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$k$-systems, $k$-networks and $k$-covers


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Abstract. The concepts of $k$-systems, $k$-networks and $k$-covers were defined by A. Arhangel’skiı in 1964, P. O’Meara in 1971 and R. McCoy, I. Ntantu in 1985, respectively. In this paper the relationships among $k$-systems, $k$-networks and $k$-covers are further discussed and are established by $mk$-systems. As applications, some new characterizations of quotients or closed images of locally compact metric spaces are given by means of $mk$-systems.

Keywords: $k$-systems, $k$-networks, $k$-covers, $k$-spaces, point-countable families, hereditarily closure-preserving families

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1. Introduction

Let $X$ be a topological space and $\mathcal{P}$ a cover of $X$. $X$ is determined by $\mathcal{P}$ if $F \subset X$ is closed in $X$ if and only if $F \cap P$ is closed in $P$ for every $P \in \mathcal{P}$ [7]. $\mathcal{P}$ is called a $k$-system (resp. $mk$-system) of $X$ [1] (resp. [10]) if $X$ is determined by $\mathcal{P}$ and each element of $\mathcal{P}$ is compact (resp. metric and compact) in $X$. $\mathcal{P}$ is called a $k$-network for $X$ if, whenever $K \subset U$ with $K$ compact and $U$ open in $X$, then $K \subset \bigcup \mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$ [14]. $\mathcal{P}$ is called a compact (resp. closed) $k$-network if $\mathcal{P}$ is a $k$-network for $X$ and each element of $\mathcal{P}$ is compact (resp. closed) in $X$. $k$-systems and $k$-networks play an important role in quotient images of metric spaces and generalized metric spaces [18]. For example, Zhaowen Li and Jinjin Li [10] partly answered the Michael-Nagami’s problem by $mk$-systems; Shou Lin [11] obtained new characterizations of generalized metric spaces by compact $k$-networks; Y. Tanaka [16] proved the following interesting result.

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**Tanaka’s Theorem.** A Hausdorff space is a closed s-image of a locally compact metric space if and only if it is a Fréchet space which is determined by a point-countable cover of metric compact subspaces.

A generalization of the concept of k-networks is the following one of k-covers introduced by McCoy and Ntantu in [12]: A family $\mathcal{P}$ of subsets of a space $X$ is called a k-cover for $X$ if whenever $K$ is compact in $X$, then $K$ is covered by some finite subset of $\mathcal{P}$. k-covers are a basic tool in the theory of convergence properties and metrization theorems on function spaces. All this shows that k-systems, k-networks and k-covers are very interesting in study of mapping theory. In this paper the relationships among $mk$-systems, k-networks and k-covers are further discussed and are established by $mk$-systems. As applications, some new characterizations of quotient or closed images of locally compact metric spaces are given by means of $mk$-systems.

We recall some basic definitions. Let $f: X \to Y$ be a map.

1. $f$ is an s-map if $f^{-1}(y)$ is separable in $X$ for any $y \in Y$;
2. $f$ is a compact-covering map [13] if each compact subset of $Y$ is an image of some compact subset of $X$ under $f$.

A space $X$ is called a k-space if it is determined by the cover consisting of all compact subsets of $X$. A space $X$ is called a Fréchet space if, whenever $x \in A \subset X$, there is a sequence $\{x_n\}$ in $A$ with $x_n \to x$. Obviously, every Fréchet space is a k-space, and a space has a k-system if and only if it is a k-space. Every k-space is preserved by quotient maps, and every Fréchet space is preserved by closed maps.

Let $\mathcal{P}$ be a family of subsets of a space $X$ and denote $\mathcal{P}$ by $\{P_\alpha\}_{\alpha \in \Lambda}$. $\mathcal{P}$ is said to be point-countable if every point of $X$ belongs to at most countably many elements of $\mathcal{P}$. $\mathcal{P}$ is said to be closure-preserving if \[ \bigcup_{\alpha \in \Lambda'} P_\alpha = \bigcup_{\alpha \in \Lambda'} P_\alpha \] for each $\Lambda' \subset \Lambda$. $\mathcal{P}$ is said to be hereditarily closure-preserving (briefly, HCP) if \[ \bigcup_{\alpha \in \Lambda} Q_\alpha = \bigcup_{\alpha \in \Lambda} Q_\alpha \] whenever $Q_\alpha \subset P_\alpha$ for each $\alpha \in \Lambda$. A $\sigma$-hereditarily closure-preserving (briefly, $\sigma$-HCP) family is a collection that is the union of countably many hereditarily closure-preserving families.

