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ISOTYPE KNICE SUBGROUPS OF GLOBAL WARFIELD GROUPS

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Abstract. If H is an isotype knice subgroup of a global Warfield group G, we introduce the notion of a k-subgroup to obtain various necessary and sufficient conditions on the quotient group G/H in order for H itself to be a global Warfield group. Our main theorem is that H is a global Warfield group if and only if G/H possesses an $H(\aleph_0)$-family of almost strongly separable k-subgroups. By an $H(\aleph_0)$-family we mean an Axiom 3 family in the strong sense of P. Hill. As a corollary to the main theorem, we are able to characterize those global k-groups of sequentially pure projective dimension $\leqslant 1$.

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1. Introduction

An abelian group is said to be *simply presented* if it has a presentation in which each relation involves at most two generators and, by definition, a *Warfield group* is a mixed abelian group that is isomorphic to a direct summand of a simply presented group. One important property of a Warfield group G is the existence of a *decomposition basis* $X = \{x_i\}_{i\in I}$, an independent subset of G in which each x_i has infinite order, $G/\langle X\rangle$ is torsion and $\langle X\rangle = \bigoplus_{i\in I}\langle x_i\rangle$ is a *valuated coproduct* in the following sense: if p is a prime, then for every finite subset J of I and all collections $\{n_j\}_{j\in J}$ of integers,

$$\Bigl|\sum_{j\in J} n_j x_j\Bigr|_p = \min_{j\in J}\{|n_j x_j|_p\}.$$

Here $|x|_p$ denotes the p-height of x (as computed in G). Warfield groups also have an infinite combinatorial characterization in terms of *knice subgroups* that parallels

P. Hill's well-known Axiom 3 description of simply presented torsion groups in terms of nice subgroups. The precise definition of a knice subgroup depends on the auxiliary notions of *primitive element* and $*$-*valuated coproduct*, which have evolved from an analysis of properties enjoyed by members of a decomposition basis. The details may be found in [7] and will not be reviewed here. The general facts we cite from [7], [8], [11] and [12] will be adequate for our purposes. Suffice it to say that the exact definitions of primitive element and $*$-valuated coproduct in a global group G can be formulated in terms of abstract height matrices and various associated fully invariant subgroups of G. We use the term *global group* for an arbitrary mixed abelian group to distinguish the groups we study from p-local groups, the simpler setting in which Warfield groups were first introduced.

Let $\mathbb{P}$ denote the set of rational primes and write $\mathscr{O}_\infty$ for the class of ordinals with the symbol ∞ adjoined as a maximal element. Given an element x in a global group G, we associate its *height matrix* $\|x\|$, a doubly infinite $\mathbb{P} \times \omega$ matrix having $|p^i x|_p$ as its (p, i) entry. The ordered class $\mathscr{O}_\infty$ induces in a pointwise manner lattice relations $\leqslant$ and $\wedge$ on the height matrices $\|x\|$ of elements of G. We shall have several occasions to employ the familiar *triangle inequality*: $\|x + y\| \geqslant \|x\| \wedge \|y\|$ for all $x, y \in G$; certainly $|x + y|_p \geqslant |x|_p \wedge |y|_p$ for all primes p. When necessary to avoid confusion, we affix superscripts to indicate the group in which height matrices and heights are computed. For example, if H is a subgroup of G, $\|x + H\|^{G/H}$ indicates the height matrix of the coset $x + H$ as computed in G/H, while if $x \in H$, $|x|_p^H$ denotes the p-height of x as computed in H. For any other unexplained notation or terminology, see the standard reference [2].

Following [7] and [8], we call a subgroup N of a global group G a *nice subgroup* provided that for each prime p and ordinal α, the cokernel of the canonical map

$$(p^\alpha G + N)/N \rightarrowtail p^\alpha (G/N)$$

contains no element of order p.

Definition 1.1. A subgroup N of the global group G is said to be a *knice* subgroup provided the following two conditions are satisfied:

(1) N is a nice subgroup of G.
(2) If S is a finite subset of G, then there exists a finite (possibly vacuous) collection of primitive elements $x_1, x_2, \ldots, x_m$ in G such that $N' = N \oplus \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \ldots \oplus \langle x_m \rangle$ is a $*$-valuated coproduct and $\langle S, N' \rangle / N'$ is finite.

We conclude this section with several frequently useful facts concerning knice subgroups.

Proposition 1.2. *The following hold for any global group G.*

(1) *If A is a knice subgroup of G and if B is a subgroup with B/A finite, then B is a knice subgroup of G.*

(2) *If $B = A \oplus \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \ldots \oplus \langle x_m \rangle$ is a $*$-valuated coproduct in G where A is a knice subgroup and $x_1, x_2, \ldots, x_m$ are primitive elements, then B is a knice subgroup of G.*

(3) *If A is a knice subgroup of G and if B/A is a knice subgroup of G/A, then B is a knice subgroup of G.*

(4) *If N is both knice and pure in G, then $p^\alpha(G/N) = (p^\alpha G + N)/N$ for all primes p and ordinals α.*

(5) *If N is a pure knice subgroup of G and A is an arbitrary subgroup of N, then N/A is a pure knice subgroup of G/A*

Parts (1), (2), and (3) in the preceding proposition are, respectively, Theorem 3.2, 3.3, and 3.7 in [7]; while (4) is Corollary 1.10 from [8] and (5) is [12, Corollary 2.3], whose proof uses both (4) and the characterization of knice subgroups given in Proposition 1.3 below.

A global group G is said to be a *k-group* if the trivial subgroup 0 is a knice subgroup. Thus, if S is a finite subset of the k-group G, then S is finite modulo a $*$-valuated coproduct $\langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \ldots \oplus \langle x_m \rangle$ where $x_1, x_2, \ldots, x_m$ are primitive elements in G. Direct summands of k-groups are themselves k-groups [8, Theorem 2.6] and, more generally, an isotype knice subgroup of a k-group is also a k-group [8, Theorem 2.8]. Recall that a subgroup H of a global group G is said to be *isotype* if $p^\alpha G \cap H = p^\alpha H$ for all primes p and ordinals α. Furthermore, G/N is a k-group whenever N is a knice subgroup of the global group G. In fact, we have the following important characterization of knice subgroups.

Proposition 1.3 ([8, Proposition 1.7])**.** *A subgroup N of the global group G is a knice subgroup if and only if the following three conditions are satisfied.*

(a) *N is a nice subgroup of G.*
(b) *To each $g \in G$ there corresponds a positive integer n such that the coset $ng + N$ contains an element x with $\|x\|^G = \|ng + N\|^{G/N}$.*
(c) *G/N is a k-group.*

Corollary 1.4 ([8, Remark 1.8])**.** *Let N be a subgroup of the global group G satisfying conditions* (a) *and* (b) *of Proposition* 1.3*. If H is a subgroup of G with $N \subseteq H$ and if $H \oplus \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \ldots \oplus \langle x_m \rangle$ is a $*$-valuated coproduct in G where $x_1, x_2, \ldots, x_m$ are primitive, then $H/N \oplus \langle x_1 + N \rangle \oplus \langle x_2 + N \rangle \oplus \ldots \oplus \langle x_m + N \rangle$ is a $*$-valuated coproduct in G/N and the $x_i + N$ are primitive in G/N.*

As noted in [8, Proposition 1.9], isotype knice subgroups possess quite special properties, and in particular, a stronger version of condition (b) of Proposition 1.3.

Proposition 1.5 ([8]). *If the subgroup H is both knice and isotype in the global group G, then for each $g \in G$ the coset $g + H$ contains a representative x such that $\|x\|^G = \|g + H\|^{G/H}$.*

2. Preliminary theorems

We propose, in this paper, to give necessary and sufficient conditions for an isotype knice subgroup H of a global Warfield group G to be itself a Warfield group. In fact, we shall establish such conditions in terms of an Axiom 3 characterization of the quotient group G/H. As a result, our Main Theorem (Theorem 4.9) will have roughly the same form as the characterization of the simply presented isotype subgroups of totally projective p-groups established in [6]. The principal result in that paper specializes to show that an isotype nice subgroup H of a totally projective p-group G is itself totally projective if and only if G/H has an Axiom 3 system of "separable" subgroups. But of course, in the global setting, the separability condition will need to be suitably modified; and moreover, each subgroup in the Axiom 3 system for G/H will be required to be a k-subgroup, a new type of subgroup whose definition appears below. We should also mention that in the case of p-local Warfield groups in [10] and of global Warfield groups in [12], the isotype subgroups that are themselves Warfield were characterized. However, those results have the unpleasant feature that one must consider properties of the quotients $(H + N)/N$ in G/N as N ranges over an entire Axiom 3 system of knice subgroups for G. In this section, we prove two theorems that will play pivotal roles in the sequel.

