Charles K. Megibben; William Ullery
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ISOTYPE KNICE SUBGROUPS OF GLOBAL WARFIELD GROUPS

CHARLES MEGIBBEN, Nashville, and WILLIAM ULLERY, Auburn

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Abstract. If $H$ is an isotype knice subgroup of a global Warfield group $G$, we introduce the notion of a $k$-subgroup to obtain various necessary and sufficient conditions on the quotient group $G/H$ in order for $H$ itself to be a global Warfield group. Our main theorem is that $H$ is a global Warfield group if and only if $G/H$ possesses an $H(\aleph_0)$-family of almost strongly separable $k$-subgroups. By an $H(\aleph_0)$-family we mean an Axiom 3 family in the strong sense of P. Hill. As a corollary to the main theorem, we are able to characterize those global $k$-groups of sequentially pure projective dimension $\leq 1$.

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1. Introduction

An abelian group is said to be simply presented if it has a presentation in which each relation involves at most two generators and, by definition, a Warfield group is a mixed abelian group that is isomorphic to a direct summand of a simply presented group. One important property of a Warfield group $G$ is the existence of a decomposition basis $X = \{x_i\}_{i \in I}$, an independent subset of $G$ in which each $x_i$ has infinite order, $G/\langle X \rangle$ is torsion and $\langle X \rangle = \bigoplus_{i \in I} \langle x_i \rangle$ is a valuated coproduct in the following sense: if $p$ is a prime, then for every finite subset $J$ of $I$ and all collections $\{n_j\}_{j \in J}$ of integers,

$$\left| \sum_{j \in J} n_j x_j \right|_p = \min_{j \in J} \{|n_j x_j|_p\}.$$  

Here $|x|_p$ denotes the $p$-height of $x$ (as computed in $G$). Warfield groups also have an infinite combinatorial characterization in terms of knice subgroups that parallels
P. Hill’s well-known Axiom 3 description of simply presented torsion groups in terms of nice subgroups. The precise definition of a knice subgroup depends on the auxiliary notions of primitive element and $*$-valuated coproduct, which have evolved from an analysis of properties enjoyed by members of a decomposition basis. The details may be found in [7] and will not be reviewed here. The general facts we cite from [7], [8], [11] and [12] will be adequate for our purposes. Suffice it to say that the exact definitions of primitive element and $*$-valuated coproduct in a global group $G$ can be formulated in terms of abstract height matrices and various associated fully invariant subgroups of $G$. We use the term global group for an arbitrary mixed abelian group to distinguish the groups we study from $p$-local groups, the simpler setting in which Warfield groups were first introduced.

Let $\mathbb{P}$ denote the set of rational primes and write $\mathcal{C}_\infty$ for the class of ordinals with the symbol $\infty$ adjoined as a maximal element. Given an element $x$ in a global group $G$, we associate its height matrix $\|x\|$, a doubly infinite $\mathbb{P} \times \omega$ matrix having $|p^i x|_p$ as its $(p, i)$ entry. The ordered class $\mathcal{C}_\infty$ induces in a pointwise manner lattice relations $\leq$ and $\land$ on the height matrices $\|x\|$ of elements of $G$. We shall have several occasions to employ the familiar triangle inequality: $\|x + y\| \geq \|x\| \land \|y\|$ for all $x, y \in G$; certainly $|x + y|_p \geq |x|_p \land |y|_p$ for all primes $p$. When necessary to avoid confusion, we affix superscripts to indicate the group in which height matrices and heights are computed. For example, if $H$ is a subgroup of $G$, $\|x + H\|_{G/H}$ indicates the height matrix of the coset $x + H$ as computed in $G/H$, while if $x \in H$, $|x|^H_p$ denotes the $p$-height of $x$ as computed in $H$. For any other unexplained notation or terminology, see the standard reference [2].

Following [7] and [8], we call a subgroup $N$ of a global group $G$ a nice subgroup provided that for each prime $p$ and ordinal $\alpha$, the cokernel of the canonical map

$$(p^\alpha G + N)/N \twoheadrightarrow p^\alpha (G/N)$$

contains no element of order $p$.

**Definition 1.1.** A subgroup $N$ of the global group $G$ is said to be a knice subgroup provided the following two conditions are satisfied:

1. $N$ is a nice subgroup of $G$.
2. If $S$ is a finite subset of $G$, then there exists a finite (possibly vacuous) collection of primitive elements $x_1, x_2, \ldots, x_m$ in $G$ such that $N' = N \oplus \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \ldots \oplus \langle x_m \rangle$ is a $*$-valuated coproduct and $\langle S, N' \rangle/N'$ is finite.

We conclude this section with several frequently useful facts concerning knice subgroups.
Proposition 1.2. The following hold for any global group $G$.

(1) If $A$ is a nice subgroup of $G$ and if $B$ is a subgroup with $B/A$ finite, then $B$ is a nice subgroup of $G$.

(2) If $B = A \oplus \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \ldots \oplus \langle x_m \rangle$ is a $*$-valuated coproduct in $G$ where $A$ is a nice subgroup and $x_1, x_2, \ldots, x_m$ are primitive elements, then $B$ is a nice subgroup of $G$.

(3) If $A$ is a nice subgroup of $G$ and if $B/A$ is a nice subgroup of $G/A$, then $B$ is a nice subgroup of $G$.

(4) If $N$ is both nice and pure in $G$, then $p^\alpha(G/N) = (p^\alpha G + N)/N$ for all primes $p$ and ordinals $\alpha$.

(5) If $N$ is a pure nice subgroup of $G$ and $A$ is an arbitrary subgroup of $N$, then $N/A$ is a pure nice subgroup of $G/A$.

Parts (1), (2), and (3) in the preceding proposition are, respectively, Theorem 3.2, 3.3, and 3.7 in [7]; while (4) is Corollary 1.10 from [8] and (5) is [12, Corollary 2.3], whose proof uses both (4) and the characterization of nice subgroups given in Proposition 1.3 below.

A global group $G$ is said to be a $k$-group if the trivial subgroup $0$ is a nice subgroup. Thus, if $S$ is a finite subset of the $k$-group $G$, then $S$ is finite modulo a $*$-valuated coproduct $\langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \ldots \oplus \langle x_m \rangle$ where $x_1, x_2, \ldots, x_m$ are primitive elements in $G$. Direct summands of $k$-groups are themselves $k$-groups [8, Theorem 2.6] and, more generally, an isotype nice subgroup of a $k$-group is also a $k$-group [8, Theorem 2.8]. Recall that a subgroup $H$ of a global group $G$ is said to be isotype if $p^\alpha G \cap H = p^\alpha H$ for all primes $p$ and ordinals $\alpha$. Furthermore, $G/N$ is a $k$-group whenever $N$ is a nice subgroup of the global group $G$. In fact, we have the following important characterization of nice subgroups.

Proposition 1.3 ([8, Proposition 1.7]). A subgroup $N$ of the global group $G$ is a nice subgroup if and only if the following three conditions are satisfied.

(a) $N$ is a nice subgroup of $G$.

(b) To each $g \in G$ there corresponds a positive integer $n$ such that the coset $ng + N$ contains an element $x$ with $\|x\|_G = \|ng + N\|_{G/N}$.

(c) $G/N$ is a $k$-group.

Corollary 1.4 ([8, Remark 1.8]). Let $N$ be a subgroup of the global group $G$ satisfying conditions (a) and (b) of Proposition 1.3. If $H$ is a subgroup of $G$ with $N \subseteq H$ and if $H \oplus \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \ldots \oplus \langle x_m \rangle$ is a $*$-valuated coproduct in $G$ where $x_1, x_2, \ldots, x_m$ are primitive, then $H/N \oplus \langle x_1 + N \rangle \oplus \langle x_2 + N \rangle \oplus \ldots \oplus \langle x_m + N \rangle$ is a $*$-valuated coproduct in $G/N$ and the $x_i + N$ are primitive in $G/N$. 

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As noted in [8, Proposition 1.9], isotype knice subgroups possess quite special properties, and in particular, a stronger version of condition (b) of Proposition 1.3.

**Proposition 1.5** ([8]). *If the subgroup $H$ is both knice and isotype in the global group $G$, then for each $g \in G$ the coset $g + H$ contains a representative $x$ such that $\|x\|^G = \|g + H\|^{G/H}$.***

2. Preliminary theorems

We propose, in this paper, to give necessary and sufficient conditions for an isotype knice subgroup $H$ of a global Warfield group $G$ to be itself a Warfield group. In fact, we shall establish such conditions in terms of an Axiom 3 characterization of the quotient group $G/H$. As a result, our Main Theorem (Theorem 4.9) will have roughly the same form as the characterization of the simply presented isotype subgroups of totally projective $p$-groups established in [6]. The principal result in that paper specializes to show that an isotype nice subgroup $H$ of a totally projective $p$-group $G$ is itself totally projective if and only if $G/H$ has an Axiom 3 system of “separable” subgroups. But of course, in the global setting, the separability condition will need to be suitably modified; and moreover, each subgroup in the Axiom 3 system for $G/H$ will be required to be a $k$-subgroup, a new type of subgroup whose definition appears below. We should also mention that in the case of $p$-local Warfield groups in [10] and of global Warfield groups in [12], the isotype subgroups that are themselves Warfield were characterized. However, those results have the unpleasant feature that one must consider properties of the quotients $(H + N)/N$ in $G/N$ as $N$ ranges over an entire Axiom 3 system of knice subgroups for $G$. In this section, we prove two theorems that will play pivotal roles in the sequel.

