) \preceq F(z)$ (respectively, $F(v) \prec F(w) \prec F(z)$). Some properties of such functions one can find in [3] and [4]. It is a simple matter to check that we also have

Lemma 2. *If $A, B \subset \mathbb{S}^1$, $\operatorname{card} A \geqslant 3$ and F is a strictly increasing function mapping A onto B, then F is invertible and $F^{-1}\colon B \to A$ is strictly increasing.*

Lemma 3. *Every increasing mapping $F\colon \mathbb{S}^1 \to \mathbb{S}^1$ such that $\operatorname{cl} F[\mathbb{S}^1] = \mathbb{S}^1$ is continuous.*

We now repeat the relevant, slightly modified, material from [5] and [7].

Lemma 4 (see [5])**.** *A disjoint iteration group $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \to \mathbb{S}^1, v \in V\}$ is discrete if and only if $\operatorname{card}\{F^v(z), v \in V\} < \aleph_0$ for $z \in \mathbb{S}^1$.*

Proposition 1 (see [5])**.** *If $\mathscr{P} = \{P^v\colon \mathbb{S}^1 \to \mathbb{S}^1, v \in V\}$ is a dense or non-dense iteration group, then there exists a unique pair (φ, c) such that $\varphi\colon \mathbb{S}^1 \to \mathbb{S}^1$ is a continuous mapping of degree 1 with $\varphi(1) = 1$ and $c\colon V \to \mathbb{S}^1$ for which*

$$\varphi(P^v(z)) = c(v)\varphi(z), \quad z \in \mathbb{S}^1,\ v \in V. \tag{1}$$

The function c is given by $c(v) = \mathrm{e}^{2\pi\mathrm{i}\alpha(P^v)}$ for $v \in V$ and it is a homomorphic mapping. The mapping φ is increasing and $\varphi[L_{\mathscr{P}}] = \mathbb{S}^1$. Moreover, φ is a homeomorphism if and only if the iteration group $\mathscr{P}$ is dense.

Given a dense or non-dense iteration group $\mathscr{P} = \{P^v\colon \mathbb{S}^1 \to \mathbb{S}^1, v \in V\}$ we write $\varphi_{\mathscr{P}}$ and $c_{\mathscr{P}}$ for the functions described by Proposition 1.

Lemma 5 (see [5] and [7])**.** *If $\mathscr{P} = \{P^v\colon \mathbb{S}^1 \to \mathbb{S}^1, v \in V\}$ is a dense or non-dense iteration group, then a pair (φ, c) such that $\varphi\colon \mathbb{S}^1 \to \mathbb{S}^1$ is a continuous mapping with $\varphi(1) = 1$ and $c\colon V \to \mathbb{S}^1$ satisfies (1) if and only if $c = (c_{\mathscr{P}})^n$ and $\varphi = (\varphi_{\mathscr{P}})^n$ for an integer n.*

If $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \to \mathbb{S}^1, v \in V\}$ is a non-dense iteration group, then its limit set is a non-empty perfect and nowhere dense subset of $\mathbb{S}^1$, and therefore

$$\mathbb{S}^1 \setminus L_{\mathscr{F}} = \bigcup_{q \in \mathbb{Q}} I_q, \tag{2}$$

where I_q for $q \in \mathbb{Q}$ are open pairwise disjoint arcs.

Lemma 6 (see [5]). *If $\mathscr{F} = \{F^v \colon \mathbb{S}^1 \to \mathbb{S}^1, v \in V\}$ is a non-dense iteration group, then:*

(i) *for every $q \in \mathbb{Q}$ the mapping $\varphi_{\mathscr{F}}$ is constant on I_q,*

(ii) *if $A \subset \mathbb{S}^1$ is an open arc and $\varphi_{\mathscr{F}}$ is constant on A, then $A \subset I_q$ for a $q \in \mathbb{Q}$,*

(iii) *for any distinct $p, q \in \mathbb{Q}$, $\varphi_{\mathscr{F}}[I_p] \cap \varphi_{\mathscr{F}}[I_q] = \emptyset$,*

(iv) *the sets $\operatorname{Im} c_{\mathscr{F}}$ and $K_{\mathscr{F}} := \varphi_{\mathscr{F}}[\mathbb{S}^1 \setminus L_{\mathscr{F}}]$ are countable and dense in $\mathbb{S}^1$,*

(v) *$K_{\mathscr{F}} \cdot \operatorname{Im} c_{\mathscr{F}} = K_{\mathscr{F}}$.*