Obviously, if $\mathcal{P}$ is an HCP-cover of closed subsets of a space $X$, then $X$ is determined by $\mathcal{P}$. In this paper, all spaces are Hausdorff spaces, and all maps are continuous and onto. $\mathbb{N}$ denotes the natural number set. Refer to [6] for terms which are not defined here.
2. Results

First of all, we discuss some relationships among $mk$-systems, $k$-networks and $k$-covers about point-countable covers. Y. Tanaka [17] proved that every point-countable $k$-system is a $k$-cover.

**Lemma 1.** Suppose $X$ is a $k$-space with a $k$-cover $\mathcal{P}$ consisting of compact subsets of $X$, then $\mathcal{P}$ is a $k$-system of $X$.

**Proof.** It is sufficient to show that $X$ is determined by the cover $\mathcal{P}$. Suppose that there exists a non-closed subset $F$ of $X$ such that $F \cap P$ is closed in $X$ for each $P \in \mathcal{P}$. Since $X$ is a $k$-space, $F \cap C$ is not closed in $X$ for some compact subset $C$ of $X$, and so $C \subset \bigcup \mathcal{P}'$ for some finite $\mathcal{P}' \subset \mathcal{P}$. However, $F \cap C = \{(F \cap P) \cap C : P \in \mathcal{P}'\}$ is closed in $X$, a contradiction. Hence $X$ is determined by $\mathcal{P}$, and $\mathcal{P}$ is a $k$-system of $X$. □

**Lemma 2.** If $X$ has a point-countable $k$-cover consisting of metric closed subspaces, then it has a point-countable closed $k$-network consisting of metric subspaces.

**Proof.** Let $\mathcal{P} = \{P_\alpha\}_{\alpha \in \Lambda}$ be a point-countable $k$-cover for $X$, where each $P_\alpha$ is a metric closed subspace of $X$. Then each $P_\alpha$ has a point-countable closed $k$-network $\mathcal{P}_\alpha$ by Nagata-Smirnov metrization theorem [6]. Put $\mathcal{P}' = \bigcup_{\alpha \in \Lambda} P_\alpha$. Then $\mathcal{P}'$ is a point-countable cover consisting of metric closed subsets of $X$. We shall show that $\mathcal{P}'$ is a $k$-network for $X$. For any $K \subset U$ with $K$ compact and $U$ open in $X$, since $\mathcal{P}$ is a $k$-cover for $X$, $K \subset \bigcup_{\alpha \in \Lambda} P_\alpha$ for some finite $\Lambda' \subset \Lambda$. For any $\alpha \in \Lambda'$, since $\mathcal{P}_\alpha$ is a $k$-network for $P_\alpha$, $K \cap P_\alpha \subset \bigcup_{\alpha' \in \Lambda'} P'_\alpha \subset U \cap P_\alpha$ for some finite $\mathcal{P}'_\alpha \subset \mathcal{P}_\alpha$. Let $\mathcal{P}'' = \bigcup_{\alpha \in \Lambda'} \mathcal{P}'_\alpha$. Then $\mathcal{P}''$ is a finite subset of $\mathcal{P}'$, and $K \subset \bigcup \mathcal{P}'' \subset U$. Thus $\mathcal{P}'$ is a $k$-network for $X$. □

The following example shows that the closedness of subsets is essential in Lemma 2.

**Example 3.** The Gillman-Jerison space $\psi(\mathbb{N})$ [2]: A locally compact space has a finite $k$-cover consisting of metric subspaces, which is not meta-Lindelöf.

**Proof.** Let $\mathcal{A}$ be a maximal almost disjoint family of $\mathbb{N}$. Let $\psi(\mathbb{N}) = \mathcal{A} \cup \mathbb{N}$ and describe a topology on $\psi(\mathbb{N})$ as follows: The points of $\mathbb{N}$ are isolated; basic neighborhoods of a point $A \in \mathcal{A}$ are sets of the form $\{A\} \cup (A \setminus F)$ where $F$ is a finite subset of $\mathbb{N}$. Then $\psi(\mathbb{N})$ is a locally compact space which is not meta-Lindelöf [2].