Theorem 2.1. *Let H be an isotype knice subgroup of the global group G and suppose that A is a pure subgroup of H. Then A is knice in G if and only if A is knice in H.*

Proof. That A is knice in H when it is knice G follows from [8, Lemma 4.2]. Conversely, assume that A is a knice subgroup of H. To establish that A is knice in G, we first show that it is a nice subgroup of G. Given a prime p and an arbitrary element $x \in G$, we apply part (4) of Proposition 1.2 to select elements $h_1 \in H$ and $a_1 \in A$ such that

$$|x + h_1|_p^G = |x + H|_p^{G/H} \text{ and } |h_1 - a_1|_p^H = |h_1 + A|_p^{H/A}.$$

It then follows that

$$(*)\qquad |x+h|_p^G = |x+h_1|_p^G \wedge |h-h_1|_p^G = |x+h_1|_p^G \wedge |h-h_1|_p^H$$

for all $h \in H$. Indeed if for some $h \in H$ we had $|x+h|_p^G > |h-h_1|_p^G$, it would follow that

$$|x+h_1|_p^G = |(x+h)+(h_1-h)|_p^G = |h-h_1|_p^G < |x+h|_p^G,$$

contrary to the choice of h_1. Moreover, from $(*)$ and our choice of a_1, we get $|x+a|_p^G \leqslant |x+a_1|_p^G$ for all $a \in A$ since

$$|x+a|_p^G = |x+h_1|_p^G \wedge |a-h_1|_p^H \leqslant |x+h_1|_p^G \wedge |a_1-h_1|_p^G \leqslant |x+a_1|_p^G.$$

That A is nice in G now follows from [2, Lemma 79.2].

In order to show that A satisfies part (2) of Definiion 1.1, let S be an arbitrary finite subset of G. Since H is a knice subgroup of G, S is finite modulo a $*$-valuated coproduct $H \oplus \langle y_1 \rangle \oplus \ldots \oplus \langle y_m \rangle$ where $y_1, \ldots, y_m$ are primitive elements of G. But then S is finite modulo a subgroup $\langle T, y_1, \ldots, y_m \rangle$ where T is a finite subset of H. Since A is assumed to be a knice subgroup of H, T is finite modulo a $*$-valuated coproduct $A \oplus \langle x_1 \rangle \oplus \ldots \oplus \langle x_n \rangle$ where $x_1, \ldots, x_n$ are primitive elements of H. Because H is both isotype and knice in G, it follows from Corollary 2.6 and Proposition 2.8 of [11], and the remark at the beginning of the proof of [11, Proposition 2.9], that $x_1, \ldots, x_n$ are primitive in G and

$$A' = A \oplus \langle x_1 \rangle \oplus \ldots \oplus \langle x_n \rangle \oplus \langle y_1 \rangle \oplus \ldots \oplus \langle y_m \rangle$$

is a $*$-valuated coproduct in G. Moreover, $\langle S, A' \rangle / A'$ is finite as desired. □

Applications of Axiom 3 characterizations of groups inevitably involve an interplay between appropriately formulated notions of *separability* and *compatibility*. We begin by recalling the notions of compatibility used by us in [12]. First, if p is a prime, we say that the subgroups H and N of G are *p-compatible* in G if for each $h \in H$ and $x \in N$ there corresponds an $x' \in H \cap N$ such that $|h+x|_p \leqslant |h+x'|_p$. (Heights and height matrices unadorned by superscripts are of course understood to be computed in the largest containing group G.) If H and N are p-compatible for all primes p, they are said to be *locally compatible*.

Definition 2.2. The subgroups H and N of a global group G are said to be *almost strongly compatible* in G provided that they are locally compatible and, for each $h \in H$ and $x \in N$, there corresponds an $x' \in H \cap N$ and a positive integer m such that $\|m(h+x)\| \leqslant \|mh+x'\|$.

Note that p-compatibility, local compatibility and almost strong compatibility are all symmetric relations. For example, $\|m(h+x)\| \leqslant \|mh+x'\|$ implies that $\|m(h+x)\| \leqslant \|mx-x'\|$. When there is no danger of confusion with previous uses of the notation (see [6] or [3]), we shall occasionally write $H \parallel N$ to indicate that H and N are almost strongly compatible. For later reference we remark that all of these compatibility relations are inductive. So, in particular, if $\{N_i\}_{i\in I}$ is a chain of subgroups of G with $H \parallel N_i$ for all $i \in I$, then $H \parallel N$ where $N = \bigcup\limits_{i\in I} N_i$.

Definition 2.3. Let N be a subgroup of a global group G. We say that N is a *k-subgroup* of G if to each finite subset S of N there corresponds a $*$-valuated coproduct

$$M = \langle y_1\rangle \oplus \langle y_2\rangle \oplus \ldots \oplus \langle y_s\rangle$$

in G with the properties that $\{y_1, y_2, \ldots, y_s\}$ is a (possibly empty) set of primitive elements of G, $M \subseteq N$ and $\langle S, M\rangle/M$ is finite.

With this new terminology, [12, Theorem 2.1] shows that any knice subgroup of a k-group is a k-subgroup.

Theorem 2.4. *Let H and N be almost strongly compatible knice subgroups of the global group G, with H isotype in G and N pure in G. Then, $H \cap N$ is a knice subgroup of G if and only if $(N+H)/H$ is a k-subgroup of G/H.*

Proof. In view of Proposition 1.3, $H \cap N$ is a knice subgroup of G if and only if the following hold:

(a) $H \cap N$ is a nice subgroup of G.
(b) For each $g \in G$ there exist a positive integer n and an $h' \in H \cap N$ such that

$$\|ng + (H\cap N)\|^{G/(H\cap N)} = \|ng + h'\|^{G}.$$

(c) The quotient group $G/(H\cap N)$ is a k-group.

It is condition (c) that hinges on the structure of $(N+H)/H$ and we shall first show that the general hypotheses imply that (a) and (b) are satisfied. To show that $H \cap N$ is a nice subgroup of G, suppose that $g + (H\cap N) \in p^{\alpha}(G/(H\cap N))$ and $pg \in p^{\alpha}G + (H\cap N)$ for some prime p and ordinal α. Since both H and N are nice subgroups of G, we can write $g = z_1 + h = z_2 + x$ where $z_1, z_2 \in p^{\alpha}G$, $h \in H$ and $x \in N$. Then, $h - x = z_2 - z_1 \in p^{\alpha}G$ and, by p-compatibility, we have an $h' \in H\cap N$ such that $\alpha \leqslant |h-x|_p \leqslant |h-h'|_p$. Thus, $g = (z_1 + (h-h')) + h' \in p^{\alpha}G + (H\cap N)$ and we conclude that $H \cap N$ is a nice subgroup of G.

Next we show that condition (b) is satisfied. Given $g \in G$, we first utilize the fact that N is a knice subgroup of G to choose a positive integer k such that $\|kg +$

$N\|^{G/N} = \|kg + x\|$ for some $x \in N$. Then, since H is isotype and knice, it follows from Proposition 1.5 that there is an $h \in H$ with $\|kg + H\|^{G/H} = \|kg + h\|$. Now, since $N \parallel H$, there is a positive integer m and an $h' \in H \cap N$ such that

$$(1) \qquad \|m(x - h)\| \leqslant \|mx - h'\| \wedge \|mh - h'\|.$$

We shall show that

$$\|kmg + h'\| = \|kmg + (H \cap N)\|^{G/(H \cap N)}$$

and thereby complete the proof that (b) is satisfied. Since

$$\begin{aligned} \|kmg + mx\| \wedge \|kmg + mh\| &= \|kmg + N\|^{G/N} \wedge \|kmg + H\|^{G/H} \\ &\geqslant \|kmg + (H \cap N)\|^{G/(H \cap N)}, \end{aligned}$$

it suffices to verify that

$$(2) \qquad \|kmg + h'\| \geqslant \|kmg + mx\| \wedge \|kmg + mh\|.$$

By three applications of the triangle inequality,

$$(3) \qquad \|m(x - h)\| \geqslant \|kmg + mx\| \wedge \|kmg + mh\|$$

and

$$\|kmg + h'\| \geqslant (\|kmg + mx\| \wedge \|mx - h'\|) \wedge (\|kmg + mh\| \wedge \|mh - h'\|).$$

Finally, applying (1) and (3) to calculate the last term, we obtain (2).

To complete the proof, we require the following observation: if $x \in N$, there is a positive integer m such that

$$(4) \qquad \|mx + H\|^{G/H} = \|mx + (H \cap N)\|^{G/(H \cap N)}.$$

To see this, first apply Proposition 1.5 to select an $h \in H$ such that $\|x + H\|^{G/H} = \|x + h\|$. Then, applying $H \parallel N$, there is a positive integer m and an $h' \in H \cap N$ such that $\|m(x + h)\| \leqslant \|mx + h'\|$. Since

$$\|mx + h'\| \leqslant \|mx + (H \cap N)\|^{G/(H \cap N)} \leqslant \|mx + H\|^{G/H} = \|m(x + h)\| \leqslant \|mx + h'\|,$$

(4) follows.