According to Lemma 6 we can correctly define the bijection $\Phi_{\mathscr{F}} \colon \mathbb{Q} \to K_{\mathscr{F}}$ and the mapping $T_{\mathscr{F}} \colon \mathbb{Q} \times V \to \mathbb{Q}$ putting

$$\{\Phi_{\mathscr{F}}(q)\} := \varphi_{\mathscr{F}}[I_q], \quad T_{\mathscr{F}}(q, v) := \Phi_{\mathscr{F}}^{-1}(\Phi_{\mathscr{F}}(q) c_{\mathscr{F}}(v)), \quad q \in \mathbb{Q},\ v \in V.$$

Proposition 2 (see [5]). *If $\mathscr{F} = \{F^v \colon \mathbb{S}^1 \to \mathbb{S}^1, v \in V\}$ is a non-dense disjoint iteration group, then there exists a unique disjoint, non-dense iteration group $\mathscr{P} = \{P^v \colon \mathbb{S}^1 \to \mathbb{S}^1, v \in V\}$ such that for any $q \in \mathbb{Q}, v \in V$, P^v is linear on I_q and $P^v[I_q] = I_{T_{\mathscr{F}}(q,v)}$. Moreover, there is a homeomorphism $\Gamma \colon \mathbb{S}^1 \to \mathbb{S}^1$ satisfying*

$$F^v = \Gamma^{-1} \circ P^v \circ \Gamma, \quad v \in V \tag{3}$$

such that $\Gamma(z) = z$ for $z \in L_{\mathscr{F}}$.

3. Main result

We are now in a position to give a general construction of non-dense disjoint iteration groups. Let us first observe that from Proposition 1 and Lemma 6 it follows that if $\mathscr{F} = \{F^v \colon \mathbb{S}^1 \to \mathbb{S}^1, v \in V\}$ is such a group, then

(H) there is a homomorphic mapping $c \colon V \to \mathbb{S}^1$ with $\operatorname{card} \operatorname{Im} c = \aleph_0$.

Therefore we assume that (H) holds true. It is obvious that if V is a finite group, then (H) is not satisfied, whereas if $V = \mathbb{Q}$, then $c := \exp|_{\mathbb{Q}}$ is the desired homomorphic mapping. From Lemma 15 in [3] it follows that (H) holds for $V = \mathbb{R}$.

Let L be a perfect nowhere dense subset of $\mathbb{S}^1$ and I_q for $q \in \mathbb{Q}$ be open pairwise disjoint arcs such that

$$\mathbb{S}^1 \setminus L = \bigcup_{q \in \mathbb{Q}} I_q. \tag{4}$$

Take an $M \subset \bigcup_{q\in\mathbb{Q}} I_q$ with $\operatorname{card}(M \cap I_q) = 1$ for $q \in \mathbb{Q}$. For any $\alpha \in M$ denote by I_α the arc I_q such that $\alpha \in I_q$. Clearly, $\operatorname{card} M = \aleph_0$, $\mathbb{S}^1 \setminus L = \bigcup_{\alpha\in M} I_\alpha$ and

$$\alpha \prec \beta \prec \gamma \quad \text{if and only if} \quad I_\alpha \prec I_\beta \prec I_\gamma,\ \alpha,\beta,\gamma \in M. \tag{5}$$

Fix a $z_M \in \mathbb{S}^1 \setminus \bigcup_{\alpha\in M} \operatorname{cl} I_\alpha$ and define

$$\alpha \preceq_M \beta \quad \text{if and only if} \quad z_M \preceq \alpha \preceq \beta,\ \alpha,\beta \in M. \tag{6}$$

Since $\alpha, \beta \in \mathbb{S}^1 \setminus L, z_M \in L$, Lemma 3 in [3] shows that $\alpha \preceq_M \beta$ if and only if $z_M \prec \alpha \preceq \beta$. Moreover, $(M, \preceq_M)$ is easily checked to be of ordered type η.

Let $c\colon V \to \mathbb{S}^1$ be a homomorphic mapping with $\operatorname{card} \operatorname{Im} c = \aleph_0$. Then, we also have $\operatorname{cl} \operatorname{Im} c = \mathbb{S}^1$.

