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CIRCUIT AND COCIRCUIT PARTITIONS OF
BINARY MATROIDS

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Abstract. We give an example of a class of binary matroids with a cocircuit partition and we give some characteristics of a set of cocircuits of such binary matroids which forms a partition of the ground set.

Keywords: binary matroid, affine matroid, cocircuit, Eulerian, circuit partition

MSC 2000: 05B35

1. INTRODUCTION

Matroid theory draws heavily on both graph theory and linear algebra for its notation and basic examples. Thus a matroid can be defined in several ways. For example we can define a matroid using the properties of its set of independent sets or its set of circuits. For further details we refer to [3]. Thus studying such type of sets as the set of circuits of a matroid enriches the field. The result of Welsh [4] below is an example of such a study. A matroid M is *Eulerian* if its ground set $E(M)$ has a partition into circuits.

Furthermore, each matroid M has a corresponding matroid M^* called the dual. The circuits of the dual matroid M^* are called the cocircuits of the matroid M . For a good introduction to duality theory, we refer to [5]. Thus studying the set of cocircuits gives information on both the matroid and its dual. But studying the set of cocircuits for a general matroid is not very easy, so in this paper we restrict our study to a well known class of binary matroids, see [3], [5].

In this paper we give a class of binary matroids whose ground set has a cocircuit partition and we give some characteristics of a set of cocircuits which forms a partition.

2. CIRCUIT AND COCIRCUIT PARTITIONS OF MATROIDS

A matroid is a collection of objects with a certain function of rank defined just like in graphs and matrices. A *matroid* $M(E)$ is a set E with a rank function r , for which the following properties hold

- (R1) If $X \subseteq E$, then $0 \leq r(X) \leq |X|$.
- (R2) If $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$.
- (R3) If X and Y are subsets of E , then

$$r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y).$$

The set E is called the *ground set* of $M(E)$. A *circuit* of $M(E)$ is a non-empty subset X of E such that for all x in X , $r(X - x) = |X| - 1 = r(X)$. For each matroid $M(E)$ there is another matroid associated with it. The *dual of a matroid* $M(E)$, denoted by $M^*(E)$, is a matroid with the rank function r^* such that for all $X \subseteq E$,

$$r^*(X) = |X| - r(M) + r(E - X).$$

The circuits of M^* are called the cocircuits of M . A function cl from 2^E into 2^E defined for all $X \subseteq E$ by $\text{cl}(X) = \{x \in E: r(X \cup x) = r(X)\}$ is called the *closure operator* of M . Let $PG(r - 1, q)$ be the projective space of rank r over a finite field $GF(q)$ as described by Oxley [3, Chapter 6]. An affine space of rank r , denoted $AG(r - 1, q)$, is obtained from the projective space $PG(r - 1, q)$ by deleting all the points of a hyperplane. A simple matroid M is *affine* over $GF(q)$ if it is isomorphic to a submatroid of $AG(r - 1, q)$. In general, a loopless matroid M is affine over $GF(q)$ if its associated simple matroid is affine over $GF(q)$. A *binary matroid* is a matroid that is representable over $GF(2)$. A *circuit partition* of a matroid M is a partition of the ground set of M into circuits. A *cocircuit partition* of a matroid M is a partition of the ground set of M into cocircuits.

If M has a loop it is clear that the ground set of M cannot be partitioned into cocircuits. Recall that $\text{si}(M)$ denotes the simple matroid associated with the matroid M . Recall that a matroid M is *Eulerian* if its ground set $E(M)$ has a partition into circuits. Also we say that a matroid M is *bipartite* if every circuit has even cardinality. The next theorem summarizes the relationship between binary Eulerian matroids and binary bipartite matroids, see Welsh [4] and for other details see Brylawski [1] and Heron [2].

Theorem 2.1 (Welsh 1969). *Let M be a binary matroid. The following are equivalent:*

- (i) M is Eulerian.
- (ii) Every cocircuit of M has even cardinality.
- (iii) M^* is bipartite.
- (iv) M^* has a partition into cocircuits.

The next theorem is well known.

Theorem 2.2. *A binary matroid M is affine over $GF(2)$ if and only if M is bipartite.*

Thus a binary affine matroid has a cocircuit partition.