**Theorem 2.1.** *Let $H$ be an isotype knice subgroup of the global group $G$ and suppose that $A$ is a pure subgroup of $H$. Then $A$ is knice in $G$ if and only if $A$ is knice in $H$.***

**Proof.** That $A$ is knice in $H$ when it is knice $G$ follows from [8, Lemma 4.2]. Conversely, assume that $A$ is a knice subgroup of $H$. To establish that $A$ is knice in $G$, we first show that it is a nice subgroup of $G$. Given a prime $p$ and an arbitrary element $x \in G$, we apply part (4) of Proposition 1.2 to select elements $h_1 \in H$ and $a_1 \in A$ such that

$$|x + h_1|^p = |x + H|^p_{G/H} \text{ and } |h_1 - a_1|^p = |h_1 + A|^p_{H/A}.$$
It then follows that

\[(*) \quad |x + h|_p^G = |x + h_1|_p^G \wedge |h - h_1|_p^G = |x + h_1|_p^G \wedge |h - h_1|_p^H\]

for all \(h \in H\). Indeed if for some \(h \in H\) we had \(|x + h|_p^G > |h - h_1|_p^G\), it would follow that

\[|x + h_1|_p^G = |(x + h) + (h_1 - h)|_p^G = |h - h_1|_p^G < |x + h|_p^G,\]

contrary to the choice of \(h_1\). Moreover, from \((*)\) and our choice of \(a_1\), we get

\[|x + a|_p^G \leq |x + a_1|_p^G\]

for all \(a \in A\) since

\[|x + a|_p^G = |x + h_1|_p^G \wedge |a - h_1|_p^H \leq |x + h_1|_p^G \wedge |a_1 - h_1|_p^G \leq |x + a_1|_p^G.\]

That \(A\) is nice in \(G\) now follows from [2, Lemma 79.2].

In order to show that \(A\) satisfies part (2) of Definiion 1.1, let \(S\) be an arbitrary finite subset of \(G\). Since \(H\) is a knice subgroup of \(G\), \(S\) is finite modulo a \(\ast\)-valuated coproduct \(H \oplus \langle y_1 \rangle \oplus \ldots \oplus \langle y_m \rangle\) where \(y_1, \ldots, y_m\) are primitive elements of \(G\). But then \(S\) is finite modulo a subgroup \(\langle T, y_1, \ldots, y_m \rangle\) where \(T\) is a finite subset of \(H\).

Since \(A\) is assumed to be a knice subgroup of \(H\), \(T\) is finite modulo a \(\ast\)-valuated coproduct \(A \oplus \langle x_1 \rangle \oplus \ldots \oplus \langle x_n \rangle\) where \(x_1, \ldots, x_n\) are primitive elements of \(H\). Because \(H\) is both isotype and knice in \(G\), it follows from Corollary 2.6 and Proposition 2.8 of [11], and the remark at the beginning of the proof of [11, Proposition 2.9], that \(x_1, \ldots, x_n\) are primitive in \(G\) and

\[A' = A \oplus \langle x_1 \rangle \oplus \ldots \oplus \langle x_n \rangle \oplus \langle y_1 \rangle \oplus \ldots \oplus \langle y_m \rangle\]

is a \(\ast\)-valuated coproduct in \(G\). Moreover, \(\langle S, A' \rangle / A'\) is finite as desired. \(\square\)

Applications of Axiom 3 characterizations of groups inevitably involve an interplay between appropriately formulated notions of separability and compatibility. We begin by recalling the notions of compatibility used by us in [12]. First, if \(p\) is a prime, we say that the subgroups \(H\) and \(N\) of \(G\) are \(p\)-compatible in \(G\) if for each \(h \in H\) and \(x \in N\) there corresponds an \(x' \in H \cap N\) such that \(|h + x|_p \leq |h + x'|_p\). (Heights and height matrices unadorned by superscripts are of course understood to be computed in the largest containing group \(G\).) If \(H\) and \(N\) are \(p\)-compatible for all primes \(p\), they are said to be locally compatible.

**Definition 2.2.** The subgroups \(H\) and \(N\) of a global group \(G\) are said to be almost strongly compatible in \(G\) provided that they are locally compatible and, for each \(h \in H\) and \(x \in N\), there corresponds an \(x' \in H \cap N\) and a positive integer \(m\) such that \(\|m(h + x)\| \leq \|mh + x'\|\).
Note that $p$-compatibility, local compatibility and almost strong compatibility are all symmetric relations. For example, $\|m(h + x)\| \leq \|mh + x'\|$ implies that $\|m(h + x)\| \leq \|mx - x'\|$. When there is no danger of confusion with previous uses of the notation (see [6] or [3]), we shall occasionally write $H \parallel N$ to indicate that $H$ and $N$ are almost strongly compatible. For later reference we remark that all of these compatibility relations are inductive. So, in particular, if $\{N_i\}_{i \in I}$ is a chain of subgroups of $G$ with $H \parallel N_i$ for all $i \in I$, then $H \parallel N$ where $N = \bigcup_{i \in I} N_i$.

**Definition 2.3.** Let $N$ be a subgroup of a global group $G$. We say that $N$ is a $k$-subgroup of $G$ if to each finite subset $S$ of $N$ there corresponds a $*$-valuated coproduct

$$M = \langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \ldots \oplus \langle y_s \rangle$$

in $G$ with the properties that $\{y_1, y_2, \ldots, y_s\}$ is a (possibly empty) set of primitive elements of $G$, $M \subseteq N$ and $\langle S, M \rangle \slash M$ is finite.

With this new terminology, [12, Theorem 2.1] shows that any knice subgroup of a $k$-group is a $k$-subgroup.

**Theorem 2.4.** Let $H$ and $N$ be almost strongly compatible knice subgroups of the global group $G$, with $H$ isotype in $G$ and $N$ pure in $G$. Then, $H \cap N$ is a knice subgroup of $G$ if and only if $(N + H) \slash H$ is a $k$-subgroup of $G \slash H$.

**Proof.** In view of Proposition 1.3, $H \cap N$ is a knice subgroup of $G$ if and only if the following hold:

(a) $H \cap N$ is a nice subgroup of $G$.

(b) For each $g \in G$ there exist a positive integer $n$ and an $h' \in H \cap N$ such that

$$\|ng + (H \cap N)\|^{G/(H \cap N)} = \|ng + h'\|^G.$$

(c) The quotient group $G \slash (H \cap N)$ is a $k$-group.

It is condition (c) that hinges on the structure of $(N + H) \slash H$ and we shall first show that the general hypotheses imply that (a) and (b) are satisfied. To show that $H \cap N$ is a nice subgroup of $G$, suppose that $g + (H \cap N) \in p^\alpha G/(H \cap N)$ and $pg \in p^\alpha G + (H \cap N)$ for some prime $p$ and ordinal $\alpha$. Since both $H$ and $N$ are nice subgroups of $G$, we can write $g = z_1 + h = z_2 + x$ where $z_1, z_2 \in p^\alpha G$, $h \in H$ and $x \in N$. Then, $h - x = z_2 - z_1 \in p^\alpha G$ and, by $p$-compatibility, we have an $h' \in H \cap N$ such that $\alpha \leq \|h - x\|_p \leq \|h - h'\|_p$. Thus, $g = (z_1 + (h - h')) + h' \in p^\alpha G + (H \cap N)$ and we conclude that $H \cap N$ is a nice subgroup of $G$.

Next we show that condition (b) is satisfied. Given $g \in G$, we first utilize the fact that $N$ is a knice subgroup of $G$ to choose a positive integer $k$ such that $\|kg +
\[ N\|^{G/N} = \|kg + x\| \text{ for some } x \in N. \] Then, since \( H \) is isotype and knice, it follows from Proposition 1.5 that there is an \( h \in H \) with \( \|kg + H\|^{G/H} = \|kg + h\|. \) Now, since \( N \| H \), there is a positive integer \( m \) and an \( h' \in H \cap N \) such that

\[ (1) \quad \|m(x - h)\| \leq \|mx - h'\| \land \|mh - h'\|. \]

We shall show that

\[ \|kmg + h'\| = \|kmg + (H \cap N)\|^{G/(H \cap N)} \]

and thereby complete the proof that (b) is satisfied. Since

\[ \|kmg + mx\| \land \|kmg + mh\| = \|kmg + N\|^{G/N} \land \|kmg + H\|^{G/H} \]

\[ \geq \|kmg + (H \cap N)\|^{G/(H \cap N)}, \]

it suffices to verify that

\[ (2) \quad \|kmg + h'\| \geq \|kmg + mx\| \land \|kmg + mh\|. \]

By three applications of the triangle inequality,

\[ (3) \quad \|m(x - h)\| \geq \|kmg + mx\| \land \|kmg + mh\| \]

and

\[ \|kmg + h'\| \geq (\|kmg + mx\| \land \|mx - h'\|) \land (\|kmg + mh\| \land \|mh - h'\|). \]

Finally, applying (1) and (3) to calculate the last term, we obtain (2).