Take a non-empty subset A of $\mathbb{S}^1$ such that $\operatorname{card} A \leqslant \aleph_0$ and put

$$K := \operatorname{Im} c \cdot A.$$

Obviously, $\operatorname{card} K = \aleph_0$ and $\operatorname{cl} K = \mathbb{S}^1$. Furthermore,

$$K \cdot \operatorname{Im} c = K. \tag{7}$$

Choose a $z_K \in \mathbb{S}^1 \setminus K$ and set

$$z_1 \preceq_K z_2 \quad \text{if and only if} \quad z_K \preceq z_1 \preceq z_2,\ z_1, z_2 \in K. \tag{8}$$

We see at once that $(K, \preceq_K)$ is of ordered type η and

$$z_1 \preceq_K z_2 \quad \text{if and only if} \quad z_K \prec z_1 \preceq z_2,\ z_1, z_2 \in K. \tag{9}$$

Let $\Phi\colon M \to K$ be an order preserving bijection. We shall show that it is strictly increasing. To do this fix $\alpha, \beta, \gamma \in M$ such that $\alpha \prec \beta \prec \gamma$ and note that according to Lemma 2 in [3] it suffices to prove that $\Phi(\alpha) \prec \Phi(\beta) \prec \Phi(\gamma)$ only in case $z_M \in \overrightarrow{(\gamma, \alpha)}$. If $z_M \in \overrightarrow{(\gamma, \alpha)}$ then, by (6) and the fact that Φ preserves order, we get $\Phi(\alpha) \preceq_K \Phi(\beta)$ and $\Phi(\beta) \preceq_K \Phi(\gamma)$. Since we also have $\Phi(\alpha) \neq \Phi(\beta)$ and $\Phi(\beta) \neq \Phi(\gamma)$, (9) together with Lemma 1(ii) now yields $\Phi(\alpha) \prec \Phi(\beta) \prec \Phi(\gamma)$.

(7) makes it possible to define the mapping $T\colon M \times V \to M$ putting

$$T(\alpha, v) := \Phi^{-1}(\Phi(\alpha)c(v)), \quad \alpha \in M,\ v \in V. \tag{10}$$

We shall now construct a piecewise linear iteration group. Let $x_0 \in [0,1)$ be such that $e^{2\pi i x_0} = z_M \in L$ and set $\nu(x) := e^{2\pi i(x+x_0)}$ for $x \in [0,1)$. Putting $L' := \nu^{-1}[L] \cap (0,1)$ we have $(0,1) \setminus L' = \bigcup_{\alpha \in M} I'_\alpha$, where $I'_\alpha := \nu^{-1}[I_\alpha]$ for $\alpha \in M$ are open pairwise disjoint intervals. Let $l_{\alpha,v}$ for $\alpha \in M$, $v \in V$ be strictly increasing linear functions with $l_{\alpha,v}[I'_\alpha] = I'_{T(\alpha,v)}$. Defining

$$B_v(z) := (\nu \circ l_{\alpha,v} \circ \nu^{-1}|_{I_\alpha})(z), \quad z \in I_\alpha,\ \alpha \in M,\ v \in V$$

we obtain

$$B_v[I_\alpha] = I_{T(\alpha,v)}, \quad \alpha \in M,\ v \in V. \tag{11}$$

Fix a $v \in V$. We claim that $B_v \colon \mathbb{S}^1 \setminus L \to \mathbb{S}^1 \setminus L$ is strictly increasing. Indeed, take $x, w, z \in \mathbb{S}^1 \setminus L$ with $x \prec w \prec z$ and assume that $\operatorname{card}(\{x,w,z\} \cap I_\alpha) \leqslant 1$ for $\alpha \in M$ (the other cases can be handled in the same way as in the proof of Lemma 13 in [3]). If $\alpha, \beta, \gamma \in M$, $\alpha \neq \beta$, $\alpha \neq \gamma$, $\beta \neq \gamma$ are such that $x \in I_\alpha$, $w \in I_\beta$, $z \in I_\gamma$, then $I_\alpha \prec I_\beta \prec I_\gamma$ and, by (5), $\alpha \prec \beta \prec \gamma$. Since Φ is strictly increasing, from Lemmas 1(i) and 2 and (7) it follows that $\Phi^{-1}(\Phi(\alpha)c(v)) \prec \Phi^{-1}(\Phi(\beta)c(v)) \prec \Phi^{-1}(\Phi(\gamma)c(v))$. This together with (10), (5) and (11) gives $B_v[I_\alpha] \prec B_v[I_\beta] \prec B_v[I_\gamma]$, which is the desired conclusion.