Let $\mathcal{P} = \{\mathcal{A}\} \cup \{\mathbb{N}\}$. Then $\mathcal{P}$ is a $k$-cover for $\psi(\mathbb{N})$ because it is finite. Since $\mathcal{A}$ is a closed discrete subset of $\psi(\mathbb{N})$, $\mathcal{P}$ is a $k$-cover consisting of metric subspaces. Since a locally compact space with a point-countable $k$-network has a point-countable base by Corollary 3.6 in [7], $\psi(\mathbb{N})$ has no point-countable $k$-network. □
Theorem 4. The following are equivalent for a space $X$:

1. $X$ has a point-countable $mk$-system;
2. $X$ is a $k$-space with a point-countable $k$-cover consisting of metric compact subspaces of $X$;
3. $X$ is a $k$-space with a point-countable compact $k$-network;
4. $X$ is a $k$-space with a point-countable closed $k$-network, and every first countable closed subspace of $X$ is locally compact;
5. $X$ is a (compact-covering and) quotient $s$-image of a locally compact metric space.

Proof. (1) $\iff$ (2) by Proposition 2.1 in [9], (2) $\implies$ (3) by Lemma 2, (3) $\iff$ (4) by Lemma 2.1 in [11] and Theorem 4.1 in [7], and (1) $\iff$ (5) by Theorem 1 in [10].

(3) $\implies$ (1). Suppose that $\mathcal{P}$ is a point-countable compact $k$-network for $X$. Each element of $\mathcal{P}$ is metrizable by Corollary 3.7 in [7]. Since every $k$-network is a $k$-cover, and $X$ is a $k$-space, $\mathcal{P}$ is a $mk$-system by Lemma 1.

The following examples show that the condition “$k$-spaces” and “metrizable properties” are essential in Theorem 4.

1. Let $\beta\mathbb{N}$ be the Stone-Čech compactification of $\mathbb{N}$, $p \in \beta\mathbb{N} \setminus \mathbb{N}$, and $X = \mathbb{N} \cup \{p\}$ with a subspace topology of $\beta\mathbb{N}$. Then every compact set of $X$ is finite, thus $X$ is a non-$k$-space with a point-countable compact $k$-network.

2. M. Sakai [15] or Huai-peng Chen [4] constructed a space $Y$ such that $Y$ has a point-countable closed $k$-network and every first countable closed subspace of $Y$ is compact, but $Y$ has no point-countable compact $k$-network.

3. $\beta\mathbb{N}$ is a $k$-space with a $k$-cover $\{\beta\mathbb{N}\}$, which is not metrizable.

By Tanaka’s theorem the following corollary holds.

Corollary 5. The following are equivalent for a space $X$:

1. $X$ is a closed $s$-image of a locally compact metric space;
2. $X$ is a Fréchet space with a point-countable $mk$-system;
3. $X$ is a Fréchet space with a point-countable compact $k$-network.

Question 6. Let $X$ be a regular and Fréchet space with a point-countable $k$-network. Is $X$ a space with a point-countable $k$-network consisting of separable subsets of $X$ if every first countable closed subspace of $X$ is locally separable?

Next, we discuss some relationships among $mk$-systems, $k$-networks and $k$-covers about HCP-families. The following example states that point-countable families cannot be replaced by $\sigma$-closure-preserving families in Lemma 2 or Theorem 4.

Example 7. There is a space $X$ with a closure-preserving $mk$-system, but $X$ having no $\sigma$-closure-preserving network.
The fact can be showed by Example 3.1 in [3]. Let \( I \) be the closed unit interval, and \( X = \mathbb{I} \times \mathbb{I} \). The set \( X \) is endowed with the following topology: each point in \( \mathbb{I} \times (0, 1) \) is isolated in \( X \); the local base of point \((s, 0) \in X\) consists of the sets of the form \( V \times \mathbb{I} \setminus \{(s) \times (0, 1)\}\) for each \( s \in \mathbb{I} \), where \( V \) is a neighborhood of \( s \) in \( \mathbb{I} \). Then \( X \) is a regular and first countable space with a closed map \( f: X \to \mathbb{I} \) with no Lindelöf fibre [3]. Thus \( X \) has no \( \sigma \)-closure-preserving network by Theorem 1.1 in [3].