To complete the proof, we require the following observation: if \( x \in N \), there is a positive integer \( m \) such that

\[ (4) \quad \|mx + H\|^{G/H} = \|mx + (H \cap N)\|^{G/(H \cap N)}. \]

To see this, first apply Proposition 1.5 to select an \( h \in H \) such that \( \|x + H\|^{G/H} = \|x + h\|. \) Then, applying \( H \| N \), there is a positive integer \( m \) and an \( h' \in H \cap N \) such that \( \|m(x + h)\| \leq \|mx + h'\|. \) Since

\[ \|mx + h'\| \leq \|mx + (H \cap N)\|^{G/(H \cap N)} \leq \|mx + H\|^{G/H} = \|m(x + h)\| \leq \|mx + h'\|, \]

(4) follows.
If \( H \cap N \) is a knice subgroup of \( G \), then \( G/(H \cap N) \) is a \( k \)-group by Proposition 1.3. Moreover, \( N/(H \cap N) \) is knice in \( G/(H \cap N) \) by Proposition 1.2(5). Therefore, as noted above, the conclusion that \( N/(H \cap N) \) is a \( k \)-subgroup of \( G/(H \cap N) \) follows from [12, Theorem 2.1]. Since the definitions of primitive elements and \(*\)-valuated coproducts depend solely on the computation of height matrices, observation (4) implies that \( (N + H)/H \) is a \( k \)-subgroup of \( G/H \). Conversely, assume that \( (N + H)/H \) is a \( k \)-subgroup of \( G/H \). Then to complete the proof that \( H \cap N \) is a knice subgroup of \( G \), it remains to show that \( (c) \ G/(H \cap N) \) is a \( k \)-group. Towards this end, consider an arbitrary finite subset \( \mathcal{S} \) of \( G/(H \cap N) \) and let \( S \) be a corresponding finite subset of \( G \) that projects onto \( \mathcal{S} \). As \( N \) is a knice subgroup of \( G \), there exist primitive elements \( x_1, x_2, \ldots, x_m \) in \( G \) such that \( S \) is finite modulo the \(*\)-valuated coproduct

\[
N \oplus \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \ldots \oplus \langle x_m \rangle.
\]

Then, by Corollary 1.4,

\[
N/(H \cap N) \oplus \langle x_1 + (H \cap N) \rangle \oplus \langle x_2 + (H \cap N) \rangle \oplus \ldots \oplus \langle x_m + (H \cap N) \rangle
\]

is a \(*\)-valuated coproduct in \( G/(H \cap N) \) with each \( x_i + (H \cap N) \) a primitive element of \( G/(H \cap N) \). Obviously then, \( \mathcal{S} \) is finite modulo this coproduct. Thus, there is a finite subset \( \mathcal{T} \) of \( N/(H \cap N) \) such that \( \mathcal{S} \) is finite modulo the subgroup

\[
\langle \mathcal{T}, x_1 + (H \cap N), x_2 + (H \cap N), \ldots, x_m + (H \cap N) \rangle.
\]

Now from the hypothesis that \( (N + H)/H \) is a \( k \)-subgroup of \( G/H \), (4) implies that \( \mathcal{T} \) is finite modulo

\[
\langle y_1 + (H \cap N) \rangle \oplus \langle y_2 + (H \cap N) \rangle \oplus \ldots \oplus \langle y_k + (H \cap N) \rangle \subseteq N/(H \cap N)
\]

where the coproduct is \(*\)-valuated in \( G/(H \cap N) \) and each \( y_j + (H \cap N) \) is a primitive element of \( G/(H \cap N) \). Finally, \( \mathcal{S} \) is clearly finite modulo the \(*\)-valuated coproduct

\[
\langle y_1 + (H \cap N) \rangle \oplus \ldots \oplus \langle y_k + (H \cap N) \rangle \oplus \langle x_1 + (H \cap N) \rangle \oplus \ldots \oplus \langle x_m + (H \cap N) \rangle
\]

and hence \( G/(H \cap N) \) is a \( k \)-group as desired. \( \square \)
If \( p \) is a prime, a subgroup \( H \) of \( G \) is said to be \( p \)-separable provided that for each \( g \in G \) there is a corresponding countable subset \( \{h_n\}_{n<\omega} \subseteq H \) with the following property: if \( h \in H \), then there exists an \( n < \omega \) such that \( |g + h|_p \leq |g + h_n|_p \). If a subgroup \( H \) of \( G \) is \( p \)-separable for all primes \( p \), then we say that \( H \) is \textit{locally separable} in \( G \). P. Hill proves in [6] that a simply presented isotype subgroup \( H \) of a torsion group \( G \) is necessarily locally separable in \( G \). On the other hand, for a local Warfield group \( H \) to appear as an isotype subgroup of a local group \( G \), \( H \) must be strongly separable in \( G \) (see [10]); in other words, to each \( g \in G \) there corresponds a countable subset \( \{h_n\}_{n<\omega} \subseteq H \) such that if \( h \in H \), then \( \|g + h\| \leq \|g + h_n\| \) for some \( n < \omega \). But for global groups, it is possible for an isotype subgroup to be a Warfield group without being strongly separable in the containing group (see [9]). The requisite necessary condition in the global context is the following intermediate form of separability first identified in [12].

\textbf{Definition 3.1.} A subgroup \( H \) of a global group \( G \) is said to be \textit{almost strongly separable} in \( G \) if \( H \) is locally separable in \( G \) and, for each \( g \in G \) there is a corresponding countable subset \( \{h_n\}_{n<\omega} \subseteq H \) with the following property: if \( h \in H \), then there exists an \( n < \omega \) and a positive integer \( m \) such that \( \|m(g+h)\| \leq \|m(g+h_n)\| \).

Thus, if \( H \) is an isotype Warfield subgroup of the global group \( G \), then \( H \) is almost strongly separable in \( G \) (see [12, Proposition 3.6]).

\textbf{Remark 3.2.} For later use, we note that every pure knice subgroup \( H \) of an arbitrary global group \( G \) is almost strongly separable in \( G \). That this is so is a routine consequence of Proposition 1.2(4) and Proposition 1.3(b). In particular, Proposition 1.2(4) implies that \( H \) is locally separable in \( G \), while Proposition 1.3(b) can be used to handle the condition on height matrices.

We shall require the next proposition several times in the sequel.

\textbf{Proposition 3.3.} Suppose that \( N \) is a knice and pure subgroup of the global group \( G \) and that \( K \) is a subgroup of \( G \) that contains \( N \). Then \( K \) is almost strongly separable in \( G \) if and only if \( K/N \) is almost strongly separable in \( G/N \).

\textbf{Proof.} We first observe that \( K \) is locally separable in \( G \) if and only if \( K/N \) is locally separable in \( G/N \). Indeed this is a routine consequence of Proposition 1.2(4) and the definitions.

Now assume that \( K \) is almost strongly separable in \( G \). Given \( g+N \in G/N \), select a corresponding countable subset \( \{c_n\}_{n<\omega} \subseteq K \) with the following property: if \( y \in K \), then \( \|m(g+y)\| \leq \|m(g+c_n)\| \) for some \( n < \omega \) and positive integer \( m \). Assume now
that $c + N$ is an arbitrary element of $K/N$. Since $N$ is a pure and knice subgroup of $G$, there is a positive integer $m$ so that $\|m(g + c) + N\|^{G/N} = \|m(g + c + z)\|$ for some $z \in N$. Moreover, replacing $m$ by a positive multiple of itself if necessary, there is an $n < \omega$ such that $\|m(g + c + z)\| \leq \|m(g + c_n)\|$. Therefore,

$$\|m(g + c) + N\|^{G/N} \leq \|m(g + c_n)\| \leq \|m(g + c_n) + N\|^{G/N}.$$ 

It is now clear that $\{c_n + N\}_{n<\omega}$ satisfies the requisite properties for $g + N$, and we conclude that $K/N$ is almost strongly separable in $G/N$.

Conversely, suppose that $K/N$ is almost strongly separable in $G/N$ and, for a given $g \in G$, select a countable subset $\{c_n + N\}_{n<\omega}$ of $K/N$ with the following property: if $c + N \in K/N$, there exists an $n < \omega$ and a positive integer $m$ such that $\|m(g + c) + N\|^{G/N} \leq \|m(g + c_n) + N\|^{G/N}$. For each $n < \omega$, choose $m_n$ to be the smallest positive integer for which there exists a $z_n \in N$ with $\|m_n(g + c_n) + N\|^{G/N} = \|m_n(g + c_n + z_n)\|$. Now given $c \in K$, select a positive integer $l$ and an $n < \omega$ such that

$$\|l(g + c) + N\|^{G/N} \leq \|l(g + c_n) + N\|^{G/N}.$$ 

Then

$$\|lm_n(g + c)\| \leq \|lm_n(g + c_n) + N\|^{G/N} = \|lm_n(g + c_n + z_n)\|$$

and we conclude that the countable subset $\{c_n + z_n\}_{n<\omega} \subseteq K$ satisfies the requisite properties for $g \in G$. \hfill \Box

In the application of Axiom 3 characterizations, there is an interplay between separability and compatibility that will be familiar to those who have studied either of the fundamental papers [6] or [3]. Such readers may indeed anticipate the following proposition (obtained by combining Propositions 4.5 and 4.6 of [12]). But it should be borne in mind that, in the present setting, $H \parallel N$ indicates that $H$ and $N$ are almost strongly compatible in the sense of Definition 2.2 above.

**Proposition 3.4** ([12]). (1) Suppose $H$ is almost strongly separable subgroup of the global group $G$. If $A$ is a countable subgroup of $G$, there is a countable subgroup $B$ of $G$ that contains $A$ and such that $H \parallel B$.

(2) Let $H$ and $N$ be subgroups of a global group $G$ where $H \parallel N$ and $N$ is pure and knice in $G$. If $M$ is any subgroup of $G$ that contains $N$, then $(H + N)/N \parallel M/N$ implies $H \parallel M$.

**Definition 3.5.** By an Axiom 3 family in the global group $G$ is meant a collection $\mathcal{C}$ of subgroups of $G$ that satisfies the following three conditions.

(H1) $\mathcal{C}$ contains the trivial subgroup $0$;
(H2) if \( N_i \in \mathcal{C} \) for each \( i \in I \), then \( \sum_{i \in I} N_i \in \mathcal{C} \);

(H3) if \( C \in \mathcal{C} \) and if \( A \) is any countable subgroup of \( G \), then there is a \( B \in \mathcal{C} \) that contains both \( C \) and \( A \) with \( B/C \) countable.