Applying Lemma 12 in [4] we see that every function B_v can be extended to a strictly increasing mapping $P^v \colon \mathbb{S}^1 \to \mathbb{S}^1$. Analysis similar to that in the proof of Lemma 13 in [3] shows that $\mathscr{P} := \{P^v \colon \mathbb{S}^1 \to \mathbb{S}^1, v \in V\}$ is a piecewise linear iteration group on $\mathbb{S}^1$.

Put

$$\varphi(z) := \begin{cases} \Phi(\alpha), & z \in I_\alpha,\ \alpha \in M, \\ z_K, & z = z_M, \end{cases}$$

$$M_z := \{\alpha \in M \colon z_M \prec \alpha \prec z\}, \quad z \in L \setminus \{z_M\}. \tag{12}$$

For any $z \in L \setminus \{z_M\}$, $\bigcup_{\alpha \in M_z} \overrightarrow{(z_K, \Phi(\alpha))}$ is an open arc of the form $\overrightarrow{(z_K, a)}$, so we define $\varphi(z) := a$. We will show that $\varphi \colon \mathbb{S}^1 \to \mathbb{S}^1$ is increasing. To do this, fix $z_1, z_2, z_3 \in \mathbb{S}^1$ with $z_1 \prec z_2 \prec z_3$ and consider the following cases:

1) $\{z_1, z_2, z_3\} \subset L$.

a) $z_M \in \{z_1, z_2, z_3\}$. By Lemma 2 and Remark 3 in [3] we can assume that $z_1 = z_M$. Then, from (12), we get $M_{z_2} \subset M_{z_3}$, which gives $z_K = \varphi(z_M) = \varphi(z_1) \prec \varphi(z_2) \preceq \varphi(z_3)$.

b) $\{z_1, z_2, z_3\} \subset L \setminus \{z_M\}$. If $z_1, z_2 \in \overrightarrow{(z_M, z_3)}$, which we may assume, then $M_{z_1} \subset M_{z_2} \subset M_{z_3}$ and

$$\varphi(z_1) \preceq \varphi(z_2) \preceq \varphi(z_3). \tag{13}$$

2) $\{z_1, z_2, z_3\} \subset \mathbb{S}^1 \setminus L$.

a) $\operatorname{card}(\{z_1, z_2, z_3\} \cap I_\alpha) \geqslant 2$ for an $\alpha \in M$. Clear.

b) $\operatorname{card}(\{z_1, z_2, z_3\} \cap I_\alpha) \leqslant 1$ for $\alpha \in M$. Let $\alpha, \beta, \gamma \in M$, $\alpha \neq \beta \neq \gamma \neq \alpha$ be such that $z_1 \in I_\alpha, z_2 \in I_\beta, z_3 \in I_\gamma$. Then $\alpha \prec \beta \prec \gamma$, $\varphi(z_1) = \Phi(\alpha)$, $\varphi(z_2) = \Phi(\beta)$ and $\varphi(z_3) = \Phi(\gamma)$, which together with the fact that Φ is strictly increasing yields $\varphi(z_1) \prec \varphi(z_2) \prec \varphi(z_3)$.

3) $\operatorname{card}(\{z_1, z_2, z_3\} \cap (\mathbb{S}^1 \setminus L)) = 2$.

Assume that $z_1, z_2 \in \mathbb{S}^1 \setminus L$, which in view of Lemma 2 and Remark 3 in [3] we may do, and consider the following cases:

a) $z_1, z_2 \in I_\alpha$ for an $\alpha \in M$. Obvious.

b) $z_1 \in I_\alpha, z_2 \in I_\beta$ for some $\alpha, \beta \in M$, $\alpha \neq \beta$.

b$_1$) $z_3 = z_M$. As $z_3 \prec z_1 \prec z_2$, we have $z_M = z_3 \prec \alpha \prec \beta$. Therefore (6) and the fact that Φ preserves order imply $\Phi(\alpha) \preceq_K \Phi(\beta)$. This, by (9), gives $z_K \prec \Phi(\alpha) \preceq \Phi(\beta)$, and $\Phi(\alpha) \neq \Phi(\beta)$ now shows that $\varphi(z_3) = z_K \prec \varphi(z_1) \prec \varphi(z_2)$.

b$_2$) $z_3 \in L \setminus \{z_M\}$.