3. A THEOREM ON BINARY AFFINE MATROIDS

In this section we prove the main theorem of this paper. This theorem is on the characteristic of a binary affine matroid with a cocircuit partition.

Throughout this section M denotes a binary affine matroid of rank r represented over $GF(2)$ by a set E of points of $PG(r-1, 2)$ and H denotes the unique hyperplane of $PG(r-1, 2)$ such that $H \cap E = \emptyset$. Refer to the definitions in Section 2. For a subset A of E , the closure of A in M is denoted by $\text{cl}_M(A)$ and the closure of A in $PG(r-1, 2)$ is denoted by $\text{cl}_P(A)$.

Theorem 3.1. *Let M be a binary affine matroid of rank r represented over $GF(2)$ by a set E of points of $PG(r-1, 2)$ and let H denote the unique hyperplane of $PG(r-1, 2)$ such that $H \cap E = \emptyset$. Let $\{C_1, C_2, \dots, C_k\}$ be a set of pairwise disjoint cocircuits of M . Then $\{C_1, C_2, \dots, C_k\}$ is a partition of E if and only if*

$$r((\text{cl}_P(C_1) \cap H) \cup (\text{cl}_P(C_2) \cap H) \cup \dots \cup (\text{cl}_P(C_k) \cap H)) = r - k.$$

To ease notation in what follows, for a subset C of E , the set $\text{cl}_P(C) \cap H$ will be denoted by C' . Before proving Theorem 3.1 we will need the following propositions and lemmas. The next claim follows immediately from the fact that all flats in a projective geometry are modular.

Claim 3.2. *Let F be a flat and H' a hyperplane of $PG(n, q)$ such that $F \not\subseteq H'$. Then*

$$r(F \cap H') = r(F) - 1.$$

The next claim follows from the fact that intersections of flats in a matroid are flats.

Claim 3.3. *Let M' be a binary matroid and suppose X is the intersection of a three point line and a hyperplane in M' . Then $|X| \in \{1, 3\}$.*

Recall that throughout this section M denotes a binary affine matroid.

Lemma 3.4. *Let C be a cocircuit of M . Then*

$$C' \subseteq \text{cl}_P(E - C).$$

Proof. We can regard elements of $PG(r - 1, 2)$ as vectors in $V(r, 2)$. Let $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m$ be vectors that form a basis of $M|C$. First, let \mathbf{b}_1 be the only such vector. In this case $\text{cl}_P(C)$ is a parallel class of \mathbf{b}_1 . Hence C' is empty. Thus $C' \subseteq \text{cl}_P(E - C)$. Now consider $\mathbf{b}_1 + \mathbf{b}_i$ for $i = 2, 3, \dots, m$. It is obvious that $\mathbf{b}_1 + \mathbf{b}_i \in \text{cl}_{PG}(C)$. It follows that $\{\mathbf{b}_1, \mathbf{b}_i, \mathbf{b}_1 + \mathbf{b}_i\}$ is a three point line. But \mathbf{b}_1 and \mathbf{b}_i are not in H , hence by applying Proposition 3.3, $\mathbf{b}_1 + \mathbf{b}_i \in H$. Hence $\mathbf{b}_1 + \mathbf{b}_i \in C'$.

We now show that the set $B = \{\mathbf{b}_1 + \mathbf{b}_2, \mathbf{b}_1 + \mathbf{b}_3, \dots, \mathbf{b}_1 + \mathbf{b}_m\}$ is independent. Consider a sum of the form $\sum_{i=2}^m a_i(\mathbf{b}_1 + \mathbf{b}_i)$ where at least one a_i is non-zero. Let $J = \{j \in \{2, 3, \dots, m\} : a_j \neq 0\}$. Then

$$\sum_{j \in J} a_j(\mathbf{b}_1 + \mathbf{b}_j) = \begin{cases} \mathbf{b}_1 + \sum_{j \in J} \mathbf{b}_j & \text{if } |J| \text{ is odd,} \\ \sum_{j \in J} \mathbf{b}_j & \text{otherwise.} \end{cases}$$

Moreover, $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m$ are independent. Hence both $\mathbf{b}_1 + \sum_{j \in J} \mathbf{b}_j \neq 0$ and $\sum_{j \in J} \mathbf{b}_j \neq 0$. Therefore, we have $\sum_{j \in J} a_j(\mathbf{b}_1 + \mathbf{b}_j) \neq 0$. Hence B is an independent set. Furthermore, $|B| = r(C) - 1$. But by Proposition 3.2, $r(C') = r(\text{cl}_P(C)) - 1 = r(C) - 1$. Hence B is a maximal independent set of C' . Hence B is a basis for C' .