The first and most famous of an Axiom 3 characterization is P. Hill’s proof that a \( p \)-primary abelian group \( G \) is simply presented if and only if there is an Axiom 3 family in \( G \) consisting of nice subgroups. The analogous result for global groups is established in [8]: a global group \( G \) is a Warfield group if and only if there is an Axiom 3 family in \( G \) consisting of knice subgroups. There are several variations on the Axiom 3 theme and we shall find it convenient to adopt the notation and terminology of [3] where an Axiom 3 family is referred to as an \( H(\aleph_0) \)-family in \( G \), and a \( G(\aleph_0) \)-family in \( G \) is a collection of subgroups \( \mathcal{C} \) that satisfies conditions (H1), (H3) and

(G2) \( \mathcal{C} \) is closed under unions of ascending chains.

For example, the family \( \mathcal{P}_G \) of all pure subgroups of \( G \) is a \( G(\aleph_0) \)-family in \( G \). Also as in [3], by an \( F(\aleph_0) \)-family \( \mathcal{C} = \{ N_\alpha \}_{\alpha < \tau} \) in \( G \) we mean a smooth ascending chain

\[
0 = N_0 \subseteq N_1 \subseteq \ldots \subseteq N_\alpha \subseteq \ldots \quad (\alpha < \tau)
\]

of subgroups of \( G \) where \( G = \bigcup_{\alpha < \tau} N_\alpha \) and \( |N_{\alpha+1}/N_\alpha| \leq \aleph_0 \) for all \( \alpha < \tau \). Obviously an \( H(\aleph_0) \)-family in \( G \) is a \( G(\aleph_0) \)-family and a \( G(\aleph_0) \)-family contains an \( F(\aleph_0) \)-family.

**Proposition 3.6 ([3]).** (1) The intersection of any two \( H(\aleph_0) \)-families (\( G(\aleph_0) \)-families) in \( G \) is again one.

(2) Let \( H \) be a subgroup of a global group \( G \). If \( \mathcal{C}_G \) and \( \mathcal{C}_H \) are \( G(\aleph_0) \)-families in \( G \) and \( H \), respectively, then \( \mathcal{C} = \{ N \in \mathcal{C}_G : N \cap H \in \mathcal{C}_H \} \) is a \( G(\aleph_0) \)-family in \( G \).

(3) Suppose \( H \) is a subgroup of a global group \( G \) and that \( \pi : G \to G/H \) is the canonical map. If \( \mathcal{C}_G \) is a \( G(\aleph_0) \)-family in \( G \) and \( \mathcal{D} \) is a \( G(\aleph_0) \)-family in \( G/H \), then there is a \( G(\aleph_0) \)-family \( \mathcal{C} \) contained in \( \mathcal{C}_G \) such that \( \pi(\mathcal{C}) \subseteq \mathcal{D} \) (where \( \pi(\mathcal{C}) = \{ \pi(N) : N \in \mathcal{C} \} \)).

**Proof.** (1) is [3, Lemma 1.2] and (2) is established in the proof of [3, Lemma 1.5]. By [3, Lemma 1.3(b)], there is a \( G(\aleph_0) \)-family family \( \mathcal{B} \) in \( G \) such that \( \pi(\mathcal{B}) = \mathcal{D} \) and we need only take \( \mathcal{C} = \mathcal{B} \cap \mathcal{C}_G \).

**Proposition 3.7.** If \( G \) is a global Warfield group, there exists a \( G(\aleph_0) \)-family \( \mathcal{C} \) in \( G \) consisting of pure knice subgroups; and furthermore, \( G/A \) is a global Warfield group whenever \( A \in \mathcal{C} \).
By [8, Theorem 3.2], there is an $G(\aleph_0)$-family $\mathcal{C}_G$ in $G$ consisting of knice subgroups and $\mathcal{C} = \mathcal{P}_G \cap \mathcal{C}_G$ has the desired property. Indeed, for each $A \in \mathcal{C}$, $\{N/A: N \in \mathcal{C} \text{ and } A \subseteq N\}$ is clearly a $G(\aleph_0)$-family in $G/A$ consisting of knice subgroups by Proposition 1.2(5). Finally, by [8, Theorem 3.2] again, $G/A$ is a global Warfield group. □

The significance of Proposition 3.4 is that it provides us with the tools required to establish the following fact which will play a prominent role in the proof of our Main Theorem.

Proposition 3.8. Let $H$ be a subgroup of the global Warfield group $G$ and suppose that $\mathcal{C}_G$ is a $G(\aleph_0)$-family in $G$ consisting of pure knice subgroups. If

$$\mathcal{C} = \{N \in \mathcal{C}_G: N \parallel H\}$$

and if $(H + N)/N$ is almost strongly separable in $G/N$ for each $N \in \mathcal{C}_G$, then $\mathcal{C}$ is a $G(\aleph_0)$-family in $G$.

Proof. First note that $\mathcal{C}$ satisfies (G2) since, as noted above, almost strong compatibility is an inductive relation. Thus, it remains only to show that $\mathcal{C}$ satisfies property (H3). To this end, let $N \in \mathcal{C}$ and suppose that $A$ is a countable subgroup of $G$. We construct inductively two ascending sequences of subgroups $\{N_n\}_{n<\omega}$ and $\{M_n\}_{n<\omega}$ such that $N + A \subseteq N_0$ and the following three conditions are satisfied for all $n < \omega$.

(i) $N_n \in \mathcal{C}_G$ and $M_n \parallel H$.
(ii) $N_n \subseteq M_n \subseteq N_{n+1}$.
(iii) $|N_n/N| \leq \aleph_0$ and $|M_n/N| \leq \aleph_0$.

Assuming that $N_n$ has been constructed, we apply part (1) of Proposition 3.4 and the hypothesis that $(H + N)/N$ is almost strongly separable in $G/N$ to obtain a countable subgroup $M_n/N$ of $G$ that contains $N_n/N$ and such that $M_n/N \parallel (H + N)/N$. Then by part (2) of Proposition 3.4, $M_n \parallel H$. Next, using the fact that $\mathcal{C}_G$ is a $G(\aleph_0)$-family in $G$, we choose $N_{n+1} \in \mathcal{C}_G$ such that $M_n \subseteq N_{n+1}$ and $N_{n+1}/N$ is countable. This completes the induction. Now take $N' = \bigcup_{n<\omega} N_n = \bigcup_{n<\omega} M_n$ and observe that both $N' \in \mathcal{C}_G$ and $N' \parallel H$. Thus, $N' \in \mathcal{C}$, $N + A \subseteq N'$ and $N'/N$ is countable. Therefore $\mathcal{C}$ satisfies property (H3). □

At this point, we have all that is needed to establish a sufficient condition for an isotype knice subgroup of a global Warfield group to be itself a Warfield group.
Theorem 3.9. Let $H$ be an isotype knice subgroup of the global Warfield group $G$. If there is a $G(\aleph_0)$-family $\mathcal{D}$ in the quotient group $G/H$ consisting of almost strongly separable $k$-subgroups, then $H$ is a Warfield group.

Proof. By Proposition 3.7, there is a $G(\aleph_0)$-family $\mathcal{C}_G$ in $G$ consisting of pure knice subgroups of $G$. In view of Proposition 3.6(3), we may assume without loss of generality that $\pi(\mathcal{C}_G) \subseteq \mathcal{D}$, where $\pi: G \to G/H$ is the canonical map. But then, $(N + H)/H$ is an almost strongly separable $k$-subgroup of $G/H$ whenever $N \in \mathcal{C}_G$. Two applications of Proposition 3.3 imply that $(H + N)/N$ is almost strongly separable in $G/N$ for each $N \in \mathcal{C}_G$. Then, by Proposition 3.8, $\{N \in \mathcal{C}_G: N \parallel H\}$ is a $G(\aleph_0)$-family in $G$. Since the family $\mathcal{P}_H$ of all pure subgroups of $H$ is a $G(\aleph_0)$-family in $H$, we conclude from Proposition 3.6 that

$$\mathcal{C} = \{N \in \mathcal{C}_G: N \parallel H \text{ and } H \cap N \text{ is pure in } H\}$$

is also a $G(\aleph_0)$-family in $G$. Recalling that $(N + H)/H$ is a $k$-subgroup of $G/H$ whenever $N \in \mathcal{C}_G$, we see from Theorem 2.4 that $H \cap N$ is knice in $G$ whenever $N \in \mathcal{C}_G$. In fact, by Theorem 2.1, each $H \cap N$ is a knice subgroup of $H$. From $\mathcal{C}$ we extract an $F(\aleph_0)$-family in $G$

$$0 = N_0 \subseteq N_1 \subseteq \ldots \subseteq N_\alpha \subseteq \ldots \quad (\alpha < \tau).$$

Then,

$$0 = H \cap N_0 \subseteq H \cap N_1 \subseteq \ldots \subseteq H \cap N_\alpha \subseteq \ldots \quad (\alpha < \tau)$$

is an $F(\aleph_0)$-family in $H$ consisting of pure knice subgroups of $H$ and, therefore, we conclude from [12, Theorem 2.5] that $H$ is a global Warfield group. \qed

The converse of Theorem 3.9 is more difficult to prove, but with the tools currently at our command, we can now obtain a strong partial converse.

Theorem 3.10. Let $H$ be an isotype knice subgroup of the global Warfield group $G$. If $H$ itself is a Warfield group, then there exists a $G(\aleph_0)$-family $\mathcal{C}$ in $G$ with the property that, for each $N \in \mathcal{C}$, $(H + N)/H$ is an almost strongly separable $k$-subgroup of $G/H$.