b$_{21}$) $z_1, z_2 \in \overrightarrow{(z_M, z_3)}$. Since $z_M \prec z_1 \prec z_2$, 3b$_1$ yields $\varphi(z_M) = z_K \prec \varphi(z_1) \prec \varphi(z_2)$. On the other hand, from (12) it follows that $\beta \in M_{z_3}$ and, according to the definition of φ, we obtain $\overrightarrow{(z_K, \varphi(z_2))} \subset \overrightarrow{(z_K, \varphi(z_3))}$. Consequently, $z_K \prec \varphi(z_2) \preceq \varphi(z_3)$, and Lemma 1(ii) now shows that $\varphi(z_1) \prec \varphi(z_2) \preceq \varphi(z_3)$.

b$_{22}$) $z_1, z_3 \in \overrightarrow{(z_M, z_2)}$. Fixing a $\gamma \in M_{z_3}$ we have $\gamma \in I_\gamma$ and $\alpha \neq \gamma \neq \beta$. Since $z_M \prec \gamma \prec z_1$ and $z_M \prec z_1 \prec z_2$, 3b$_1$ gives $z_K = \varphi(z_M) \prec \varphi(\gamma) \prec \varphi(z_1)$ and $z_K \prec \varphi(z_1) \prec \varphi(z_2)$. Therefore from Lemma 1(ii) it follows that $\Phi(\gamma) = \varphi(\gamma) \prec \varphi(z_1) \prec \varphi(z_2)$, which together with $\gamma \in M_{z_3}$ and the definition of φ implies $\varphi(z_3) \in \overrightarrow{(\varphi(z_2), \varphi(z_1))} \cup \{\varphi(z_1)\}$. Thus $\varphi(z_3) \preceq \varphi(z_1) \prec \varphi(z_2)$, and (13) follows.

b$_{23}$) $z_2, z_3 \in \overrightarrow{(z_M, z_1)}$. As $z_M \prec z_2 \prec z_3$ and $z_2, \beta \in I_\beta$, we have $z_M \prec \beta \prec z_3$, and (12) leads to $\beta \in M_{z_3}$. The definition of φ and the equality $\Phi(\beta) = \varphi(z_2)$ now give $z_K \prec \varphi(z_2) \preceq \varphi(z_3)$. Fix a $\gamma \in M_{z_3}$. Then, by (12), we obtain $z_3 \prec z_M \prec \gamma$. Since we also have $z_3 \prec z_1 \prec z_M$, Lemma 1(ii) yields $z_M \prec \gamma \prec z_1$. Moreover, $\gamma \in I_\gamma$ for $\gamma \neq \alpha$. 3b$_1$ now shows that $z_K \prec \varphi(\gamma) \prec \varphi(z_1)$ and therefore $z_K \prec \varphi(z_3) \preceq \varphi(z_1)$. From this, $z_K \prec \varphi(z_2) \preceq \varphi(z_3)$ and Lemma 1(ii) we conclude that $\varphi(z_2) \preceq \varphi(z_3) \preceq \varphi(z_1)$.

4) $\operatorname{card}(\{z_1, z_2, z_3\} \cap (\mathbb{S}^1 \setminus L)) = 1$. Assume that $z_1 \in I_\alpha$ for an $\alpha \in M$, which in view of Lemma 2 and Remark 3 in [3] we may do, and consider the following cases:

a) $z_M \in \{z_2, z_3\}$.

a$_1$) $z_3 = z_M$. As $z_1, \alpha \in I_\alpha$, we have $z_3 = z_M \prec \alpha \prec z_2$ and, by (12), $\alpha \in M_{z_2}$. Using the definition of φ we thus get $\varphi(z_3) \prec \varphi(\alpha) = \varphi(z_1) \preceq \varphi(z_2)$, and (13) follows.

a$_2$) $z_2 = z_M$. Since $(M, \preceq_M)$ has no first element, there exists a $\gamma \in M_{z_3}$ for which $\varphi(\gamma) \neq \varphi(z_3)$. Clearly, $\gamma \neq \alpha$. On account of 3b$_1$, we have $\varphi(\gamma) \prec \varphi(z_1) \prec \varphi(z_2)$.