If $H \cap N$ is a knice subgroup of G, then $G/(H \cap N)$ is a k-group by Proposition 1.3. Moreover, $N/(H \cap N)$ is knice in $G/(H \cap N)$ by Proposition 1.2(5). Therefore, as noted above, the conclusion that $N/(H \cap N)$ is a k-subgroup of $G/(H \cap N)$ follows from [12, Theorem 2.1]. Since the definitions of primitive elements and $*$-valuated coproducts depend solely on the computation of height matrices, observation (4) implies that $(N+H)/H$ is a k-subgroup of G/H. Conversely, assume that $(N+H)/H$ is a k-subgroup of G/H. Then to complete the proof that $H \cap N$ is a knice subgroup of G, it remains to show that (c) $G/(H \cap N)$ is a k-group. Towards this end, consider an arbitrary finite subset $\overline{S}$ of $G/(H \cap N)$ and let S be a corresponding finite subset of G that projects onto $\overline{S}$. As N is a knice subgroup of G, there exist primitive elements $x_1, x_2, \ldots, x_m$ in G such that S is finite modulo the $*$-valuated coproduct

$$N \oplus \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \ldots \oplus \langle x_m \rangle.$$

Then, by Corollary 1.4,

$$N/(H \cap N) \oplus \langle x_1 + (H \cap N) \rangle \oplus \langle x_2 + (H \cap N) \rangle \oplus \ldots \oplus \langle x_m + (H \cap N) \rangle$$

is a $*$-valuated coproduct in $G/(H \cap N)$ with each $x_i + (H \cap N)$ a primitive element of $G/(H \cap N)$. Obviously then, $\overline{S}$ is finite modulo this coproduct. Thus, there is a finite subset $\overline{T}$ of $N/(H \cap N)$ such that $\overline{S}$ is finite modulo the subgroup

$$\langle \overline{T}, x_1 + (H \cap N), x_2 + (H \cap N), \ldots, x_m + (H \cap N) \rangle.$$

Now from the hypothesis that $(N + H)/H$ is a k-subgroup of G/H, (4) implies that $\overline{T}$ is finite modulo

$$\langle y_1 + (H \cap N) \rangle \oplus \langle y_2 + (H \cap N) \rangle \oplus \ldots \oplus \langle y_k + (H \cap N) \rangle \subseteq N/(H \cap N)$$

where the coproduct is $*$-valuated in $G/(H \cap N)$ and each $y_j + (H \cap N)$ is a primitive element of $G/(H \cap N)$. Finally, $\overline{S}$ is clearly finite modulo the $*$-valuated coproduct

$$\langle y_1 + (H \cap N) \rangle \oplus \ldots \oplus \langle y_k + (H \cap N) \rangle \oplus \langle x_1 + (H \cap N) \rangle \oplus \ldots \oplus \langle x_m + (H \cap N) \rangle$$

and hence $G/(H \cap N)$ is a k-group as desired. □

If p is a prime, a subgroup H of G is said to be *p-separable* provided that for each $g \in G$ there is a corresponding countable subset $\{h_n\}_{n<\omega} \subseteq H$ with the following property: if $h \in H$, then there exists an $n < \omega$ such that $|g+h|_p \leqslant |g+h_n|_p$. If a subgroup H of G is p-separable for all primes p, then we say that H is *locally separable* in G. P. Hill proves in [6] that a simply presented isotype subgroup H of a torsion group G is necessarily locally separable in G. On the other hand, for a local Warfield group H to appear as an isotype subgroup of a local group G, H must be strongly separable in G (see [10]); in other words, to each $g \in G$ there corresponds a countable subset $\{h_n\}_{n<\omega} \subseteq H$ such that if $h \in H$, $\|g+h\| \leqslant \|g+h_n\|$ for some $n < \omega$. But for global groups, it is possible for an isotype subgroup to be a Warfield group without being strongly separable in the containing group (see [9]). The requisite necessary condition in the global context is the following intermediate form of separability first identified in [12].

Definition 3.1. A subgroup H of a global group G is said to be *almost strongly separable* in G if H is locally separable in G and, for each $g \in G$ there is a corresponding countable subset $\{h_n\}_{n<\omega} \subseteq H$ with the following property: if $h \in H$, then there exists an $n < \omega$ and a positive integer m such that $\|m(g+h)\| \leqslant \|m(g+h_n)\|$.

Thus, if H is an isotype Warfield subgroup of the global group G, then H is almost strongly separable in G (see [12, Proposition 3.6]).

Remark 3.2. For later use, we note that every pure knice subgroup H of an arbitrary global group G is almost strongly separable in G. That this is so is a routine consequence of Proposition 1.2(4) and Proposition 1.3(b). In particular, Proposition 1.2(4) implies that H is locally separable in G, while Proposition 1.3(b) can be used to handle the condition on height matrices.

We shall require the next proposition several times in the sequel.

Proposition 3.3. *Suppose that N is a knice and pure subgroup of the global group G and that K is a subgroup of G that contains N. Then K is almost strongly separable in G if and only if K/N is almost strongly separable in G/N.*

Proof. We first observe that K is locally separable in G if and only if K/N is locally separable in G/N. Indeed this is a routine consequence of Proposition 1.2(4) and the definitions.

Now assume that K is almost strongly separable in G. Given $g+N \in G/N$, select a corresponding countable subset $\{c_n\}_{n<\omega} \subseteq K$ with the following property: if $y \in K$, then $\|m(g+y)\| \leqslant \|m(g+c_n)\|$ for some $n < \omega$ and positive integer m. Assume now that $c+N$ is an arbitrary element of K/N. Since N is a pure and knice subgroup of G, there is a positive integer m so that $\|m(g+c)+N\|^{G/N} = \|m(g+c+z)\|$ for some $z \in N$. Moreover, replacing m by a positive muliple of itself if necessary, there is an $n < \omega$ such that $\|m(g+c+z)\| \leqslant \|m(g+c_n)\|$. Therefore,

$$\|m(g+c)+N\|^{G/N} \leqslant \|m(g+c_n)\| \leqslant \|m(g+c_n)+N\|^{G/N}.$$

It is now clear that $\{c_n+N\}_{n<\omega}$ satisfies the requisite properties for $g+N$, and we conclude that K/N is almost strongly separable in G/N.

Conversely, suppose that K/N is almost strongly separable in G/N and, for a given $g \in G$, select a countable subset $\{c_n+N\}_{n<\omega}$ of K/N with the following property: if $c+N \in K/N$, there exists an $n < \omega$ and a positive integer m such that $\|m(g+c)+N\|^{G/N} \leqslant \|m(g+c_n)+N\|^{G/N}$. For each $n < \omega$, choose m_n to be the smallest positive integer for which there exists a $z_n \in N$ with $\|m_n(g+c_n)+N\|^{G/N} = \|m_n(g+c_n+z_n)\|$. Now given $c \in K$, select a positive integer l and an $n < \omega$ such that

$$\|l(g+c)+N\|^{G/N} \leqslant \|l(g+c_n)+N\|^{G/N}.$$

Then

$$\|lm_n(g+c)\| \leqslant \|lm_n(g+c_n)+N\|^{G/N} = \|lm_n(g+c_n+z_n)\|$$

and we conclude that the countable subset $\{c_n+z_n\}_{n<\omega} \subseteq K$ satisfies the requisite properties for $g \in G$. □

In the application of Axiom 3 characterizations, there is an interplay between separability and compatibility that will be familiar to those who have studied either of the fundamental papers [6] or [3]. Such readers may indeed anticipate the following proposition (obtained by combining Propositions 4.5 and 4.6 of [12]). But it should be borne in mind that, in the present setting, $H \parallel N$ indicates that H and N are almost strongly compatible in the sense of Definition 2.2 above.

Proposition 3.4 ([12]). (1) *Suppose H is almost strongly separable subgroup of the global group G. If A is a countable subgroup of G, there is a countable subgroup B of G that contains A and such that $H \parallel B$.*

(2) *Let H and N be subgroups of a global group G where $H \parallel N$ and N is pure and knice in G. If M is any subgroup of G that contains N, then $(H+N)/N \parallel M/N$ implies $H \parallel M$.*

Definition 3.5. By an *Axiom 3* family in the global group G is meant a collection $\mathscr{C}$ of subgroups of G that satisfies the following three conditions.

(H1) $\mathscr{C}$ contains the trivial subgroup 0;

(H2) if $N_i \in \mathscr{C}$ for each $i \in I$, then $\sum\limits_{i\in I} N_i \in \mathscr{C}$;

(H3) if $C \in \mathscr{C}$ and if A is any countable subgroup of G, then there is a $B \in \mathscr{C}$ that contains both C and A with B/C countable.

The first and most famous of an Axiom 3 characterization is P. Hill's proof that a p-primary abelian group G is simply presented if and only if there is an Axiom 3 family in G consisting of nice subgroups. The analogous result for global groups is established in [8]: a global group G is a Warfield group if and only if there is an Axiom 3 family in G consisting of knice subgroups. There are several variations on the Axiom 3 theme and we shall find it convenient to adopt the notation and terminology of [3] where an Axiom 3 family is referred to as an $H(\aleph_0)$-family in G, and a $G(\aleph_0)$-family in G is a collection of subgroups $\mathscr{C}$ that satisfies conditions (H1), (H3) and

(G2) $\mathscr{C}$ is closed under unions of ascending chains.