Proof. By the hypotheses and Proposition 3.7, there exists a $G(\aleph_0)$-family $\mathcal{C}_G$ of pure knice subgroups in $G$ and a $G(\aleph_0)$-family $\mathcal{C}_H$ of pure knice subgroups in $H$. Since $H$ is an isotype Warfield subgroup of the global Warfield group $G$, our characterization of this situation in [12, Theorem 5.2] establishes the existence of a $G(\aleph_0)$-family $\mathcal{C}''$ in $G$ such that $(H + N)/N$ is almost strongly separable in $G/N$ for all $N \in \mathcal{C}''$. Set $\mathcal{C}' = \mathcal{C}'' \cap \mathcal{C}_G$. By Proposition 3.6(1), $\mathcal{C}'$ is a $G(\aleph_0)$-family in $G$ so
that \( \{N \in \mathcal{C}': N \parallel H\} \) is a \( G(\aleph_0) \)-family in \( G \) by Proposition 3.8. Therefore, by an application of Proposition 3.6(2),

\[
\mathcal{C} = \{N \in \mathcal{C}' : N \parallel H \text{ and } H \cap N \in \mathcal{C}_H\}
\]
is a \( G(\aleph_0) \)-family in \( G \). By two applications of Proposition 3.3, \( (H + N)/H \) is almost strongly separable in \( G/H \) for all \( N \in \mathcal{C} \). Also, if \( N \in \mathcal{C} \), \( N \cap H \) is a pure knice subgroup of \( H \) and therefore Theorem 2.1 implies that \( N \cap H \) is knice in \( G \). Finally, since all the hypotheses of Theorem 2.4 are satisfied when \( N \in \mathcal{C} \), we conclude that \( (H + N)/H \) is also a \( k \)-subgroup of \( G/H \) whenever \( N \in \mathcal{C} \).

\[\square\]

It is tempting to speculate that, when \( \mathcal{C} \) is as in Theorem 3.10, \( \mathcal{D} = \{(H + N)/H : N \in \mathcal{C}\} \) might constitute the appropriate family of subgroups of \( G/H \) yielding the converse of Theorem 3.9. However, \( \mathcal{D} \) need not be a \( G(\aleph_0) \)-family in \( G/H \) since chains of subgroups in \( \mathcal{D} \) need not be induced by chains of subgroups in \( \mathcal{C} \). Nonetheless, we can extract from \( \mathcal{C} \) an \( F(\aleph_0) \)-family in \( G \) that will project to an \( F(\aleph_0) \)-family in \( G/H \), yielding a proof of the following.

**Corollary 3.11.** Let \( H \) be an isotype knice subgroup of the global Warfield group \( G \). If \( H \) is itself a global Warfield group, then there exists an \( F(\aleph_0) \)-family in \( G/H \) consisting of almost strongly separable \( k \)-subgroups.

4. Axiom 3, closed sets and the main theorem

There is a standard method introduced by P. Hill in [5] for converting \( F(\aleph_0) \)-families of appropriate sorts of subgroups into \( H(\aleph_0) \)-families of the same sort of subgroups. This technique, called the *closed set method*, is a powerful tool that we shall require frequently in this section. The method is, of course, not universally applicable and hence each application requires *ad hoc* arguments. Since it leads to cleaner and less obscure proofs, we shall use the version of the method introduced in [1]. The common core in all our applications is the existence of an \( F(\aleph_0) \)-family \( \{G_\alpha\}_{\alpha<\tau} \) in the global group \( G \) consisting of pure subgroups. Since each quotient \( G_{\alpha+1}/G_\alpha \) is countable, we also have an associated family \( \{B_\alpha\}_{\alpha<\tau} \) of countable pure subgroups with \( G_{\alpha+1} = G_\alpha + B_\alpha \) for each \( \alpha < \tau \). Since the set \( \mathcal{P}_G \) of all pure subgroups of \( G \) is a \( G(\aleph_0) \)-family in \( G \), each global group \( G \) possesses such an \( F(\aleph_0) \)-family. A routine induction leads to a proof of the fundamental fact that \( G_\alpha = \sum_{\beta<\alpha} B_\beta \) for each \( \alpha < \tau \). Following [4], for each \( g \in G \) we let \( \nu(g) \) denote the least ordinal such that \( g \in G_{\nu(g)+1} \). All our applications of the closed set method involve inductions on \( \nu(g) \).
A subset $S$ of $\tau$ is called a closed subset if it enjoys the following property: for each $\lambda \in S$,

$$B_\lambda \cap G_\lambda \subseteq \sum \{B_\alpha : \alpha \in S \text{ and } \alpha < \lambda\}.$$ 

The relevant facts, established in [1], are the following:

1. the empty set is closed,
2. arbitrary unions of closed subsets are closed, and
3. each countable subset of $\tau$ is contained in a countable closed subset.

With each subset $S \subseteq \tau$, we associate the subgroup $G(S) = \sum \{B_\alpha : \alpha \in S\}$. Noting that $\sum_{i \in I} G(S_i) = G\left(\bigcup_{i \in I} S_i\right)$, we see that (1), (2) and (3) imply that $\mathcal{C} = \{G(S) : S \text{ is a closed subset of } \tau\}$ is an $H(\aleph_0)$-family in $G$.

Each nonzero $x \in G(S)$ has a standard representation

$$x = b_{\mu(1)} + b_{\mu(2)} + \ldots + b_{\mu(m)}$$

where $\mu(i) \in S$ and $b_{\mu(i)}$ is a nonzero element of $B_{\mu(i)}$ for $i = 1, 2, \ldots, m$,

$$\mu(1) < \mu(2) < \ldots < \mu(m)$$

and $\mu(m)$ is minimal. The following fundamental fact is implicit in [4].

**Lemma 4.1.** If $x = b_{\mu(1)} + b_{\mu(2)} + \ldots + b_{\mu(m)}$ is a standard representation of a nonzero $x \in G(S)$ with $S$ a closed subset of $\tau$, then $\nu(x) = \mu(m)$. In particular, $\nu(x) \in S$.

**Proof.** We first prove that $\nu(x) \in S$. Assume by way of contradiction that $\mu = \nu(x) \notin S$. Then we may write $x = x_0 + x_1$ where $x_0 \in G(S) \cap G_\mu$ and $0 \neq x_1 \in \sum \{B_\alpha : \mu < \alpha \text{ and } \alpha \in S\}$. Select a standard representation $x_1 = b'_{\lambda(1)} + \ldots + b'_{\lambda(n)}$. Since $x_1 \in G_{\mu+1}$ and $\mu + 1 \leq \lambda(n)$,

$$b'_{\lambda(n)} = x_1 - (b'_{\lambda(1)} + \ldots + b'_{\lambda(n-1)})$$

is an element of $G_{\lambda(n)} \cap B_{\lambda(n)} \subseteq \sum \{B_\alpha : \alpha \in S \text{ and } \alpha < \lambda(n)\}$. This, however, contradicts the minimality of $\lambda(n)$, forcing us to conclude that $\nu(x) \in S$.

Clearly, $\mu = \nu(x) \leq \mu(m)$ and we assume that equality has been established for all nonzero $y \in G(S)$ with $\nu(y) < \nu(x)$. Write $x = y + b$ where $y \in G_\mu$ and $b \in B_\mu$. Since $x \in G(S)$ and $\mu \in S$ implies that $b \in G(S)$, we conclude that $y \in G(S)$ and we may assume that $y \neq 0$. Thus, we have a standard representation $y = b'_{\lambda(1)} + \ldots + b'_{\lambda(n)}$ where $\nu(y) < \mu$. Therefore, our induction hypothesis yields $\lambda(n) = \nu(y) < \mu$. If we were to have $\mu < \mu(m)$, then the representation $x = b'_{\lambda(1)} + \ldots + b'_{\lambda(n)} + b$ would contradict the minimality of $\mu(m)$.

\[\square\]
The following result was first established in the special case of torsion-free groups by Hill [5].

**Proposition 4.2.** If $G$ is an arbitrary global group, then there is an $H(\aleph_0)$-family in $G$ consisting of pure subgroups.

**Proof.** By the discussion above, it suffices to show that $G(S)$ is a pure subgroup of $G$ whenever $S$ is a closed subset. Let $0 \neq x \in nG \cap G(S)$ where $n$ is a positive integer and suppose $\mu = \nu(x)$. Since $G_{\mu+1}$ is a pure subgroup of $G$, $x = ny$ where $y = z + b$ with $z \in G_{\mu}$ and $b \in B_\mu \subseteq G(S)$, with the latter containment holding by Lemma 4.1. Thus $nz = x - nb = x_1 \in G(S)$ with $\nu(x_1) < \nu(x)$. Proceeding by induction, there is a $y_1 \in G(S)$ such that $ny_1 = x_1$ and therefore $x = n(y_1 + b)$ where $y_1 + b \in G(S)$. □

**Corollary 4.3.** If $G$ is a global Warfield group, there is an $H(\aleph_0)$-family in $G$ consisting of almost strongly separable pure knice subgroups.

**Proof.** As shown in [8], $G$ has an $H(\aleph_0)$-family $\mathcal{C}$ consisting of knice subgroups. If we now take $\mathcal{C}'$ to be an $H(\aleph_0)$-family in $G$ of pure subgroups, then Proposition 3.6(1) and Remark 3.2 imply that $\mathcal{C} \cap \mathcal{C}'$ is an $H(\aleph_0)$-family in $G$ with the desired properties. □

When the $F(\aleph_0)$-family in $G$ consists of pure subgroups with additional special properties, the countable pure subgroups $B_\alpha$ must be selected carefully if there is any hope of applying the closed set method to obtain an $H(\aleph_0)$-family consisting of subgroups with the same special properties. We illustrate this phenomenon in our next proof. Although the following proposition is known in the case of $p$-primary groups (see [3] or [6]), we give a detailed demonstration which will serve as a model for our subsequent proofs. See [4, Theorem 5.1] for the genesis of this technique of applying the closed set method.