The fact that $\varphi(z_3) \neq \varphi(\gamma) \neq \varphi(z_1)$ and $3b_{23}$ now give $\varphi(\gamma) \prec \varphi(z_3) \preceq \varphi(z_1)$, which together with $\varphi(\gamma) \prec \varphi(z_1) \prec \varphi(z_2)$ and Lemma 1(ii) shows that $\varphi(z_2) \prec \varphi(z_3) \preceq \varphi(z_1)$.

b) $\{z_2, z_3\} \subset L \setminus \{z_M\}$.

b$_1$) $z_1, z_2 \in \overrightarrow{(z_M, z_3)}$. By (12) we obtain $\alpha \in M_{z_2} \subset M_{z_3}$, and consequently (13) holds true.

b$_2$) $z_2, z_3 \in \overrightarrow{(z_M, z_1)}$. Since from 1a and $4a_2$ we see that $z_K \prec \varphi(z_2) \preceq \varphi(z_3)$ and $\varphi(z_M) = z_K \prec \varphi(z_3) \preceq \varphi(z_1)$, Lemma 1(ii) implies (13).

b$_3$) $z_3, z_1 \in \overrightarrow{(z_M, z_2)}$. Using $4a_1$ and $4a_2$ we obtain $z_K \prec \varphi(z_1) \preceq \varphi(z_2)$ and $z_K \prec \varphi(z_3) \preceq \varphi(z_1)$, and Lemma 1(ii) now leads to (13).

We have thus proved that $\varphi\colon \mathbb{S}^1 \to \mathbb{S}^1$ is increasing. As we also have $K \subset \operatorname{Im}\varphi$ and K is dense in $\mathbb{S}^1$, Lemma 3 shows that φ is continuous.

Fix $v \in V$, $\alpha \in M$, $z \in I_\alpha$. Then, by (11), $P^v(z) \in P^v[I_\alpha] = I_{T(\alpha,v)}$ and the definition of φ and (10) give

$$\varphi(P^v(z)) = \Phi(T(\alpha, v)) = \Phi(\alpha)c(v) = \varphi(z)c(v).$$

Therefore from the continuity of φ and P^v and the density of $\mathbb{S}^1 \setminus L$ in $\mathbb{S}^1$ it follows that (1) holds true. Analysis similar to that in the proof of Lemma 13 in [3] now shows that the iteration group $\mathscr{P}$ is disjoint. Moreover, since c satisfies (1) with $b \cdot \varphi$ for $b \in \mathbb{S}^1$, we may assume that $\varphi(1) = 1$.

For any $v \in V$ denote by $a(v)$ the number from $[0, 1)$ with $c(v) = \mathrm{e}^{2\pi \mathrm{i} a(v)}$. Let us first assume that

$$\text{(14)} \qquad \text{there exists a } v_0 \in V \text{ for which } a(v_0) \notin \mathbb{Q}.$$

If it were true that $(P^{v_0})^{n_0}(z_0) = z_0$ for a positive integer n_0 and a $z_0 \in \mathbb{S}^1$, from (1) we would have $c(n_0 v_0) = 1$, and consequently $1 = c(v_0)^{n_0} = \mathrm{e}^{2\pi \mathrm{i} n_0 a(v_0)}$, contrary to (14). Therefore the iteration group $\mathscr{P}$ is non-singular.

Next, assume that

$$\text{(15)} \qquad a(v) \in \mathbb{Q}, \ \ v \in V.$$

If there existed a $v_0 \in V$ with $\alpha(P^{v_0}) \notin \mathbb{Q}$, from Lemma 5 and the fact that $\operatorname{card} \operatorname{Im} c = \aleph_0$ we would have $c = (c_{\mathscr{P}})^n$ for an $n \in \mathbb{Z} \setminus \{0\}$ and, consequently, $a(v_0) = n \cdot \alpha(P^{v_0}) (\mathrm{mod} 1)$, which contradicts (15). Thus, the iteration group $\mathscr{P}$ is singular. Moreover, it is not discrete. Indeed, if it were true that $L_{\mathscr{P}} = \emptyset$, from Lemma 4 and (1) it would follow that

$$\operatorname{card} \operatorname{Im} c = \operatorname{card}\{\varphi(z)c(v), \ v \in V\} = \operatorname{card}\{\varphi(P^v(z)), \ v \in V\} < \aleph_0$$

for $z \in \mathbb{S}^1$, which is impossible.