For example, the family $\mathscr{P}_G$ of all pure subgroups of G is a $G(\aleph_0)$-family in G. Also as in [3], by an $F(\aleph_0)$-family $\mathscr{C} = \{N_\alpha\}_{\alpha<\tau}$ in G we mean a smooth ascending chain

$$0 = N_0 \subseteq N_1 \subseteq \ldots \subseteq N_\alpha \subseteq \ldots \qquad (\alpha < \tau)$$

of subgroups of G where $G = \bigcup\limits_{\alpha<\tau} N_\alpha$ and $|N_{\alpha+1}/N_\alpha| \leqslant \aleph_0$ for all $\alpha < \tau$. Obviously an $H(\aleph_0)$-family in G is a $G(\aleph_0)$-family and a $G(\aleph_0)$-family contains an $F(\aleph_0)$-family.

Proposition 3.6 ([3]). (1) *The intersection of any two $H(\aleph_0)$-families ($G(\aleph_0)$-families) in G is again one.*

(2) *Let H be a subgroup of a global group G. If $\mathscr{C}_G$ and $\mathscr{C}_H$ are $G(\aleph_0)$-families in G and H, respectively, then $\mathscr{C} = \{N \in \mathscr{C}_G\colon\ N \cap H \in \mathscr{C}_H\}$ is a $G(\aleph_0)$-family in G.*

(3) *Suppose H is a subgroup of a global group G and that $\pi\colon\ G \to G/H$ is the canonical map. If $\mathscr{C}_G$ is a $G(\aleph_0)$-family in G and $\mathscr{D}$ is a $G(\aleph_0)$-family in G/H, then there is a $G(\aleph_0)$-family $\mathscr{C}$ contained in $\mathscr{C}_G$ such that $\pi(\mathscr{C}) \subseteq \mathscr{D}$ (where $\pi(\mathscr{C}) = \{\pi(N)\colon\ N \in \mathscr{C}\}$).*

Proof. (1) is [3, Lemma 1.2] and (2) is established in the proof of [3, Lemma 1.5]. By [3, Lemma 1.3(b)], there is a $G(\aleph_0)$-family family $\mathscr{B}$ in G such that $\pi(\mathscr{B}) = \mathscr{D}$ and we need only take $\mathscr{C} = \mathscr{B} \cap \mathscr{C}_G$. □

Proposition 3.7. *If G is a global Warfield group, there exists a $G(\aleph_0)$-family $\mathscr{C}$ in G consisting of pure knice subgroups; and furthermore, G/A is a global Warfield group whenever $A \in \mathscr{C}$.*

P r o o f. By [8, Theorem 3.2], there is an $G(\aleph_0)$-family $\mathscr{C}_G$ in G consisting of knice subgroups and $\mathscr{C} = \mathscr{P}_G \cap \mathscr{C}_G$ has the desired property. Indeed, for each $A \in \mathscr{C}$, $\{N/A\colon N \in \mathscr{C}$ and $A \subseteq N\}$ is clearly a $G(\aleph_0)$-family in G/A consisting of knice subgroups by Proposition 1.2(5). Finally, by [8, Theorem 3.2] again, G/A is a global Warfield group. □

The significance of Proposition 3.4 is that it provides us with the tools required to establish the following fact which will play a prominent role in the proof of our Main Theorem.

Proposition 3.8. *Let H be a subgroup of the global Warfield group G and suppose that $\mathscr{C}_G$ is a $G(\aleph_0)$-family in G consisting of pure knice subgroups. If*

$$\mathscr{C} = \{N \in \mathscr{C}_G\colon\ N \parallel H\}$$

and if $(H+N)/N$ is almost strongly separable in G/N for each $N \in \mathscr{C}_G$, then $\mathscr{C}$ is a $G(\aleph_0)$-family in G.

P r o o f. First note that $\mathscr{C}$ satisfies (G2) since, as noted above, almost strong compatibility is an inductive relation. Thus, it remains only to show that $\mathscr{C}$ satisfies property (H3). To this end, let $N \in \mathscr{C}$ and suppose that A is a countable subgroup of G. We construct inductively two ascending sequences of subgroups $\{N_n\}_{n<\omega}$ and $\{M_n\}_{n<\omega}$ such that $N + A \subseteq N_0$ and the following three conditions are satisfied for all $n < \omega$.

(i) $N_n \in \mathscr{C}_G$ and $M_n \parallel H$.
(ii) $N_n \subseteq M_n \subseteq N_{n+1}$.
(iii) $|N_n/N| \leqslant \aleph_0$ and $|M_n/N| \leqslant \aleph_0$.

Assuming that N_n has been constructed, we apply part (1) of Proposition 3.4 and the hypothesis that $(H+N)/N$ is almost strongly separable in G/N to obtain a countable subgroup M_n/N of G/N that contains N_n/N and such that $M_n/N \parallel (H+N)/N$. Then by part (2) of Proposition 3.4, $M_n \parallel H$. Next, using the fact that $\mathscr{C}_G$ is a $G(\aleph_0)$-family in G, we choose $N_{n+1} \in \mathscr{C}_G$ such that $M_n \subseteq N_{n+1}$ and N_{n+1}/N is countable. This completes the induction. Now take $N' = \bigcup\limits_{n<\omega} N_n = \bigcup\limits_{n<\omega} M_n$ and observe that both $N' \in \mathscr{C}_G$ and $N' \parallel H$. Thus, $N' \in \mathscr{C}$, $N + A \subseteq N'$ and N'/N is countable. Therefore $\mathscr{C}$ satisfies property (H3). □

At this point, we have all that is needed to establish a sufficient condition for an isotype knice subgroup of a global Warfield group to be itself a Warfield group.

Theorem 3.9. *Let H be an isotype knice subgroup of the global Warfield group G. If there is a $G(\aleph_0)$-family $\mathscr{D}$ in the quotient group G/H consisting of almost strongly separable k-subgroups, then H is a Warfield group.*

P r o o f. By Proposition 3.7, there is a $G(\aleph_0)$-family $\mathscr{C}_G$ in G consisting of pure knice subgroups of G. In view of Proposition 3.6(3), we may assume without loss of generality that $\pi(\mathscr{C}_G) \subseteq \mathscr{D}$, where $\pi\colon G \to G/H$ is the canonical map. But then, $(N+H)/H$ is an almost strongly separable k-subgroup of G/H whenever $N \in \mathscr{C}_G$. Two applications of Proposition 3.3 imply that $(H+N)/N$ is almost strongly separable in G/N for each $N \in \mathscr{C}_G$. Then, by Proposition 3.8, $\{N \in \mathscr{C}_G\colon\ N \parallel H\}$ is a $G(\aleph_0)$-family in G. Since the family $\mathscr{P}_H$ of all pure subgroups of H is a $G(\aleph_0)$-family in H, we conclude from Proposition 3.6 that

$$\mathscr{C} = \{N \in \mathscr{C}_G\colon\ N \parallel H \text{ and } H \cap N \text{ is pure in } H\}$$

is also a $G(\aleph_0)$-family in G. Recalling that $(N+H)/H$ is a k-subgroup of G/H whenever $N \in \mathscr{C}_G$, we see from Theorem 2.4 that $H \cap N$ is knice in G whenever $N \in \mathscr{C}$. In fact, by Theorem 2.1, each $H \cap N$ is a knice subgroup of H. From $\mathscr{C}$ we extract an $F(\aleph_0)$-family in G

$$0 = N_0 \subseteq N_1 \subseteq \ldots \subseteq N_\alpha \subseteq \ldots \qquad (\alpha < \tau).$$

Then,

$$0 = H \cap N_0 \subseteq H \cap N_1 \subseteq \ldots \subseteq H \cap N_\alpha \subseteq \ldots \qquad (\alpha < \tau)$$

is an $F(\aleph_0)$-family in H consisting of pure knice subgroups of H and, therefore, we conclude from [12, Theorem 2.5] that H is a global Warfield group. □

The converse of Theorem 3.9 is more difficult to prove, but with the tools currently at our command, we can now obtain a strong partial converse.