**Proposition 4.4.** If $\{G_\alpha\}_{\alpha<\tau}$ is an $F(\aleph_0)$-family in the global group $G$ with each $G_\alpha$ a locally separable pure subgroup, then there is an $H(\aleph_0)$-family in $G$ consisting of locally separable subgroups.

**Proof.** First we define the appropriate countable pure subgroups $B_\alpha$. Let $\alpha < \tau$ and select a countable set $\{x_n\}_{n<\omega}$ of representatives for the cosets of $G_\alpha$ in $G_{\alpha+1}$. Since $G_\alpha$ is locally separable in $G$, we have for each $n < \omega$ a countable subset $\{b_{n,k}\}_{k<\omega} \subseteq G_\alpha$ such that, for each $y \in G_\alpha$ and each prime $p$, there exists a $k < \omega$ such that $|x_n + y|_p \leq |x_n + b_{n,k}|_p$. Then select $B_\alpha$ to be a countable pure subgroup of $G_{\alpha+1}$ that contains all the $x_n$ and all the $b_{n,k}$. Clearly, $G_{\alpha+1} = G_\alpha + B_\alpha$. We
maintain that the following special condition is satisfied: if $x \in G_{\alpha + 1}$, $y \in G_{\alpha}$ and $p$ is a prime, then there exists a $z \in G_{\alpha}$ with $x + z \in B_{\alpha}$ and $|x + y|_p \leq |x + z|_p$. Indeed, for appropriate $n < \omega$ and $y_1 \in G_{\alpha}$, $x + y = x_n + y_1$ and there is a $k < \omega$ with $|x_n + y_1|_p \leq |x_n + b_{n,k}|_p$. Then, for $z = (y - y_1) + b_{n,k}$, $z \in G_{\alpha}$, $x + z = x_n + b_{n,k} \in B_{\alpha}$ and $|x + y|_p \leq |x + z|_p$, as claimed.

Given a closed subset $S$ of $\tau$, it suffices to show that $G(S)$ is a locally separable subgroup of $G$. Thus, for $g \in G \setminus G(S)$, we are required to find a countable set \( \{a_n\}_{n<\omega} \subseteq G(S) \) such that for each $x \in G(S)$ and prime $p$, there is an $n$ such that $|g + x|_p \leq |g + a_n|_p$. We shall establish this by induction on $\nu(g)$, where we may assume that $g$ is selected in the coset $g + G(S)$ with $\mu = \nu(g)$ minimal. Notice then that $\mu \notin S$, for if $\mu$ were in $S$, we would have $x = y + b$ with $y \in G_\mu$ and $b \in B_\mu$. Hence, $y$ would be an element of $g + G(S)$ with $\nu(y) < \nu(g)$, contradicting the choice of $g$.

Since $G_\mu$ is locally separable in $G$, there is a countable set $\{b_n\} \subseteq G_\mu$ satisfying the following condition: for each $x \in G_\mu$ and each prime $p$, there is an $n < \omega$ such that $|g + x|_p \leq |g + b_n|_p$. Notice that $\nu(b_n) < \mu = \nu(g)$ for each $n$ and therefore our implicit induction hypothesis implies that we have countable subsets $\{a_{n,k}\}_{k<\omega} \subseteq G(S)$ such that for each nonzero $x \in G(S)$ and each prime $p$, there exists $k < \omega$ with $|b_n + x|_p \leq |b_n - a_{n,k}|_p$.

We maintain that the countable set $\{a_{n,k}\}_{n<\omega,k<\omega}$ has the desired property that, for each nonzero $x \in G(S)$ and prime $p$, there exists an $(n, k) \in \omega \times \omega$ such that $|g + x|_p \leq |g + a_{n,k}|_p$. The proof of this fact, however, requires a secondary induction on the ordinal $\lambda = \nu(x)$. We shall first deal with the case $\lambda < \mu$ which implies that $x \in G_\mu$. Thus, for each prime $p$, there is an $n < \omega$ such that $|g + x|_p \leq |g + b_n|_p$, from which $|g + x|_p \leq |b_n - x|_p$ by the triangle inequality. The choice of the $a_{n,k}$ yields $|b_n - x|_p \leq |b_n - a_{n,k}|_p$ for some $k$. Therefore, since $g + a_{n,k} = (g + x) - (x - b_n) - (b_n - a_{n,k})$, the triangle inequality implies the desired conclusion $|g + x|_p \leq |g + a_{n,k}|_p$. By Lemma 4.1, $\lambda \in S$ so that the case $\lambda = \mu$ is excluded. It remains to deal with the case $\lambda > \mu$. Under this assumption, $g \in G_\lambda$ and by the choice of the $B_\alpha$, for each prime $p$, there is a $z \in G_\lambda$ such that $x + z \in B_\lambda$ and $|x + g|_p \leq |x + z|_p$. Then, by the triangle inequality, $|g + x|_p \leq |g - z|_p$. Observe that since $\lambda \in S$, $z \in G(S)$ with $\nu(z) < \lambda$ and our induction hypothesis implies that there is an $(n, k) \in \omega \times \omega$ such that $|g + x|_p \leq |g - z|_p \leq |g + a_{n,k}|_p$.

We, of course, need the generalization of the above proposition to $F(\mathbb{N}_0)$-families of almost strongly separable subgroups. Since the intersection of two $H(\mathbb{N}_0)$-families is an $H(\mathbb{N}_0)$-family, we limit the proof that follows to establishing the existence of
an $H(\aleph_0)$-family that satisfies the requisite conditions in Definition 3.1 that relate to height matrices.

**Proposition 4.5.** If $\{G_\alpha\}_{\alpha<\tau}$ is an $F(\aleph_0)$-family in the global group $G$ with each $G_\alpha$ an almost strongly separable pure subgroup of $G$, then there is an $H(\aleph_0)$-family in $G$ of almost strongly separable subgroups.

**Proof.** For each $\alpha < \tau$, select a countable set $\{x_n\}_{n<\omega}$ of representatives of the cosets in $G_{\alpha+1}/G_\alpha$. Since $G_\alpha$ is almost strongly separable in $G$, we have for each $n$ a countable set $\{b_{n,k}\}_{k<\omega} \subseteq G_\alpha$ such that for each $y \in G_\alpha$ there exists a $k$ and a positive integer $m$ with

$$\|m(x_n + y)\| \leq \|m(x_n + b_{n,k})\|.$$

Now take $B_\alpha$ to be a countable pure subgroup of $G_{\alpha+1}$ that contains all $x_n$ and $b_{n,k}$. Clearly then, $G_{\alpha+1} = G_\alpha + B_\alpha$. We maintain that the following special condition is satisfied: if $x \in G_{\alpha+1}$, $y \in G_\alpha$, then there is a $z \in G_\alpha$ with $x + z \in B_\alpha$ and a positive integer $m$ such that $\|m(x + y)\| \leq \|m(x + z)\|$. Indeed, for appropriate $n$ and $y_1 \in G_\alpha$, $x + y = x_n + y_1$ and we select a $k < \omega$ such that

$$\|m(x_n + y_1)\| \leq \|m(x_n + b_{n,k})\|$$

for some positive integer $m$. If we now set $z = (y - y_1) + b_{n,k} \in G_\alpha$, then $x + z = x_n + b_{n,k} \in B_\alpha$ and $\|m(x + y)\| \leq \|m(x + z)\|$, as claimed.

In view of the remarks preceding the statement of our proposition, it suffices to establish the following: if $S$ is a closed subset of $\tau$ and if $g \in G \setminus G(S)$, then there is a countable subset $\{a_n\}_{n<\omega} \subseteq G(S)$ such that for each $x \in G(S)$ there exist an $n < \omega$ and a positive integer $m$ such that $\|m(g + x)\| \leq \|m(g + a_n)\|$. As in the proof of Proposition 4.4, we establish this by induction on $\nu(g)$, where $g$ is selected in the coset $g + G(S)$ with $\mu = \nu(g)$ minimal. Consequently, $\mu \notin S$. Since $G_\mu$ is almost strongly separable in $G$, there is a countable subset $\{a_n\}_{n<\omega} \subseteq G_\mu$ satisfying the following condition: for each $x \in G_\mu$ there is an $n < \omega$ and a positive integer $m$ such that $\|m(g + x)\| \leq \|m(g + b_n)\|$. Since $\nu(b_n) < \mu = \nu(g)$ for all $n$, our implicit induction hypothesis implies that we have countable subsets $\{a_{n,k}\}_{k<\omega} \subseteq G(S)$ such that for each nonzero $x \in G(S)$ there is a $k < \omega$ and a positive integer $m$ such that $\|m(b_{n,k} + x)\| \leq \|m(b_n - a_{k,n})\|$

We claim that the countable set $\{a_{n,k}\}_{n<\omega,k<\omega}$ has the desired property that, for each $x \in G(S)$, there exists $(n, k) \in \omega \times \omega$ and a positive integer $m$ such that $\|m(g+x)\| \leq \|m(g+a_{n,k})\|$. As in the previous proposition, a secondary induction on the ordinal $\lambda = \nu(x)$ is required. Assume first that $\lambda < \mu$ and so $x \in G_\mu$. Then there
is an \( n < \omega \) and a positive integer \( m \) such that \( \|m(g+x)\| \leq \|m(g+b_n)\| \), from which the triangle inequality implies \( \|m(g+x)\| \leq \|m(b_n-x)\| \). The choice of the \( a_{n,k} \) yields a \( k < \omega \) and a positive multiple \( m_1 \) of \( m \) such that \( \|m_1(b_n-x)\| \leq \|m_1(b_n-a_{n,k})\| \). Therefore, since

\[
g + a_{n,k} = (g + x) + (b_n - x) - (b_n - a_{n,k}),
\]

the triangle inequality yields the desired conclusion \( \|m_1(g+x)\| \leq \|m_1(g+a_{n,k})\| \). Since \( \lambda \in S \) by Lemma 4.1, the case \( \lambda = \mu \) is once again excluded. Assuming that \( \lambda > \mu \), \( g \in G_\lambda \) and, by the choice of the \( B_\alpha \), there is a \( z \in G_\lambda \) such that \( x + z \in B_\lambda \) and \( \|m(x+g)\| \leq \|m(x+z)\| \) for some positive integer \( m \). Then, by the triangle inequality, \( \|m(g+x)\| \leq \|m(g-z)\| \). Since \( \lambda \in S \) and \( z \in G(S) \) with \( \nu(z) < \lambda \), our induction hypothesis implies that there exists an \( (n,k) \in \omega \times \omega \) and a positive multiple \( m_1 \) of \( m \) such that \( \|m_1(g+x)\| \leq \|m_1(g+z)\| \leq \|m_1(g+a_{n,k})\| \).  