Thus the iteration group $\mathscr{P}$ is dense or non-dense, and therefore, by Lemma 5, $c = (c_{\mathscr{P}})^n$ and $\varphi = (\varphi_{\mathscr{P}})^n$ for an $n \in \mathbb{Z} \setminus \{0\}$. Since φ is constant on each arc I_α, the mapping $\varphi_{\mathscr{P}}$ is not invertible and Proposition 1 now leads to $L_{\mathscr{P}} \neq \mathbb{S}^1$. Let J_α for $\alpha \in M$ be open pairwise disjoint arcs with $\mathbb{S}^1 \setminus L_{\mathscr{P}} = \bigcup_{\alpha \in M} J_\alpha$. From Lemma 6 it follows that they are the maximal open arcs of constancy of $\varphi_{\mathscr{P}}$. We show that they also have this property for φ. To do this, let us note that φ is constant on each J_α and suppose, contrary to our claim, that there exists an $\alpha \in M$ and an open arc J such that $J_\alpha \subsetneq J$ and φ is constant on J. Then there are an infinite number of $\beta \in M$ with $J_\beta \subset J$. On these J_β the mapping $\varphi_{\mathscr{P}}$ assumes only a finite number of values, which contradicts Lemma 6. Since from the definition of φ it follows that I_α for $\alpha \in M$ are also the maximal open arcs of constancy of φ, we obtain $L_{\mathscr{P}} = L$.

The above constructed piecewise linear, disjoint and non-dense iteration group $\mathscr{P}$ has been determined uniquely by the sequence $(L, M, z_M, c, A, z_K, \Phi)$, and therefore will be denoted by $P(L, M, z_M, c, A, z_K, \Phi)$.

Theorem 1. *Assume that* $\Gamma\colon \mathbb{S}^1 \to \mathbb{S}^1$ *is a homeomorphism with* $\Gamma(z) = z$ *for* $z \in L$ *and let* $\{P^v\colon \mathbb{S}^1 \to \mathbb{S}^1, v \in V\} = P(L, M, z_M, c, A, z_K, \Phi)$. *Then formula* (3) *defines a disjoint non-dense iteration group* $\mathscr{F} := \{F^v\colon \mathbb{S}^1 \to \mathbb{S}^1,\ v \in V\}$ *with* $L_{\mathscr{F}} = L$, *which is non-singular if and only if* (14) *holds true. Moreover, every disjoint non-dense iteration group can be obtained in this way.*

Proof. We see at once that $\mathscr{F}$ is an iteration group, which, according to Remarks 2, 3 and Lemma 2 in [6], has the desired properties.

Now, assume that $\mathscr{F} = \{F^v\colon \mathbb{S}^1 \to \mathbb{S}^1, v \in V\}$ is a disjoint non-dense iteration group and let I_q for $q \in \mathbb{Q}$ be open pairwise disjoint arcs for which (2) holds true.

Put $L := L_{\mathscr{F}}$.

Of course, L is a perfect nowhere dense subset of $\mathbb{S}^1$ and we have (4).

Take a $\Phi_0\colon \mathbb{Q} \to \bigcup_{q \in \mathbb{Q}} I_q$ with $\Phi_0(q) \in I_q$ for $q \in \mathbb{Q}$ and set $M := \Phi_0[\mathbb{Q}]$. It is evident that $\Phi_0\colon \mathbb{Q} \to M$ is a bijection and $M \subset \bigcup_{q \in \mathbb{Q}} I_q$ satisfies $\operatorname{card}(M \cap I_q) = 1$ for $q \in \mathbb{Q}$. For any $\alpha \in M$ denote by I_α the arc I_q such that $\alpha \in I_q$ and observe that $I_\alpha = I_{\Phi_0^{-1}(\alpha)}$ for $\alpha \in M$. Since from Proposition 1 it follows that $\varphi_{\mathscr{F}}[L_{\mathscr{F}}] = \mathbb{S}^1$, we check at once that $\operatorname{card} \varphi_{\mathscr{F}}[\mathbb{S}^1 \setminus \bigcup_{\alpha \in M} \operatorname{cl} I_\alpha] > \aleph_0$. This together with Lemma 6(iv) shows that $\varphi_{\mathscr{F}}[\mathbb{S}^1 \setminus \bigcup_{\alpha \in M} \operatorname{cl} I_\alpha]$ is not contained in $K_{\mathscr{F}}$.

Choose a $z_M \in \mathbb{S}^1 \setminus \bigcup_{\alpha \in M} \operatorname{cl} I_\alpha$ for which $\varphi_{\mathscr{F}}(z_M) \in \mathbb{S}^1 \setminus K_{\mathscr{F}}$ and let an order relation "$\preceq_M$" be given by (6).

Put $c := c_{\mathscr{F}}$.

From Proposition 1 and Lemma 6(iv) we conclude that $c\colon V \to \mathbb{S}^1$ is a homomorphic mapping with $\operatorname{card} \operatorname{Im} c = \aleph_0$.