Theorem 3.10. *Let H be an isotype knice subgroup of the global Warfield group G. If H itself is a Warfield group, then there exists a $G(\aleph_0)$-family $\mathscr{C}$ in G with the property that, for each $N \in \mathscr{C}$, $(H+N)/H$ is an almost strongly separable k-subgroup of G/H*

P r o o f. By the hypotheses and Proposition 3.7, there exists a $G(\aleph_0)$-family $\mathscr{C}_G$ of pure knice subgroups in G and a $G(\aleph_0)$-family $\mathscr{C}_H$ of pure knice subgroups in H. Since H is an isotype Warfield subgroup of the global Warfield group G, our characterization of this situation in [12, Theorem 5.2] establishes the existence of a $G(\aleph_0)$-family $\mathscr{C}''$ in G such that $(H+N)/N$ is almost strongly separable in G/N for all $N \in \mathscr{C}''$. Set $\mathscr{C}' = \mathscr{C}'' \cap \mathscr{C}_G$. By Proposition 3.6(1), $\mathscr{C}'$ is a $G(\aleph_0)$-family in G so that $\{N \in \mathscr{C}' \colon N \parallel H\}$ is a $G(\aleph_0)$-family in G by Proposition 3.8. Therefore, by an application of Proposition 3.6(2),

$$\mathscr{C} = \{N \in \mathscr{C}' \colon N \parallel H \text{ and } H \cap N \in \mathscr{C}_H\}$$

is a $G(\aleph_0)$-family in G. By two applications of Proposition 3.3, $(H+N)/H$ is almost strongly separable in G/H for all $N \in \mathscr{C}$. Also, if $N \in \mathscr{C}$, $N \cap H$ is a pure knice subgroup of H and therefore Theorem 2.1 implies that $N \cap H$ is knice in G. Finally, since all the hypotheses of Theorem 2.4 are satisfied when $N \in \mathscr{C}$, we conclude that $(H + N)/H$ is also a k-subgroup of G/H whenever $N \in \mathscr{C}$. □

It is tempting to speculate that, when $\mathscr{C}$ is as in Theorem 3.10, $\mathscr{D} = \{(H+N)/H \colon N \in \mathscr{C}\}$ might constitute the appropriate family of subgroups of G/H yielding the converse of Theorem 3.9. However, $\mathscr{D}$ need not be a $G(\aleph_0)$-family in G/H since chains of subgroups in $\mathscr{D}$ need not be induced by chains of subgroups in $\mathscr{C}$. Nonetheless, we can extract from $\mathscr{C}$ an $F(\aleph_0)$-family in G that will project to an $F(\aleph_0)$-family in G/H, yielding a proof of the following.

Corollary 3.11. *Let H be an isotype knice subgroup of the global Warfield group G. If H is itself a global Warfield group, then there exists an $F(\aleph_0)$-family in G/H consisting of almost strongly separable k-subgroups.*

4. Axiom 3, closed sets and the main theorem

There is a standard method introduced by P. Hill in [5] for converting $F(\aleph_0)$-families of appropriate sorts of subgroups into $H(\aleph_0)$-families of the same sort of subgroups. This technique, called the *closed set method*, is a powerful tool that we shall require frequently in this section. The method is, of course, not universally applicable and hence each application requires *ad hoc* arguments. Since it leads to cleaner and less obscure proofs, we shall use the version of the method introduced in [1]. The common core in all our applications is the existence of an $F(\aleph_0)$-family $\{G_\alpha\}_{\alpha<\tau}$ in the global group G consisting of pure subgroups. Since each quotient $G_{\alpha+1}/G_\alpha$ is countable, we also have an associated family $\{B_\alpha\}_{\alpha<\tau}$ of countable pure subgroups with $G_{\alpha+1} = G_\alpha + B_\alpha$ for each $\alpha < \tau$. Since the set $\mathscr{P}_G$ of all pure subgroups of G is a $G(\aleph_0)$-family in G, each global group G possesses such an $F(\aleph_0)$-family. A routine induction leads to a proof of the fundamental fact that $G_\alpha = \sum_{\beta<\alpha} B_\beta$ for each $\alpha < \tau$. Following [4], for each $g \in G$ we let $\nu(g)$ denote the least ordinal such that $g \in G_{\nu(g)+1}$. All our applications of the closed set method involve inductions on $\nu(g)$.

A subset S of τ is called a *closed subset* if it enjoys the following property: for each $\lambda \in S$,

$$B_\lambda \cap G_\lambda \subseteq \sum\{B_\alpha\colon\ \alpha \in S \text{ and } \alpha < \lambda\}.$$

The relevant facts, established in [1], are the following:

(1) the empty set is closed,

(2) arbitrary unions of closed subsets are closed, and

(3) each countable subset of τ is contained is a countable closed subset.

With each subset $S \subseteq \tau$, we associate the subgroup $G(S) = \sum\{B_\alpha\colon\ \alpha \in S\}$. Noting that $\sum\limits_{i\in I} G(S_i) = G\Big(\bigcup\limits_{i\in I} S_i\Big)$, we see that (1), (2) and (3) imply that $\mathscr{C} = \{G(S)\colon$ S is a closed subset of $\tau\}$ is an $H(\aleph_0)$-family in G.

Each nonzero $x \in G(S)$ has a *standard representation*

$$x = b_{\mu(1)} + b_{\mu(2)} + \ldots + b_{\mu(m)}$$

where $\mu(i) \in S$ and $b_{\mu(i)}$ is a nonzero element of $B_{\mu(i)}$ for $i = 1, 2, \ldots, m$,

$$\mu(1) < \mu(2) < \ldots < \mu(m)$$

and $\mu(m)$ is minimal. The following fundamental fact is implicit in [4].

Lemma 4.1. *If $x = b_{\mu(1)} + b_{\mu(2)} + \ldots + b_{\mu(m)}$ is a standard representation of a nonzero $x \in G(S)$ with S a closed subset of τ, then $\nu(x) = \mu(m)$. In particular, $\nu(x) \in S$.*

Proof. We first prove that $\nu(x) \in S$. Assume by way of contradiction that $\mu = \nu(x) \notin S$. Then we may write $x = x_0 + x_1$ where $x_0 \in G(S) \cap G_\mu$ and $0 \neq x_1 \in \sum\{B_\alpha\colon\ \mu < \alpha \text{ and } \alpha \in S\}$. Select a standard representation $x_1 = b'_{\lambda(1)} + \ldots + b'_{\lambda(n)}$. Since $x_1 \in G_{\mu+1}$ and $\mu + 1 \leqslant \lambda(n)$,

$$b'_{\lambda(n)} = x_1 - (b'_{\lambda(1)} + \ldots + b'_{\lambda(n-1)})$$

is an element of $G_{\lambda(n)} \cap B_{\lambda(n)} \subseteq \sum\{B_\alpha\colon\ \alpha \in S \text{ and } \alpha < \lambda(n)\}$. This, however, contradicts the minimality of $\lambda(n)$, forcing us to conclude that $\nu(x) \in S$.

Clearly, $\mu = \nu(x) \leqslant \mu(m)$ and we assume that equality has been established for all nonzero $y \in G(S)$ with $\nu(y) < \nu(x)$. Write $x = y + b$ where $y \in G_\mu$ and $b \in B_\mu$. Since $x \in G(S)$ and $\mu \in S$ implies that $b \in G(S)$, we conclude that $y \in G(S)$ and we may assume that $y \neq 0$. Thus, we have a standard representation $y = b'_{\lambda(1)} + \ldots + b'_{\lambda(n)}$ where $\nu(y) < \mu$. Therefore, our induction hypothesis yields $\lambda(n) = \nu(y) < \mu$. If we were to have $\mu < \mu(m)$, then the representation $x = b'_{\lambda(1)} + \ldots + b'_{\lambda(n)} + b$ would contradict the minimality of $\mu(m)$. $\square$

The following result was first established in the special case of torsion-free groups by Hill [5].

Proposition 4.2. *If G is an arbitrary global group, then there is an $H(\aleph_0)$-family in G consisting of pure subgroups.*

Proof. By the discussion above, it suffices to show that $G(S)$ is a pure subgroup of G whenever S is a closed subset. Let $0 \neq x \in nG \cap G(S)$ where n is a positive integer and suppose $\mu = \nu(x)$. Since $G_{\mu+1}$ is a pure subgroup of G, $x = ny$ where $y = z + b$ with $z \in G_\mu$ and $b \in B_\mu \subseteq G(S)$, with the latter containment holding by Lemma 4.1. Thus $nz = x - nb = x_1 \in G(S)$ with $\nu(x_1) < \nu(x)$. Proceeding by induction, there is a $y_1 \in G(S)$ such that $ny_1 = x_1$ and therefore $x = n(y_1 + b)$ where $y_1 + b \in G(S)$. □

Corollary 4.3. *If G is a global Warfield group, there is an $H(\aleph_0)$-family in G consisting of almost strongly separable pure knice subgroups.*

Proof. As shown in [8], G has an $H(\aleph_0)$-family $\mathscr{C}$ consisting of knice subgroups. If we now take $\mathscr{C}'$ to be an $H(\aleph_0)$-family in G of pure subgroups, then Proposition 3.6(1) and Remark 3.2 imply that $\mathscr{C} \cap \mathscr{C}'$ is an $H(\aleph_0)$-family in G with the desired properties. □

When the $F(\aleph_0)$-family in G consists of pure subgroups with additional special properties, the countable pure subgroups B_α must be selected carefully if there is any hope of applying the closed set method to obtain an $H(\aleph_0)$-family consisting of subgroups with the same special properties. We illustrate this phenomenon in our next proof. Although the following proposition is known in the case of p-primary groups (see [3] or [6]), we give a detailed demonstration which will serve as a model for our subsequent proofs. See [4, Theorem 5.1] for the genesis of this technique of applying the closed set method.