In view of Corollary 3.11 and the preceding proposition, we may now conclude that if the isotype knie subgroup \( H \) of the global Warfield group \( G \) is itself a Warfield group, then there is an \( H(\aleph_0) \)-family in \( G/H \) consisting of almost strongly separable subgroups. Most of the remainder of this section is devoted to proving, under these hypotheses, that \( G/H \) also has an \( H(\aleph_0) \)-family consisting of \( k \)-subgroups. As Theorem 2.4 will be indispensable in the endeavor, the reader will appreciate the significance of the following specialized result.

**Lemma 4.6.** If \( \{G_\alpha\}_{\alpha < \tau} \) is an \( F(\aleph_0) \)-family in the global group \( G \) with each \( G_\alpha \) an almost strongly separable pure subgroup, and if \( H \) is a subgroup of \( G \) such that \( H \parallel G_\alpha \) for all \( \alpha < \tau \), then there is an \( H(\aleph_0) \)-family \( \mathcal{C} \) in \( G \) such that \( H \parallel N \) for all \( N \in \mathcal{C} \).

**Proof.** Given the example of the preceding proof, we shall limit ourselves to establishing local compatibility. Thus, we choose the \( B_\alpha \) as in the proof of Proposition 4.4 and show that \( H \) and \( G(S) \) are locally compatible whenever \( S \) is a closed subset of \( \tau \).

For \( h \in H \), \( x \in G(S) \) and \( p \) a prime, we are required to find an element \( y \in H \cap G(S) \) such that \( |h + x|_p \leq |h + y|_p \). First, we may assume that \( h \) is selected in the coset \( h + G(S) \) with \( \mu = \nu(h) \) minimal. As in the proof of Proposition 4.4, \( \mu \notin S \); in particular, \( \lambda = \nu(x) \neq \mu \).

If \( \lambda < \mu \), then \( x \in G_\mu \) and we proceed by induction on \( \mu \). Since \( G_\mu \) is locally separable in \( G \), there is a countable subset \( \{b_n\}_{n < \omega} \subseteq G_\mu \) satisfying the following condition: for each prime \( p \), there is an \( n < \omega \) such that \( |h + x|_p \leq |h + b_n|_p \). But each \( b_n \) is in \( G_\mu \), and since \( H \) and \( G_\mu \) are locally compatible in \( G \), there is a countable set
\{y_n\}_{n<\omega} \subseteq H \cap G_\mu \text{ such that for each prime } p, \text{ there is there is an } n < \omega \text{ such that } |h+b_n|_p \leq |h+y_n|_p. \text{ Thus, } |h+x|_p \leq |h+y_n|_p \text{ and consequently } |h+x|_p \leq |y_n-x|_p. \text{ But, for all } n, y_n \in H \text{ with } \nu(y_n) < \mu \text{ and we have, by the induction hypothesis, an element } y \in H \cap G(S) \text{ with } |y_n-x|_p \leq |y_n-y|_p. \text{ Since then }

\begin{align*}
h + y &= (h + x) + (y_n - x) + (y - y_n),
\end{align*}

the triangle inequality yields the desired relation \(|h+x|_p \leq |h+y|_p\) with \(y \in H \cap G(S)\).

If \(\lambda > \mu\), then \(h \in G_\lambda\) and we proceed by induction on \(\lambda\). As in the proof of Proposition 4.4, for each prime \(p\) there is a \(z \in G_\lambda\) such that \(|x+h|_p \leq |x+z|_p\) and \(x+z \in B_\lambda\). By the triangle inequality, \(|h+x|_p \leq |h-z|_p\) where \(\nu(z) < \lambda\) and \(z \in G(S)\) since \(\lambda \in S\). Finally, by induction, for each prime \(p\) there is a \(y \in H \cap G(S)\) with \(|h+x|_p \leq |h-z|_p \leq |h+y|_p\).

\textbf{Lemma 4.7.} Suppose that \(\{G_\alpha\}_{\alpha<\tau}\) is an \(F(\aleph_0)\)-family in the global group \(G\) consisting of pure subgroups and that \(\{B_\alpha\}_{\alpha<\tau}\) is an associated family of countable pure subgroups with \(G_{\alpha+1} = G_\alpha + B_\alpha\) for all \(\alpha < \tau\). If \(H\) is a subgroup of \(G\) such that \(H \cap G_{\alpha+1} = (H \cap G_\alpha) + (H \cap B_\alpha)\) for all \(\alpha < \tau\) and if \(S\) is a closed subset of \(\tau\), then \(H \cap G(S) = \sum_{\alpha \in S} (H \cap B_\alpha)\).

\textbf{Proof.} Since the reverse inclusion is obvious, it suffices to show that \(H \cap G(S) \subseteq \sum_{\alpha \in S} (H \cap B_\alpha)\). The proof, of course, is by induction on \(\nu(x)\) where \(x \in H \cap G(S)\). Assuming without loss that \(x \neq 0\), choose a standard representation \(x = b_{\mu(1)} + b_{\mu(2)} + \ldots + b_{\mu(m)}\) where, by Lemma 4.1, \(\nu(x) = \mu(m) \in S\). Notice that \(x \in H \cap G_{\mu(m)+1} = (H \cap G_{\mu(m)}) + (H \cap B_{\mu(m)})\). Consequently, we may write \(x = y + b\) where \(y \in H \cap G_{\mu(m)}\) and \(b \in H \cap B_{\mu(m)}\). Thus

\begin{align*}
b_{\mu(1)} + \ldots + b_{\mu(m-1)} - y &= b - b_{\mu(m)} \in B_{\mu(m)}.
\end{align*}

Because \(\mu(m) \in S\), \(b_{\mu(1)} + \ldots + b_{\mu(m-1)} \in G(S)\) and we conclude that \(y \in H \cap G(S)\). Moreover, \(y \in G_{\mu(m)}\) implies that \(\nu(y) < \mu(m) = \nu(x)\). Therefore, by induction, \(y \in \sum_{\alpha \in S} (H \cap B_\alpha)\). Finally, since \(b \in H \cap B_{\mu(m)}\), and since once again \(\mu(m) \in S\), it follows that \(x = y + b \in \sum_{\alpha \in S} (H \cap B_\alpha)\), as desired \(\square\)

In order to apply Lemma 4.7, we require the following technical result.
Lemma 4.8. Suppose that $H$ is a subgroup of the global group $G$ and that $\mathcal{C}_H$ and $\mathcal{C}_G$ are $G(\aleph_0)$-families in $H$ and $G$, respectively. If $C \in \mathcal{C}_G$ with $C \cap H \in \mathcal{C}_H$ and if $X$ is a countable subset of $G$, then there is a countable subgroup $B \in \mathcal{C}_G$ such that $X \subseteq B$, $B \cap H \in \mathcal{C}_H$, $C + B \in \mathcal{C}_G$, and $(C + B) \cap H = (C \cap H) + (B \cap H)$. □

Proof. First we observe that any $G(\aleph_0)$-family $\mathcal{C}$ in a global group $G$ satisfies the following stronger version of (H3): if $C \in \mathcal{C}$ and if $X$ is a countable subset of $G$, then there is a countable $B \in \mathcal{C}$ such that $X \subseteq B$ and $C + B \in \mathcal{C}$. Indeed, by repeated applications of (H3), there are two ascending families $\{C_n\}_{n<\omega} \subseteq \mathcal{C}$ and $\{B_n\}_{n<\omega} \subseteq \mathcal{C}$ such that $X \subseteq B_0$, each $B_n$ is countable and, for all $n < \omega$, the two conditions

\begin{itemize}
  \item[(a)] $C + B_n \subseteq C_n$ with $C_n/C$ countable, and
  \item[(b)] $C_n \subseteq C + B_{n+1}$
\end{itemize}

are satisfied. Then $B = \bigcup_{n<\omega} B_n$ is a countable member of $\mathcal{C}$ with $C + B = \bigcup_{n<\omega} C_n \in \mathcal{C}$, as desired.

We now construct inductively two sequences $\{B_n\}_{n<\omega}$ and $\{A_n\}_{n<\omega}$ where, for each $n < \omega$, the following conditions hold:

\begin{itemize}
  \item[(i)] $B_n \in \mathcal{C}_G$ and $A_n \in \mathcal{C}_H$ with $B_n$ and $A_n$ countable.
  \item[(ii)] $(C + B_n) \cap H \subseteq (C \cap H) + A_n$ and $B_n \cap H \subseteq A_n$.
  \item[(iii)] $C + B_n \in \mathcal{C}_G$.
  \item[(iv)] $B_n + A_n \subseteq B_{n+1}$.
\end{itemize}

By the above observation, we may begin the induction with a countable $B_0 \in \mathcal{C}_G$ such that $X \subseteq B_0$ and $C + B_0 \in \mathcal{C}_G$. Assuming that we have, for some $n < \omega$, a countable $B_n \in \mathcal{C}_G$ satisfying condition (iii), we demonstrate how to construct suitable $A_n$ and $B_{n+1}$.