Next we show that for all $i \in \{1, 2, \dots, m\}$ we have $\mathbf{b}_1 + \mathbf{b}_i \in \text{cl}_P(E - C)$. We know $E - C$ is a hyperplane of M , so $\text{cl}_P(E - C)$ is a hyperplane of $PG(r - 1, 2)$. But $\text{cl}_P(E - C) \cap E = E - C$. Hence $\mathbf{b}_1, \mathbf{b}_i \notin \text{cl}_P(E - C)$. So by applying Proposition 3.3 we deduce that $\mathbf{b}_1 + \mathbf{b}_i \in \text{cl}_P(E - C)$. Hence $B \subseteq \text{cl}_P(E - C)$. It follows that $\text{cl}_P(B) \subseteq \text{cl}_P(E - C)$. Hence $C' \subseteq \text{cl}_P(E - C)$. \square

Lemma 3.5. *Let C_1, C_2, \dots, C_k be a set of pairwise disjoint cocircuits of M , let F be the set $C'_1 \cup C'_2 \cup \dots \cup C'_k$ and let $M_1 = PG(r - 1, 2)|_{E \cup F}$. Then C_i is a cocircuit of M_1 for $i = 1, 2, \dots, k$.*

Proof. Consider $\text{cl}_P(E - C_i)$. For all $j \neq i$ and $C_j \subseteq (E - C_i)$ we have $\text{cl}_P(C_j) \subseteq \text{cl}_P(E - C_i)$. Hence $C'_j \subseteq \text{cl}_P(E - C_i)$. But by Lemma 3.4, $C'_i \subseteq \text{cl}_P(E - C_i)$. Thus $\text{cl}_{M_1}(E - C_i) = (E - C_i) \cup F$. Therefore C_i is a cocircuit of M_1 . \square

Evidently if C is a circuit of M and $A \cap C = \emptyset$, then C is a circuit of $M \setminus A$. By duality we therefore have the following claim.

Claim 3.6. *Let C be a cocircuit of a matroid M , let $H = E - C$, and $A \subseteq H$. Then C is a cocircuit in M/A*

The next proposition is a direct application of Claim 3.6 in our context.

Proposition 3.7. *Let C_1, C_2, \dots, C_k be cocircuits of M , let F be the set $C'_1 \cup C'_2 \cup \dots \cup C'_k$ and let $M_1 = PG(r-1, 2) \upharpoonright E \cup F$. Then C_i is a cocircuit of M_1/F for $i = 1, 2, \dots, k$.*

Proof. By Lemma 3.5, C_i is a cocircuit of M_1 . Thus $(E - C_i) \cup F$ is a hyperplane of M_1 . But $F \subseteq (E - C_i) \cup F$. Hence the result follows by applying Proposition 3.6. \square

Lemma 3.8. *Let C_1, C_2, \dots, C_k be cocircuits of M such that $r(C'_1 \cup C'_2 \cup \dots \cup C'_k) = r - k$. Let $M_1 = PG(r-1, 2) \upharpoonright (E \cup C'_1 \cup C'_2 \cup \dots \cup C'_k)$. Then*

- (i) $r(M_1/(C'_1 \cup C'_2 \cup \dots \cup C'_k)) = k$;
- (ii) C_i is a parallel class of $M_1/(C'_1 \cup C'_2 \cup \dots \cup C'_k)$ for $i \in \{1, 2, \dots, k\}$.

Proof. Let $F = C'_1 \cup C'_2 \cup \dots \cup C'_k$. Consider (i). $r(M_1)/F = r(M_1) - r(F)$. But $r(M_1) = r(M)$ and $r(F) = r(M) - k$. Hence $r(M_1/F) = k$.