Proposition 4.4. *If $\{G_\alpha\}_{\alpha<\tau}$ is an $F(\aleph_0)$-family in the global group G with each G_α a locally separable pure subgroup, then there is an $H(\aleph_0)$-family in G consisting of locally separable subgroups.*

Proof. First we define the appropriate countable pure subgroups B_α. Let $\alpha < \tau$ and select a countable set $\{x_n\}_{n<\omega}$ of representatives for the cosets of G_α in $G_{\alpha+1}$. Since G_α is locally separable in G, we have for each $n < \omega$ a countable subset $\{b_{n,k}\}_{k<\omega} \subseteq G_\alpha$ such that, for each $y \in G_\alpha$ and each prime p, there exists a $k < \omega$ such that $|x_n + y|_p \leqslant |x_n + b_{n,k}|_p$. Then select B_α to be a countable pure subgroup of $G_{\alpha+1}$ that contains all the x_n and all the $b_{n,k}$. Clearly, $G_{\alpha+1} = G_\alpha + B_\alpha$. We maintain that the following special condition is satisfied: if $x \in G_{\alpha+1}$, $y \in G_\alpha$ and p is a prime, then there exists a $z \in G_\alpha$ with $x + z \in B_\alpha$ and $|x + y|_p \leqslant |x + z|_p$. Indeed, for approptiate $n < \omega$ and $y_1 \in G_\alpha$, $x+y = x_n+y_1$ and there is a $k < \omega$ with $|x_n+y_1|_p \leqslant |x_n+b_{n,k}|_p$. Then, for $z = (y-y_1)+b_{n,k}$, $z \in G_\alpha$, $x+z = x_n+b_{n,k} \in B_\alpha$ and $|x + y|_p \leqslant |x + z|_p$, as claimed.

Given a closed subset S of τ, it suffices to show that $G(S)$ is a locally separable subgroup of G. Thus, for $g \in G \setminus G(S)$, we are required to find a countable set $\{a_n\}_{n<\omega} \subseteq G(S)$ such that for each $x \in G(S)$ and prime p, there is an n such that $|g + x|_p \leqslant |g + a_n|_p$. We shall establish this by induction on $\nu(g)$, where we may assume that g is selected in the coset $g + G(S)$ with $\mu = \nu(g)$ minimal. Notice then that $\mu \notin S$, for if μ were in S, we would have $x = y + b$ with $y \in G_\mu$ and $b \in B_\mu$. Hence, y would be an element of $g+G(S)$ with $\nu(y) < \nu(g)$, contradicting the choice of g.

Since G_μ is locally separable in G, there is a countable set $\{b_n\} \subseteq G_\mu$ satisfying the following condition: for each $x \in G_\mu$ and each prime p, there is an $n < \omega$ such that $|g + x|_p \leqslant |g + b_n|_p$. Notice that $\nu(b_n) < \mu = \nu(g)$ for each n and therefore our implicit induction hypothesis implies that we have countable subsets $\{a_{n,k}\}_{k<\omega} \subseteq G(S)$ such that for each nonzero $x \in G(S)$ and each prime p, there exists $k < \omega$ with $|b_n + x|_p \leqslant |b_n - a_{n,k}|_p$.

We maintain that the countable set $\{a_{n,k}\}_{n<\omega,k<\omega}$ has the desired property that, for each nonzero $x \in G(S)$ and prime p, there exists an $(n, k) \in \omega \times \omega$ such that $|g+x|_p \leqslant |g+a_{n,k}|_p$. The proof of this fact, however, requires a secondary induction on the ordinal $\lambda = \nu(x)$. We shall first deal with the case $\lambda < \mu$ which implies that $x \in G_\mu$. Thus, for each prime p, there is an $n < \omega$ such that $|g + x|_p \leqslant |g + b_n|_p$, from which $|g + x|_p \leqslant |b_n - x|_p$ by the triangle inequality. The choice of the $a_{n,k}$ yields $|b_n - x|_p \leqslant |b_n - a_{n,k}|_p$ for some k. Therefore, since

$$g + a_{n,k} = (g + x) - (x - b_n) - (b_n - a_{n,k}),$$

the triangle inequality implies the desired conclusion $|g + x|_p \leqslant |g + a_{n,k}|_p$. By Lemma 4.1, $\lambda \in S$ so that the case $\lambda = \mu$ is excluded. It remains to deal with the case $\lambda > \mu$. Under this assumption, $g \in G_\lambda$ and by the choice of the B_α, for each prime p, there is a $z \in G_\lambda$ such that $x + z \in B_\lambda$ and $|x + g|_p \leqslant |x + z|_p$. Then, by the triangle inequality, $|g + x|_p \leqslant |g - z|_p$. Observe that since $\lambda \in S$, $z \in G(S)$ with $\nu(z) < \lambda$ and our induction hypothesis implies that there is an $(n, k) \in \omega \times \omega$ such that $|g + x|_p \leqslant |g - z|_p \leqslant |g + a_{n,k}|_p$. □

We, of course, need the generalization of the above proposition to $F(\aleph_0)$-families of almost strongly separable subgroups. Since the intersection of two $H(\aleph_0)$-families is an $H(\aleph_0)$-family, we limit the proof that follows to establishing the existence of an $H(\aleph_0)$-family that satisfies the requisite conditions in Definition 3.1 that relate to height matrices.

Proposition 4.5. *If $\{G_\alpha\}_{\alpha<\tau}$ is an $F(\aleph_0)$-family in the global group G with each G_α an almost strongly separable pure subgroup of G, then there is an $H(\aleph_0)$-family in G of almost strongly separable subgroups.*

Proof. For each $\alpha < \tau$, select a countable set $\{x_n\}_{n<\omega}$ of representatives of the cosets in $G_{\alpha+1}/G_\alpha$. Since G_α is almost strongly separable in G, we have for each n a countable set $\{b_{n,k}\}_{k<\omega} \subseteq G_\alpha$ such that for each $y \in G_\alpha$ there exists a k and a positive integer m with

$$\|m(x_n + y)\| \leqslant \|m(x_n + b_{n,k})\|.$$

Now take B_α to be a countable pure subgroup of $G_{\alpha+1}$ that contains all x_n and $b_{n,k}$. Clearly then, $G_{\alpha+1} = G_\alpha + B_\alpha$. We maintain that the following special condition is satisfied: if $x \in G_{\alpha+1}$, $y \in G_\alpha$, then there is a $z \in G_\alpha$ with $x + z \in B_\alpha$ and a positive integer m such that $\|m(x + y)\| \leqslant \|m(x + z)\|$. Indeed, for appropriate n and $y_1 \in G_\alpha$, $x + y = x_n + y_1$ and we select a $k < \omega$ such that

$$\|m(x_n + y_1)\| \leqslant \|m(x_n + b_{n,k})\|$$

for some positive integer m. If we now set $z = (y - y_1) + b_{n,k} \in G_\alpha$, then $x + z = x_n + b_{n,k} \in B_\alpha$ and $\|m(x + y)\| \leqslant \|m(x + z)\|$, as claimed.

In view of the remarks preceding the statement of our proposition, it suffices to establish the following: if S is a closed subset of τ and if $g \in G \setminus G(S)$, then there is a countable subset $\{a_n\}_{n<\omega} \subseteq G(S)$ such that for each $x \in G(S)$ there exist an $n < \omega$ and a positive integer m such that $\|m(g + x)\| \leqslant \|m(g + a_n)\|$. As in the proof of Proposition 4.4, we establish this by induction on $\nu(g)$, where g is selected in the coset $g + G(S)$ with $\mu = \nu(g)$ minimal. Consequently, $\mu \notin S$. Since G_μ is almost strongly separable in G, there is a countable subset $\{b_n\}_{n<\omega}$ of G_μ satisfying the following condition: for each $x \in G_\mu$ there is an $n < \omega$ and a positive integer m such that $\|m(g + x)\| \leqslant \|m(g + b_n)\|$. Since $\nu(b_n) < \mu = \nu(g)$ for all n, our implicit induction hypothesis implies that we have countable subsets $\{a_{n,k}\}_{k<\omega} \subseteq G(S)$ such that for each nonzero $x \in G(S)$ there is a $k < \omega$ and a positive integer m such that $\|m(b_n + x)\| \leqslant \|m(b_n - a_{k,n})\|$.

