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k-SYSTEMS, k-NETWORKS AND k-COVERS

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Abstract. The concepts of k-systems, k-networks and k-covers were defined by A. Arhangel'skiĭ in 1964, P. O'Meara in 1971 and R. McCoy, I. Ntantu in 1985, respectively. In this paper the relationships among k-systems, k-networks and k-covers are further discussed and are established by mk-systems. As applications, some new characterizations of quotients or closed images of locally compact metric spaces are given by means of mk-systems.

Keywords: k-systems, k-networks, k-covers, k-spaces, point-countable families, hereditarily closure-preserving families

MSC 2000: 54D50, 54E45, 54C10

1. Introduction

Let X be a topological space and $\mathscr{P}$ a cover of X. X is determined by $\mathscr{P}$ if $F \subset X$ is closed in X if and only if $F \cap P$ is closed in P for every $P \in \mathscr{P}$ [7]. $\mathscr{P}$ is called a k-system (resp. mk-system) of X [1] (resp. [10]) if X is determined by $\mathscr{P}$ and each element of $\mathscr{P}$ is compact (resp. metric and compact) in X. $\mathscr{P}$ is called a k-network for X if, whenever $K \subset U$ with K compact and U open in X, then $K \subset \bigcup \mathscr{P}' \subset U$ for some finite $\mathscr{P}' \subset \mathscr{P}$ [14]. $\mathscr{P}$ is called a compact (resp. closed) k-network if $\mathscr{P}$ is a k-network for X and each element of $\mathscr{P}$ is compact (resp. closed) in X. k-systems and k-networks play an important role in quotient images of metric spaces and generalized metric spaces [18]. For example, Zhaowen Li and Jinjin Li [10] partly answered the Michael-Nagami's problem by mk-systems; Shou Lin [11] obtained new characterizations of generalized metric spaces by compact k-networks; Y. Tanaka [16] proved the following interesting result.

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Tanaka's Theorem. *A Hausdorff space is a closed s-image of a locally compact metric space if and only if it is a Fréchet space which is determined by a point-countable cover of metric compact subspaces.*

A generalization of the concept of k-networks is the following one of k-covers introduced by McCoy and Ntantu in [12]: A family $\mathscr{P}$ of subsets of a space X is called a k-cover for X if whenever K is compact in X, then K is covered by some finite subset of $\mathscr{P}$. k-covers are a basic tool in the theory of convergence properties and metrization theorems on function spaces. All this shows that k-systems, k-networks and k-covers are very interesting in study of mapping theory. In this paper the relationships among mk-systems, k-networks and k-covers are further discussed and are established by mk-systems. As applications, some new characterizations of quotient or closed images of locally compact metric spaces are given by means of mk-systems.

We recall some basic definitions. Let $f\colon X \to Y$ be a map.

(1) f is an s-map if $f^{-1}(y)$ is separable in X for any $y \in Y$;

(2) f is a compact-covering map [13] if each compact subset of Y is an image of some compact subset of X under f.

A space X is called a k-space if it is determined by the cover consisting of all compact subsets of X. A space X is called a Fréchet space if, whenever $x \in \bar{A} \subset X$, there is a sequence $\{x_n\}$ in A with $x_n \to x$. Obviously, every Fréchet space is a k-space, and a space has a k-system if and only if it is a k-space. Every k-space is preserved by quotient maps, and every Fréchet space is preserved by closed maps.

Let $\mathscr{P}$ be a family of subsets of a space X and denote $\mathscr{P}$ by $\{P_\alpha\}_{\alpha\in\Lambda}$. $\mathscr{P}$ is said to be point-countable if every point of X belongs to at most countably many elements of $\mathscr{P}$. $\mathscr{P}$ is said to be closure-preserving if $\bigcup\limits_{\alpha\in\Lambda'} \overline{P}_\alpha = \overline{\bigcup\limits_{\alpha\in\Lambda'} P_\alpha}$ for each $\Lambda' \subset \Lambda$. $\mathscr{P}$ is said to be hereditarily closure-preserving (briefly, HCP) if $\bigcup\limits_{\alpha\in\Lambda} \bar{Q}_\alpha = \overline{\bigcup\limits_{\alpha\in\Lambda} Q_\alpha}$ whenever $Q_\alpha \subset P_\alpha$ for each $\alpha \in \Lambda$. A σ-hereditarily closure-preserving (briefly, σ-HCP) family is a collection that is the union of countably many hereditarily closure-preserving families.