Since the countability of $B_n$ implies that $((C + B_n) \cap H)/(C \cap H)$ is countable, we can certainly select a countable $A_n \in \mathcal{C}_H$ satisfying the first part of condition (ii). But then, if necessary, $A_n$ can be enlarged within $\mathcal{C}_H$ to ensure that $B_n \cap H \subseteq A_n$. We complete the induction by applying our previous observation to select a countable subgroup $B_{n+1} \in \mathcal{C}_G$ that contains $A_n + B_n$ and has the property that $C + B_{n+1} \in \mathcal{C}_G$. Finally, take $B = \bigcup_{n<\omega} B_n$ and observe that $B$ is countable with $X \subseteq B \in \mathcal{C}_G$, $B \cap H = \bigcup_{n<\omega} A_n \in \mathcal{C}_H$ and

\[ C + B = C + \bigcup_{n<\omega} B_n = \bigcup_{n<\omega} (C + B_n) \in \mathcal{C}_G. \]

Moreover, $(C + B) \cap H = (C \cap H) + (B \cap H)$. □

We now have all the necessary ingredients to prove our main result.
Theorem 4.9 (Main Theorem). Suppose that $H$ is an isotype knice subgroup of a global Warfield group $G$. Then, $H$ itself is a global Warfield group if and only if there is an $H(\aleph_0)$-family $\mathcal{D}$ in $G/H$ consisting of almost strongly separable $k$-subgroups.

Proof. Since an $H(\aleph_0)$-family in $G/H$ is a $G(\aleph_0)$-family, Theorem 3.9 implies that if $G/H$ has an $H(\aleph_0)$-family of almost strongly separable $k$-subgroups, then $H$ is a Warfield group.

Conversely, suppose that $H$ is a Warfield group. As in the proof of Theorem 3.10, we apply [12, Theorem 5.2] to obtain a $G(\aleph_0)$-family $\mathcal{C}_G$ in $G$ with the property that $(H + N)/N$ is almost strongly separable in $G/N$ for all $N \in \mathcal{C}_G$. Moreover, from Corollary 4.3 we conclude that $\mathcal{C}_G$ may be chosen so that each $N \in \mathcal{C}_G$ is also a pure knice subgroup of $G$ that is almost strongly separable in $G$. In particular, $\mathcal{C} = \{ N \in \mathcal{C}_G : H \parallel N \}$ is a $G(\aleph_0)$-family in $G$ by Proposition 3.8. Furthermore, by Corollary 4.3, there exists an $H(\aleph_0)$-family $\mathcal{C}_H$ in $H$ consisting of pure knice subgroups of $H$.

We shall first establish the existence of an $F(\aleph_0)$-family $\{G_\alpha\}_{\alpha<\tau}$ of pure subgroups in $G$ with an associated family $\{B_\alpha\}_{\alpha<\tau}$ of countable pure subgroups where the following are satisfied for all $\alpha < \tau$:

1. $G_\alpha \in \mathcal{C}$.
2. $H \cap G_\alpha \in \mathcal{C}_H$.
3. $G_{\alpha+1} = G_\alpha + B_\alpha$ with $B_\alpha \in \mathcal{C}$.
4. $H \cap G_{\alpha+1} = (H \cap G_\alpha) + (H \cap B_\alpha)$ with $H \cap B_\alpha \in \mathcal{C}_H$.
5. $H \parallel G_\alpha$.

Notice that (5) is a consequence of (1). We begin with a well-ordering $\{x_\beta\}_{\beta<\tau}$ of the elements of $G$. Proceeding by induction, assume that for some $\mu < \tau$ we have constructed $\{G_\alpha\}_{\alpha<\mu} \subseteq \mathcal{C}$ together with the corresponding family $\{B_\alpha\}_{\alpha+1<\mu} \subseteq \mathcal{C}$ such that conditions (1)–(4) are satisfied and $\{x_\beta\}_{\beta<\alpha} \subseteq G_\alpha$ whenever $\alpha < \mu$. It suffices to construct $G_\mu \in \mathcal{C}$ such that the enlarged family $\{G_\alpha\}_{\alpha<\mu}$ continues to satisfy all the requisite conditions. If $\mu$ is a limit ordinal, it suffices to take $G_\mu = \bigcup_{\alpha<\mu} G_\alpha \in \mathcal{C}$ and note that $H \cap G_\mu = \bigcup_{\alpha<\mu} (H \cap G_\alpha) \in \mathcal{C}_H$ with the remaining conditions (3) and (4) vacuously satisfied. On the other hand, if $\mu = \beta + 1$, then apply Lemma 4.8 to the the singleton $X = \{x_\beta\}$ with $C = G_\beta$ to obtain $B_\beta \in \mathcal{C}$ so that $G_{\beta+1} = G_\beta + B_\beta$ possesses all the desired properties.

By the remark preceding the statement of Lemma 4.6, there is already an $H(\aleph_0)$-family in $G/H$ consisting of almost strongly separable subgroups. Thus, since the intersection of two $H(\aleph_0)$-families is an $H(\aleph_0)$-family, it suffices now to exhibit an $H(\aleph_0)$-family $\mathcal{D}$ in $G/H$ consisting of $k$-subgroups of $G/H$. Using the $G_\alpha$ and $B_\alpha$.
constructed in the previous paragraph, set

\[ \mathcal{C}_1 = \{ G(S) : S \text{ is a closed subset of } \tau \}. \]

We know that \( \mathcal{C}_1 \) is an \( H(\aleph_0) \)-family in \( G \). By condition (4) and Lemma 4.7, for each closed set \( S \) in \( \tau \), \( H \cap G(S) = \sum_{\alpha \in S} (H \cap B_\alpha) \). Moreover, by the second part of condition (4), each \( H \cap B_\alpha \) is in the \( H(\aleph_0) \)-family \( \mathcal{C}_H \). Thus, \( H \cap G(S) \in \mathcal{C}_H \), and in particular, \( H \cap G(S) \) is a pure knice subgroup of \( H \). It now follows from Theorem 2.1 that \( H \cap G(S) \) is a knice subgroup of \( G \). Next, by conditions (1) and (5), each \( G_\alpha \) is an almost strongly separable pure subgroup of \( G \) such that \( H \parallel G_\alpha \). Hence, from Lemma 4.6, there is an \( H(\aleph_0) \)-family \( \mathcal{C}_2 \) in \( G \) such that \( H \parallel N \) for all \( N \in \mathcal{C}_2 \).

To complete the proof, set \( \mathcal{C}_0 = \mathcal{C}_1 \cap \mathcal{C}_2 \) and \( \mathcal{D} = \{ (N + H)/H : N \in \mathcal{C}_0 \} \). Since \( \mathcal{C}_0 \) is an \( H(\aleph_0) \)-family in \( G \), it follows easily that \( \mathcal{D} \) is an \( H(\aleph_0) \)-family in \( G/H \) (for example, see [3, Lemma 1.3(a)]). Further observe that each \( N \in \mathcal{C}_0 \) is a pure knice subgroup of \( G \) with \( H \parallel N \) and \( H \cap N \) knice in \( G \). Therefore, by Theorem 2.4, \( \mathcal{D} \) consists of \( k \)-subgroups of \( G/H \). □

From results in [8] and [13], it follows that every global \( k \)-group \( K \) has a **sequentially pure projective resolution**; that is, an exact sequence

\[
\ldots \longrightarrow A_{n+1} \xrightarrow{\varphi_{n+1}} A_n \longrightarrow \ldots \longrightarrow A_1 \xrightarrow{\varphi_1} A_0 \xrightarrow{\varphi_0} K \longrightarrow 0
\]

where, for each \( n < \omega \), \( A_n \) is a global Warfield group and \( \varphi_{n+1}(A_{n+1}) \) is a \( k \)-group that is an isotype knice subgroup of \( A_n \). If there exists a nonnegative integer \( k \) such that the resolution (†) can be chosen with \( A_n = 0 \) for all \( n > k \), the smallest such \( k \) is called the **sequentially pure projective dimension** of \( K \) and we write \( \dim K = k \). If no such \( k \) exists, we set \( \dim K = \infty \). (By an obvious version of Schanuel’s Lemma, \( \dim K \) is well defined for each \( k \)-group \( K \).) Thus, \( K \) is a global Warfield group if and only if \( \dim K = 0 \), while \( \dim K \leq 1 \) if and only if there is a short exact sequence

\[
0 \longrightarrow A_1 \xrightarrow{\varphi_1} A_0 \xrightarrow{\varphi_0} K \longrightarrow 0
\]

with \( \varphi_1(A_1) \) an isotype knice Warfield subgroup of the global Warfield group \( A_0 \). Therefore, Theorem 4.9 can be reformulated as follows.

**Corollary 4.10.** If \( K \) is a global \( k \)-group, then \( \dim K \leq 1 \) if and only if there is an \( H(\aleph_0) \)-family in \( K \) consisting of almost strongly separable \( k \)-subgroups.

In a forthcoming paper, we intend to apply the results and techniques of this paper to the study of global \( k \)-groups \( K \) with \( \dim K > 1 \).
References


Author’s address: C. Megibben, Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37240, e-mail: charles.k.megibben@math.vanderbilt.edu; W. Ullery, Department of Mathematics, Auburn University, Auburn, Alabama 36849, e-mail: ullery@math.auburn.edu.