We claim that the countable set $\{a_{n,k}\}_{n<\omega,k<\omega}$ has the desired property that, for each $x \in G(S)$, there exists $(n, k) \in \omega \times \omega$ and a positive integer m such that $\|m(g+x)\| \leqslant \|m(g+a_{n,k})\|$. As in the previous proposition, a secondary induction on the ordinal $\lambda = \nu(x)$ is required. Assume first that $\lambda < \mu$ and so $x \in G_\mu$. Then there is an $n < \omega$ and a positive integer m such that $\|m(g+x)\| \leqslant \|m(g+b_n)\|$, from which the triangle inequality implies $\|m(g+x)\| \leqslant \|m(b_n-x)\|$. The choice of the $a_{n,k}$ yields a $k < \omega$ and a positive multiple m_1 of m such that $\|m_1(b_n - x)\| \leqslant \|m_1(b_n - a_{n,k})\|$. Therefore, since

$$g + a_{n,k} = (g + x) + (b_n - x) - (b_n - a_{n,k}),$$

the triangle inequality yields the desired conclusion $\|m_1(g + x)\| \leqslant \|m_1(g + a_{n,k})\|$. Since $\lambda \in S$ by Lemma 4.1, the case $\lambda = \mu$ is once again excluded. Assuming that $\lambda > \mu$, $g \in G_\lambda$ and, by the choice of the B_α, there is a $z \in G_\lambda$ such that $x + z \in B_\lambda$ and $\|m(x + g)\| \leqslant \|m(x + z)\|$ for some positive integer m. Then, by the triangle inequality, $\|m(g + x)\| \leqslant \|m(g - z)\|$. Since $\lambda \in S$ and $z \in G(S)$ with $\nu(z) < \lambda$, our induction hypothesis implies that there exists an $(n, k) \in \omega \times \omega$ and a positive multiple m_1 of m such that $\|m_1(g + x)\| \leqslant \|m_1(g - z)\| \leqslant \|m_1(g + a_{n,k})\|$. □

In view of Corollary 3.11 and the preceding proposition, we may now conclude that if the isotype knice subgroup H of the global Warfield group G is itself a Warfield group, then there is an $H(\aleph_0)$-family in G/H consisting of almost strongly separable subgroups. Most of the remainder of this section is devoted to proving, under these hypotheses, that G/H also has an $H(\aleph_0)$-family consisting of k-subgroups. As Theorem 2.4 will be indispensable in ths endeavor, the reader will appreciate the significance of the following specialized result.

Lemma 4.6. *If $\{G_\alpha\}_{\alpha<\tau}$ is an $F(\aleph_0)$-family in the global group G with each G_α an almost strongly separable pure subgroup, and if H is a subgroup of G such that $H \parallel G_\alpha$ for all $\alpha < \tau$, then there is an $H(\aleph_0)$-family $\mathscr{C}$ in G such that $H \parallel N$ for all $N \in \mathscr{C}$.*

Proof. Given the example of the preceding proof, we shall limit ourselves to establishing local compatibility. Thus, we choose the B_α as in the proof of Proposition 4.4 and show that H and $G(S)$ are locally compatible whenever S is a closed subset of τ.

For $h \in H$, $x \in G(S)$ and p a prime, we are required to find an element $y \in H \cap G(S)$ such that $|h + x|_p \leqslant |h + y|_p$. First, we may assume that h is selected in the coset $h + G(S)$ with $\mu = \nu(h)$ minimal. As in the proof of Proposition 4.4, $\mu \notin S$; in particular, $\lambda = \nu(x) \neq \mu$.

If $\lambda < \mu$, then $x \in G_\mu$ and we proceed by induction on μ. Since G_μ is locally separable in G, there is a countable subset $\{b_n\}_{n<\omega} \subseteq G_\mu$ satisfying the following condition: for each prime p, there is an $n < \omega$ such that $|h+x|_p \leqslant |h+b_n|_p$. But each b_n is in G_μ, and since H and G_μ are locally compatible in G, there is a countable set

$\{y_n\}_{n<\omega} \subseteq H \cap G_\mu$ such that for each prime p, there is there is an $n < \omega$ such that $|h+b_n|_p \leqslant |h+y_n|_p$. Thus, $|h+x|_p \leqslant |h+y_n|_p$ and consequently $|h+x|_p \leqslant |y_n - x|_p$. But, for all n, $y_n \in H$ with $\nu(y_n) < \mu$ and we have, by the induction hypothesis, an element $y \in H \cap G(S)$ with $|y_n - x|_p \leqslant |y_n - y|_p$. Since then

$$h + y = (h + x) + (y_n - x) + (y - y_n),$$

the triangle inequality yields the desired relation $|h+x|_p \leqslant |h+y|_p$ with $y \in H \cap G(S)$.

If $\lambda > \mu$, then $h \in G_\lambda$ and we proceed by induction on λ. As in the proof of Proposition 4.4, for each prime p there is a $z \in G_\lambda$ such that $|x + h|_p \leqslant |x + z|_p$ and $x + z \in B_\lambda$. By the triangle inequality, $|h + x|_p \leqslant |h - z|_p$ where $\nu(z) < \lambda$ and $z \in G(S)$ since $\lambda \in S$. Finally, by induction, for each prime p there is a $y \in H \cap G(S)$ with $|h + x|_p \leqslant |h - z|_p \leqslant |h + y|_p$. □

Lemma 4.7. *Suppose that $\{G_\alpha\}_{\alpha<\tau}$ is an $F(\aleph_0)$-family in the global group G consisting of pure subgroups and that $\{B_\alpha\}_{\alpha<\tau}$ is an associated family of countable pure subgroups with $G_{\alpha+1} = G_\alpha + B_\alpha$ for all $\alpha < \tau$. If H is a subgroup of G such that $H \cap G_{\alpha+1} = (H \cap G_\alpha) + (H \cap B_\alpha)$ for all $\alpha < \tau$ and if S is a closed subset of τ, then $H \cap G(S) = \sum_{\alpha \in S} (H \cap B_\alpha)$.*

P r o o f. Since the reverse inclusion is obvious, it suffices to show that $H \cap G(S) \subseteq \sum_{\alpha \in S} (H \cap B_\alpha)$. The proof, of course, is by induction on $\nu(x)$ where $x \in H \cap G(S)$. Assuming without loss that $x \neq 0$, choose a standard representation $x = b_{\mu(1)} + b_{\mu(2)} + \ldots + b_{\mu(m)}$ where, by Lemma 4.1, $\nu(x) = \mu(m) \in S$. Notice that $x \in H \cap G_{\mu(m)+1} = (H \cap G_{\mu(m)}) + (H \cap B_{\mu(m)})$. Consequently, we may write $x = y + b$ where $y \in H \cap G_{\mu(m)}$ and $b \in H \cap B_{\mu(m)}$. Thus

$$b_{\mu(1)} + \ldots + b_{\mu(m-1)} - y = b - b_{\mu(m)} \in B_{\mu(m)}.$$

Because $\mu(m) \in S$, $b_{\mu(1)} + \ldots + b_{\mu(m-1)} \in G(S)$ and we conclude that $y \in H \cap G(S)$. Moreover, $y \in G_{\mu(m)}$ implies that $\nu(y) < \mu(m) = \nu(x)$. Therefore, by induction, $y \in \sum_{\alpha \in S} (H \cap B_\alpha)$. Finally, since $b \in H \cap B_{\mu(m)}$, and since once again $\mu(m) \in S$, it follows that $x = y + b \in \sum_{\alpha \in S} (H \cap B_\alpha)$, as desired □

In order to apply Lemma 4.7, we require the following technical result.

Lemma 4.8. *Suppose that H is a subgroup of the global group G and that $\mathscr{C}_H$ and $\mathscr{C}_G$ are $G(\aleph_0)$-families in H and G, respectively. If $C \in \mathscr{C}_G$ with $C \cap H \in \mathscr{C}_H$ and if X is a countable subset of G, then there is a countable subgroup $B \in \mathscr{C}_G$ such that $X \subseteq B$, $B \cap H \in \mathscr{C}_H$, $C + B \in \mathscr{C}_G$ and $(C + B) \cap H = (C \cap H) + (B \cap H)$.*

Proof. First we observe that any $G(\aleph_0)$-family $\mathscr{C}$ in a global group G satisfies the following stronger version of (H3): if $C \in \mathscr{C}$ and if X is a countable subset of G, then there is a countable $B \in \mathscr{C}$ such that $X \subseteq B$ and $C + B \in \mathscr{C}$. Indeed, by repeated applications of (H3), there are two ascending families $\{C_n\}_{n<\omega} \subseteq \mathscr{C}$ and $\{B_n\}_{n<\omega} \subseteq \mathscr{C}$ such that $X \subseteq B_0$, each B_n is countable and, for all $n < \omega$, the two conditions

(a) $C + B_n \subseteq C_n$ with C_n/C countable, and
(b) $C_n \subseteq C + B_{n+1}$

are satisfied. Then $B = \bigcup_{n<\omega} B_n$ is a countable member of $\mathscr{C}$ with $C + B = \bigcup_{n<\omega} C_n \in \mathscr{C}$, as desired.

We now construct inductively two sequences $\{B_n\}_{n<\omega}$ and $\{A_n\}_{n<\omega}$ where, for each $n < \omega$, the following conditions hold:

(i) $B_n \in \mathscr{C}_G$ and $A_n \in \mathscr{C}_H$ with B_n and A_n countable.
(ii) $(C + B_n) \cap H \subseteq (C \cap H) + A_n$ and $B_n \cap H \subseteq A_n$.
(iii) $C + B_n \in \mathscr{C}_G$.
(iv) $B_n + A_n \subseteq B_{n+1}$.