Let \( \mathcal{F} = \{(x_n, y_n): n \in \mathbb{N}\}: \{x_n\} \) is a convergent sequence in \( \mathbb{I} \) with all \( x_n \)'s distinct and \( y_n \in (0, 1) \), \( Y = \mathbb{I} \times \{0\} \), and \( \mathcal{P} = \{Y\} \cup \{Y \cup S: S \in \mathcal{F}\} \).

For each \( S \in \mathcal{F} \), then \( S \) is metric and compact in \( X \), thus \( Y \cup S \) is a compact and metric subspace of \( X \), hence \( \mathcal{P} \) is a compact and metric cover of \( X \). If \( \mathcal{P}' \) is a subset of \( \mathcal{P} \), then \( Y \subset \bigcup \mathcal{P}' \), so \( \bigcup \mathcal{P}' \) is closed in \( X \), hence \( \mathcal{P} \) is closure-preserving in \( X \). Suppose a subset \( A \) of \( X \) is such that \( P \cap A \) is closed in \( P \) for each \( P \in \mathcal{P} \), we shall show that \( A \) is closed in \( X \). Let \( z \in X \setminus A \). If \( z \notin Y \), then \( \{z\} \) is open and \( \{z\} \cap A = \emptyset \). If \( z = (s, 0) \in Y \), put \( Z = A \cap Y \), then \( Z \) is closed, and \( z \notin Z \), thus there exists an open neighborhood \( V \) of \( s \) in \( \mathbb{I} \) with \( V \times \{0\} \cap Z = \emptyset \). Let \( D = \{x \in \mathbb{I}: \text{there is } y \in \mathbb{I} \text{ such that } (x, y) \in A \cap (V \times \mathbb{I})\} \), then \( D \) is finite. If not, there is a sequence \( \{(x_n, y_n)\} \) in \( A \) such that each \( x_n \in V \), all \( x_n \)'s are distinct and \( y_n \in (0, 1) \) because \( V \times \{0\} \cap Z = \emptyset \). We can assume that the sequence \( \{x_n\} \) is convergent to \( x_0 \in \mathbb{I} \), then \( x_0 \in V \), thus the sequence \( \{(x_n, y_n)\} \) converges to \( (x_0, 0) \) in \( X \). Take \( S = \{(x_n, y_n): n \in \mathbb{N}\} \), then \( S \in \mathcal{F} \) and \( (Y \cup S) \cap A = Z \cup S \). Since \( (x_0, 0) \notin Z \), \( (Y \cup S) \cap A \) is not closed, a contradiction. This shows that \( D \) is finite, so there exists an open neighborhood \( V' \) of \( s \) in \( \mathbb{I} \) with \( V' \subset V \) and \( V' \times \mathbb{I} \setminus \{(s) \times (0, 1)\} \cap A = \emptyset \), hence \( A \) is closed in \( X \). Therefore, \( X \) is determined by \( \mathcal{P} \), and \( X \) has a closure-preserving \( mk \)-system. \( \square \)

**Lemma 8.** If \( X \) has a \( \sigma \)-HCP \( k \)-cover consisting of metric closed subspaces, then it has a \( \sigma \)-HCP closed \( k \)-network consisting of metric subspaces.