Define $A := K_{\mathscr{F}}$.

Clearly, $\operatorname{card} A = \aleph_0$. Putting $K := \operatorname{Im} c \cdot A$ we deduce from Lemma 6(v) that $K = \operatorname{Im} c_{\mathscr{F}} \cdot K_{\mathscr{F}} = K_{\mathscr{F}}$.

Set $z_K := \varphi_{\mathscr{F}}(z_M) \in \mathbb{S}^1 \setminus K$ and let an order relation "$\preceq_K$" be given by (8).

Define $\Phi := \Phi_{\mathscr{F}} \circ \Phi_0^{-1}$.

Obviously, $\Phi\colon M \to K$ is a bijection. We show that it also preserves order. To do this, fix $\alpha, \beta \in M$ with $\alpha \preceq_M \beta$ and note that (6) and the fact that $\varphi_{\mathscr{F}}$ is increasing give $\varphi_{\mathscr{F}}(z_M) \preceq \varphi_{\mathscr{F}}(\alpha) \preceq \varphi_{\mathscr{F}}(\beta)$. Since $\alpha \in I_\alpha = I_{\Phi_0^{-1}(\alpha)}$ for $\alpha \in M$, Lemma 6(i) together with the definitions of $\Phi_{\mathscr{F}}$ and Φ shows that

$$\{\varphi_{\mathscr{F}}(\alpha)\} = \varphi_{\mathscr{F}}[I_{\Phi_0^{-1}(\alpha)}] = \{\Phi_{\mathscr{F}}[\Phi_0^{-1}(\alpha)]\} = \{\Phi(\alpha)\},\ \alpha \in M.$$

Therefore $\varphi_{\mathscr{F}}(z_M) \preceq \varphi_{\mathscr{F}}(\alpha) \preceq \varphi_{\mathscr{F}}(\beta)$ and (8) imply $\Phi(\alpha) \preceq_K \Phi(\beta)$.

Consider the iteration group $P(L, M, z_M, c, A, z_K, \Phi) = \{P^v\colon \mathbb{S}^1 \to \mathbb{S}^1, v \in V\}$ and let us first note that $P^v[I_\alpha] = I_{T(\alpha,v)}$ for $\alpha \in M$, $v \in V$, where $T\colon M \times V \to M$ is given by (10). Fix $q \in \mathbb{Q}, v \in V$. Using the definitions of T, Φ, c and $T_{\mathscr{F}}$ we have $T(\Phi_0(q), v) = \Phi_0(T_{\mathscr{F}}(q, v))$, which together with the equalities $I_q = I_{\Phi_0(q)}$ and $P^v[I_{\Phi_0(q)}] = I_{T(\Phi_0(q),v)}$ gives $P^v[I_q] = I_{T_{\mathscr{F}}(q,v)}$. Proposition 2 now completes the proof. □

References

[1] *J. S. Bae, K. J. Min, D. H. Sung and S. K. Yang*: Positively equicontinuous flows are topologically conjugate to rotation flows. Bull. Korean Math. Soc. *36* (1999), 707–716.

[2] *M. Bajger*: On the structure of some flows on the unit circle. Aequationes Math. *55* (1998), 106–121.

[3] *K. Ciepliński*: On the embeddability of a homeomorphism of the unit circle in disjoint iteration groups. Publ. Math. Debrecen *55* (1999), 363–383.

[4] *K. Ciepliński*: On conjugacy of disjoint iteration groups on the unit circle. European Conference on Iteration Theory (Muszyna-Złockie, 1998). Ann. Math. Sil. *13* (1999), 103–118.

[5] *K. Ciepliński*: The structure of disjoint iteration groups on the circle. Czechoslovak Math. J. *54* (2004), 131–153.

[6] *K. Ciepliński*: Topological conjugacy of disjoint flows on the circle. Bull. Korean Math. Soc. *39* (2002), 333–346.

[7] *K. Ciepliński and M. C. Zdun*: On a system of Schröder equations on the circle. Internat. J. Bifur. Chaos Appl. Sci. Engrg. *13* (2003), 1883–1888.

[8] *M. C. Zdun*: The structure of iteration groups of continuous functions. Aequationes Math. *46* (1993), 19–37.

Author's address: Institute of Mathematics, Pedagogical University, Podchorążych 2, 30-084 Kraków, Poland, e-mail: kc@wsp.krakow.pl.