Now consider (ii). For any cocircuit, C_i of M_1 , we have $r_{M_1/C_i}(C_i) = r_{M_1}(C_i \cup C'_i) - r_{M_1}(C'_i)$. But by definition we know that $C'_i \subseteq \text{cl}_{M_1}(C_i)$. Hence $r_{M_1}(C_i \cup C'_i) = r_{M_1}(C_i)$. We also know that $r_{M_1}(C'_i) = r_{M_1}(C_i) - 1$ by applying Proposition 3.2. Hence $r_{M_1/C_i}(C_i) = 1$. It follows that $r_{M_1/F}(C_i) \leq 1$. But no element of C_i is in $\text{cl}_{PG}(F)$. Thus no element of C_i is a loop of M_1/F . Therefore $r_{M_1/F}(C_i) = 1$. Thus C_i is a parallel class of M_1/F . \square

Lemma 3.9. *Let M be a rank- k loopless matroid containing k parallel classes P_1, P_2, \dots, P_k of which each P_i is a cocircuit of M . Then $E(M) = P_1 \cup P_2 \cup \dots \cup P_k$.*

Proof. For each P_i there is a corresponding hyperplane H_i such that $E(M) - H_i = P_i$. Thus $\bigcup_i (E(M) - H_i) = P_1 \cup P_2 \cup \dots \cup P_k$. But $\bigcup_i (E(M) - H_i) = E(M) - \bigcap_i (H_i)$.

Next we show that $\bigcap_i (H_i) = \emptyset$. Consider a set of hyperplanes $\{H_1, H_2, \dots, H_k\}$. The cocircuit P_k is contained in all hyperplanes, H_j , such that $k \neq j$. Hence $P_k \subseteq (H_1 \cap H_2 \cap \dots \cap H_{k-1})$. Moreover, we know that $P_k \not\subseteq H_k$. Hence

$$(H_1 \cap H_2 \cap \dots \cap H_{k-1} \cap H_k) \subset (H_1 \cap H_2 \cap \dots \cap H_{k-1}).$$

But we know that any set which is an intersection of flats is a flat. Thus $(H_1 \cap H_2 \cap \dots \cap H_{k-1} \cap H_k)$ and $(H_1 \cap H_2 \cap \dots \cap H_{k-1})$ are flats. We also know that for any two flats F_1 and F_2 such that $F_1 \subset F_2$ we have $r(F_1) < r(F_2)$. Without loss of generality, it follows that $r(H_1 \cap H_2) < r(H_1)$. Hence $r(H_1 \cap H_2) < k - 1 \leq k - 2$. It follows by recursion that $r(H_1 \cap H_2 \cap \dots \cap H_{k-1} \cap H_k) \leq k - k = 0$. But the rank of a set can not be negative. Thus $r(H_1 \cap H_2 \cap \dots \cap H_{k-1} \cap H_k) = 0$. Since M is loopless $(H_1 \cap H_2 \cap \dots \cap H_{k-1} \cap H_k) = \emptyset$. Hence $E(M) = P_1 \cup P_2 \cup \dots \cup P_k$. \square

Proposition 3.10. *Let M be a binary affine matroid with a partition C_1, C_2, \dots, C_k of $E(M)$ into cocircuits. Then C_i is a flat of M for all $i \in \{1, 2, \dots, k\}$.*

Proof. Consider the following set of hyperplanes, $\{H_i: H_i = E(M) - C_i, i \in 1, 2, \dots, k\}$, of M . Without loss of generality let $i = k$. Then $H_k = C_1 \cup C_2 \cup \dots \cup C_{k-1}$. Moreover for any $k \neq j$, we have $C_k \subseteq H_j$. Thus $C_k \subseteq H_1 \cap H_2 \cap \dots \cap H_{k-1}$. Furthermore, we know by definition that no element of C_j is contained in H_j . Hence $H_1 \cap H_2 \cap \dots \cap H_{k-1} = C_k$. Hence C_k is a flat since it is an intersection of flats. \square

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Let C_1, C_2, \dots, C_k be a set of pairwise disjoint cocircuits of M and let $M_1 = PG(r - 1, 2) \mid E \cup (C'_1 \cup C'_2 \cup \dots \cup C'_k)$.