By the above observation, we may begin the induction with a countable $B_0 \in \mathscr{C}_G$ such that $X \subseteq B_0$ and $C + B_0 \in \mathscr{C}_G$. Assuming that we have, for some $n < \omega$, a countable $B_n \in \mathscr{C}_G$ satisfying condition (iii), we demonstrate how to construct suitable A_n and B_{n+1}.

Since the countability of B_n implies that $((C + B_n) \cap H)/(C \cap H)$ is countable, we can certainly select a countable $A_n \in \mathscr{C}_H$ satisfying the first part of condition (ii). But then, if necessary, A_n can be enlarged within $\mathscr{C}_H$ to ensure that $B_n \cap H \subseteq A_n$. We complete the induction by applying our previous observation to select a countable subgroup $B_{n+1} \in \mathscr{C}_G$ that contains $A_n + B_n$ and has the property that $C + B_{n+1} \in \mathscr{C}_G$. Finally, take $B = \bigcup_{n<\omega} B_n$ and observe that B is countable with $X \subseteq B \in \mathscr{C}_G$, $B \cap H = \bigcup_{n<\omega} A_n \in \mathscr{C}_H$ and

$$C + B = C + \bigcup_{n<\omega} B_n = \bigcup_{n<\omega} (C + B_n) \in \mathscr{C}_G.$$

Moreover, $(C + B) \cap H = (C \cap H) + (B \cap H)$. □

We now have all the necessary ingredients to prove our main result.

Theorem 4.9 (Main Theorem). *Suppose that H is an isotype knice subgroup of a global Warfield group G. Then, H itself is a global Warfield group if and only if there is an $H(\aleph_0)$-family $\mathscr{D}$ in G/H consisting of almost strongly separable k-subgroups.*

Proof. Since an $H(\aleph_0)$-family in G/H is a $G(\aleph_0)$-family, Theorem 3.9 implies that if G/H has an $H(\aleph_0)$-family of almost strongly separable k-subgroups, then H is a Warfield group.

Conversely, suppose that H is a Warfield group. As in the proof of Theorem 3.10, we apply [12, Theorem 5.2] to obtain a $G(\aleph_0)$-family $\mathscr{C}_G$ in G with the property that $(H + N)/N$ is almost strongly separable in G/N for all $N \in \mathscr{C}_G$. Moreover, from Corollary 4.3 we conclude that $\mathscr{C}_G$ may be chosen so that each $N \in \mathscr{C}_G$ is also a pure knice subgroup of G that is almost strongly separable in G. In particular, $\mathscr{C} = \{N \in \mathscr{C}_G \colon H \parallel N\}$ is a $G(\aleph_0)$-family in G by Proposition 3.8. Furthermore, by Corollary 4.3, there exists an $H(\aleph_0)$-family $\mathscr{C}_H$ in H consisting of pure knice subgroups of H.

We shall first establish the existence of an $F(\aleph_0)$-family $\{G_\alpha\}_{\alpha<\tau}$ of pure subgroups in G with an associated family $\{B_\alpha\}_{\alpha<\tau}$ of countable pure subgroups where the following are satisfied for all $\alpha < \tau$:

(1) $G_\alpha \in \mathscr{C}$.
(2) $H \cap G_\alpha \in \mathscr{C}_H$.
(3) $G_{\alpha+1} = G_\alpha + B_\alpha$ with $B_\alpha \in \mathscr{C}$.
(4) $H \cap G_{\alpha+1} = (H \cap G_\alpha) + (H \cap B_\alpha)$ with $H \cap B_\alpha \in \mathscr{C}_H$.
(5) $H \parallel G_\alpha$.

Notice that (5) is a consequence of (1). We begin with a well-ordering $\{x_\beta\}_{\beta<\tau}$ of the elements of G. Proceeding by induction, assume that for some $\mu < \tau$ we have constructed $\{G_\alpha\}_{\alpha<\mu} \subseteq \mathscr{C}$ together with the corresponding family $\{B_\alpha\}_{\alpha+1<\mu} \subseteq \mathscr{C}$ such that conditions (1)–(4) are satisfied and $\{x_\beta\}_{\beta<\alpha} \subseteq G_\alpha$ whenever $\alpha < \mu$. It suffices to construct $G_\mu \in \mathscr{C}$ such that the enlarged family $\{G_\alpha\}_{\alpha\leqslant\mu}$ continues to satisfy all the requisite conditions. If μ is a limit ordinal, it suffices to take $G_\mu = \bigcup\limits_{\alpha<\mu} G_\alpha \in \mathscr{C}$ and note that $H \cap G_\mu = \bigcup\limits_{\alpha<\mu} (H \cap G_\alpha) \in \mathscr{C}_H$ with the remaining conditions (3) and (4) vacuously satisfied. On the other hand, if $\mu = \beta + 1$, then apply Lemma 4.8 to the the singleton $X = \{x_\beta\}$ with $C = G_\beta$ to obtain $B_\beta \in \mathscr{C}$ so that $G_{\beta+1} = G_\beta + B_\beta$ possesses all the desired properties.

By the remark preceding the statement of Lemma 4.6, there is already an $H(\aleph_0)$-family in G/H consisting of almost strongly separable subgroups. Thus, since the intersection of two $H(\aleph_0)$-families is an $H(\aleph_0)$-family, it suffices now to exhibit an $H(\aleph_0)$-family $\mathscr{D}$ in G/H consisting of k-subgroups of G/H. Using the G_α and B_α

constructed in the previous paragraph, set

$$\mathscr{C}_1 = \{G(S)\colon\ S \text{ is a closed subset of } \tau\}.$$

We know that $\mathscr{C}_1$ is an $H(\aleph_0)$-family in G. By condition (4) and Lemma 4.7, for each closed set S in τ, $H \cap G(S) = \sum_{\alpha\in S} (H \cap B_\alpha)$. Moreover, by the second part of condition (4), each $H \cap B_\alpha$ is in the $H(\aleph_0)$-family $\mathscr{C}_H$. Thus, $H \cap G(S) \in \mathscr{C}_H$, and in particular, $H \cap G(S)$ is a pure knice subgroup of H. It now follows from Theorem 2.1 that $H \cap G(S)$ is a knice subgroup of G. Next, by conditions (1) and (5), each G_α is an almost strongly separable pure subgroup of G such that $H \parallel G_\alpha$. Hence, from Lemma 4.6, there is an $H(\aleph_0)$-family $\mathscr{C}_2$ in G such that $H \parallel N$ for all $N \in \mathscr{C}_2$.

To complete the proof, set $\mathscr{C}_0 = \mathscr{C}_1 \cap \mathscr{C}_2$ and $\mathscr{D} = \{(N + H)/H\colon\ N \in \mathscr{C}_0\}$. Since $\mathscr{C}_0$ is an $H(\aleph_0)$-family in G, it follows easily that $\mathscr{D}$ is an $H(\aleph_0)$-family in G/H (for example, see [3, Lemma 1.3(a)]). Further observe that each $N \in \mathscr{C}_0$ is a pure knice subgroup of G with $H \parallel N$ and $H \cap N$ knice in G. Therefore, by Theorem 2.4, $\mathscr{D}$ consists of k-subgroups of G/H. □

From results in [8] and [13], it follows that every global k-group K has a *sequentially pure projective resolution*; that is, an exact sequence

$$(\dagger) \qquad \dots \longrightarrow A_{n+1} \xrightarrow{\varphi_{n+1}} A_n \longrightarrow \dots \longrightarrow A_1 \xrightarrow{\varphi_1} A_0 \xrightarrow{\varphi_0} K \longrightarrow 0$$

where, for each $n < \omega$, A_n is a global Warfield group and $\varphi_{n+1}(A_{n+1})$ is a k-group that is an isotype knice subgroup of A_n. If there exists a nonnegative integer k such that the resolution (†) can be chosen with $A_n = 0$ for all $n > k$, the smallest such k is called the *sequentially pure projective dimension* of K and we write $\dim K = k$. If no such k exists, we set $\dim K = \infty$. (By an obvious version of Schanuel's Lemma, $\dim K$ is well defined for each k-group K.) Thus, K is a global Warfield group if and only if $\dim K = 0$, while $\dim K \leqslant 1$ if and only if there is a short exact sequence

$$0 \longrightarrow A_1 \xrightarrow{\varphi_1} A_0 \xrightarrow{\varphi_0} K \longrightarrow 0$$

with $\varphi_1(A_1)$ an isotype knice Warfield subgroup of the global Warfield group A_0. Therefore, Theorem 4.9 can be reformulated as follows.

Corollary 4.10. *If K is a global k-group, then $\dim K \leqslant 1$ if and only if there is an $H(\aleph_0)$-family in K consisting of almost strongly separable k-subgroups.*

In a forthcoming paper, we intend to apply the results and techniques of this paper to the study of global k-groups K with $\dim K > 1$.

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