**Proof.** Suppose \( \mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n \) is a \( \sigma \)-HCP \( k \)-cover consisting of metric closed subspaces of \( X \), where each \( \mathcal{P}_n \) is HCP. We can assume that each \( \mathcal{P}_n \subset \mathcal{P}_{n+1} \), and put \( X_n = \bigcup \mathcal{P}_n \), \( Z_n = \bigoplus \mathcal{P}_n \), and let \( f_n: Z_n \to X_n \) be the natural map. Then \( Z_n \) is a metric space, and \( f_n \) is a closed map because \( \mathcal{P}_n \) is HCP. By the Nagata-Smirnov metrization theorem, \( Z_n \) has a \( \sigma \)-locally finite closed \( k \)-network \( \mathcal{Q}_n \). Put \( \mathcal{R} = \bigcup_{n \in \mathbb{N}} f_n(\mathcal{Q}_n) \). Then \( \mathcal{R} \) is a \( \sigma \)-HCP cover consisting of closed subsets of \( X \) by the closeness of the map \( f_n \). If \( K \) is compact in \( X \), then \( K \subset X_m \) for some \( m \in \mathbb{N} \). In fact, suppose not, then \( K \setminus X_n \neq \emptyset \) for each \( n \in \mathbb{N} \), and so there exists a sequence \( \{x_i\} \) in \( K \) such that each \( x_i \in X_{n_{i+1}} \setminus X_{n_i} \) and \( n_i < n_{i+1} \). If \( D \) is a subset of \( \{x_i: i \in \mathbb{N}\} \) and \( P \in \mathcal{P} \), then \( P \in \mathcal{P}_n \) for some \( k \in \mathbb{N} \), thus \( D \cap P \subset \{x_i: i < k\} \) is finite.
By Lemma 1, $K$ is determined by $\mathcal{R}|_K = \{P \cap K : P \in \mathcal{R}\}$, $D$ is closed in $K$, thus $\{x_i : i \in \mathbb{N}\}$ is an infinite discrete subset of $K$, a contradiction to the compactness of $K$. We shall show that $\mathcal{R}$ is a $k$-network for $X$. For each $K \subset V$ with $K$ compact and $V$ open in $X$, then $K \subset X_m$ for some $m \in \mathbb{N}$. Since $f_m$ is a closed map, $f_m$ is compact-covering [13], i.e., there exists a compact subset $L$ in $Z_m$ such that $f_m(L) = K$. Because $\mathcal{D}_m$ is a $k$-network for $Z_m$, so $L \subset \bigcup \mathcal{D}'_m \subset f_m^{-1}(X_m \cap V)$ for some finite subset $\mathcal{D}'_m$ of $\mathcal{D}_m$. Thus $K \subset \bigcup f_m(\mathcal{D}'_m) \subset V$. Hence $\mathcal{R}$ is a $\sigma$-HCP closed $k$-network consisting of metric subspaces. □

The Gillman-Jerison space $\psi(\mathbb{N})$ in Example 3 shows that the closedness of subsets is essential in Lemma 8 because $\psi(\mathbb{N})$ has not any $\sigma$-HCP $k$-network by Corollary 6 in [5].

**Theorem 9.** The following are equivalent for a space $X$:

1. $X$ has a $\sigma$-HCP $mk$-system;
2. $X$ is a $k$-space with a $\sigma$-HCP $k$-cover consisting of metric compact subspaces of $X$;
3. $X$ is a $k$-space with a $\sigma$-HCP compact $k$-network;
4. $X$ is a $k$-space with a $\sigma$-HCP closed $k$-network, and every first countable closed subspace of $X$ is locally compact.

**Proof.** (3) ⇒ (1). Suppose $\mathcal{P}$ is a $\sigma$-HCP compact $k$-network for a $k$-space $X$. By Lemma 1, $\mathcal{P}$ is a $k$-system for $X$. Since $X$ has a $\sigma$-HCP $k$-network, $X$ is a $\sigma$-space (i.e., a regular space with a $\sigma$-locally finite network), and so each compact subset of $X$ is metrizable [6]. Thus $\mathcal{P}$ is a $\sigma$-HCP $mk$-system for $X$.

(1) ⇒ (2). Suppose $\mathcal{P}$ is a $\sigma$-HCP $mk$-system for $X$, then $X$ is a $k$-space. $\mathcal{P}$ is a $\sigma$-HCP $k$-cover consisting of metric compact subspaces of $X$ by Proposition 2.1 in [8].

(2) ⇒ (3) by Lemma 8, and (3) ⇔ (4) by Theorem 3.1 in [11]. □

**Corollary 10.** The following are equivalent for a space $X$:

1. $X$ is a closed image of a locally compact metric space;
2. $X$ is a Fréchet space with a $\sigma$-HCP $mk$-system;
3. $X$ has a HCP $mk$-system;
4. $X$ is a Fréchet space with a $\sigma$-HCP compact $k$-network.

**Proof.** (2) ⇔ (4) by Theorem 9, (1) ⇔ (4) by Corollary 3.2 in [11], and (2) ⇔ (3) by the proof of Theorem 2.5 in [8]. □
References


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