Assume that $r(C'_1 \cup C'_2 \cup \dots \cup C'_k) = r(M) - k$. By Lemma 3.7, each C_i for $i \in \{1, 2, \dots, k\}$ is a cocircuit of $M_1 / (C'_1 \cup C'_2 \cup \dots \cup C'_k)$. By Lemma 3.8, each C_i for $i \in \{1, 2, \dots, k\}$ is a parallel class of $M_1 / (C'_1 \cup C'_2 \cup \dots \cup C'_k)$. Moreover, the ground set of $M_1 / (C'_1 \cup C'_2 \cup \dots \cup C'_k)$ is E . Thus $E = C_1 \cup C_2 \cup \dots \cup C_k$ by Lemma 3.9. But we know that C_1, C_2, \dots, C_k are pairwise disjoint cocircuits of M and E is the ground set of M . Hence C_1, C_2, \dots, C_k is a partition of $E(M)$ into cocircuits.

Assume that C_1, C_2, \dots, C_k is a partition of $E(M)$ into cocircuits. Then M is a loopless matroid since a cocircuit does not contain loops. We use induction on the number of cocircuits of M . Let M be a matroid with one cocircuit C_1 . Then M is a matroid with a single parallel class. Thus $H = \emptyset$. Therefore $C'_1 = \emptyset$. Hence $r(C'_1) = 0 = r - 1$. Therefore it holds for a matroid with a single cocircuit.

Assume that for every binary matroid M with a cocircuit partition C_1, C_2, \dots, C_n of $E(M)$ we have $r(C'_1 \cup C'_2 \cup \dots \cup C'_n) = r - n$ for $n \geq 1$.

Now let M be a binary affine matroid with a partition $C_1, C_2, \dots, C_n, C_{n+1}$ of $E(M)$. Then by Claim 3.6, C_1, C_2, \dots, C_n is a cocircuit partition of M / C_{n+1} . Let $M_1 = PG(r - 1, 2) \mid E \cup C'_1 \cup C'_2 \cup \dots \cup C'_{n+1}$. Hence by the induction assumption,

$$r_{M_1 / C_{n+1}}(C'_1 \cup C'_2 \cup \dots \cup C'_n) = r(M_1 / C_{n+1}) - n.$$

It is clear that C'_{n+1} is a collection of loops of M_1/C_{n+1} . Let X denote $C'_1 \cup C'_2 \cup \dots \cup C'_{n+1}$. Hence

$$(3.1) \quad r_{M_1/C_{n+1}}(X) = r(M_1/C_{n+1}) - n.$$

But

$$(3.2) \quad r_{M_1/C_{n+1}}(X) = r_{M_1}(X \cup C_{n+1}) - r_{M_1}(C_{n+1}).$$

We now show that

$$(3.3) \quad r_{M_1}(X \cup C_{n+1}) = r_{M_1}(X) + 1.$$

Certainly, $r_{M_1}(X \cup C_{n+1}) > r_{M_1}(X)$ because $C_{n+1} \not\subseteq \text{cl}_{M_1}(X)$. But $r_{M_1}(C_{n+1} \cup C'_{n+1}) = r_{M_1}(C'_{n+1}) + 1$ thus C_{n+1} and C'_{n+1} is a modular pair of flats. Thus

$$r_{M_1}(X \cup C_{n+1}) + r_{M_1}(X \cap C_{n+1}) \leq r_{M_1}(X) + r_{M_1}(C_{n+1}).$$

Hence

$$r_{M_1}(X \cup C_{n+1}) \leq r_{M_1}(X) + 1.$$

Hence Equation 3.3 holds. Using Equations 3.1, 3.2, 3.3, we see that

$$\begin{aligned} r(C'_1 \cup C'_2 \cup \dots \cup C'_{n+1}) &= r_{M_1}(X \cup C_{n+1}) - 1 \\ &= r_{M_1/C_{n+1}}(X) + r_{M_1}(C_{n+1}) - 1 \\ &= r(M_1/C_{n+1}) - n + r_{M_1}(C_{n+1}) - 1 \\ &= r(M_1) - n - 1 = r - 1 - n = r - (n + 1). \end{aligned}$$

Therefore any binary affine matroid M with a cocircuit partition C_1, C_2, \dots, C_k has $r(C'_1 \cup C'_2 \cup \dots \cup C'_k) = r - k$. \square

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