Obviously, if $\mathscr{P}$ is an HCP-cover of closed subsets of a space X, then X is determined by $\mathscr{P}$. In this paper, all spaces are *Hausdorff* spaces, and all maps are continuous and onto. $\mathbb{N}$ denotes the natural number set. Refer to [6] for terms which are not defined here.

2. Results

First of all, we discuss some relationships among mk-systems, k-networks and k-covers about point-countable covers. Y. Tanaka [17] proved that every point-countable k-system is a k-cover.

Lemma 1. *Suppose X is a k-space with a k-cover $\mathscr{P}$ consisting of compact subsets of X, then $\mathscr{P}$ is a k-system of X.*

Proof. It is sufficient to show that X is determined by the cover $\mathscr{P}$. Suppose that there exists a non-closed subset F of X such that $F \cap P$ is closed in X for each $P \in \mathscr{P}$. Since X is a k-space, $F \cap C$ is not closed in X for some compact subset C of X, and so $C \subset \bigcup \mathscr{P}'$ for some finite $\mathscr{P}' \subset \mathscr{P}$. However, $F \cap C = \{(F \cap P) \cap C \colon P \in \mathscr{P}'\}$ is closed in X, a contradiction. Hence X is determined by $\mathscr{P}$, and $\mathscr{P}$ is a k-system of X. $\square$

Lemma 2. *If X has a point-countable k-cover consisting of metric closed subspaces, then it has a point-countable closed k-network consisting of metric subspaces.*

Proof. Let $\mathscr{P} = \{P_\alpha\}_{\alpha \in \Lambda}$ be a point-countable k-cover for X, where each P_α is a metric closed subspace of X. Then each P_α has a point-countable closed k-network $\mathscr{P}_\alpha$ by Nagata-Smirnov metrization theorem [6]. Put $\mathscr{P}' = \bigcup\limits_{\alpha \in \Lambda} \mathscr{P}_\alpha$. Then $\mathscr{P}'$ is a point-countable cover consisting of metric closed subsets of X. We shall show that $\mathscr{P}'$ is a k-network for X. For any $K \subset U$ with K compact and U open in X, since $\mathscr{P}$ is a k-cover for X, $K \subset \bigcup\limits_{\alpha \in \Lambda'} P_\alpha$ for some finite $\Lambda' \subset \Lambda$. For any $\alpha \in \Lambda'$, since $\mathscr{P}_\alpha$ is a k-network for P_α, $K \cap P_\alpha \subset \bigcup \mathscr{P}'_\alpha \subset U \cap P_\alpha$ for some finite $\mathscr{P}'_\alpha \subset \mathscr{P}_\alpha$. Let $\mathscr{P}'' = \bigcup\limits_{\alpha \in \Lambda'} \mathscr{P}'_\alpha$. Then $\mathscr{P}''$ is a finite subset of $\mathscr{P}'$, and $K \subset \bigcup \mathscr{P}'' \subset U$. Thus $\mathscr{P}'$ is a k-network for X. $\square$

The following example shows that the closedness of subsets is essential in Lemma 2.

Example 3. The Gillman-Jerison space $\psi(\mathbb{N})$ [2]: A locally compact space has a finite k-cover consisting of metric subspaces, which is not meta-Lindelöf.

Proof. Let $\mathscr{A}$ be a maximal almost disjoint family of $\mathbb{N}$. Let $\psi(\mathbb{N}) = \mathscr{A} \cup \mathbb{N}$ and describe a topology on $\psi(\mathbb{N})$ as follows: The points of $\mathbb{N}$ are isolated; basic neighborhoods of a point $A \in \mathscr{A}$ are sets of the form $\{A\} \cup (A \setminus F)$ where F is a finite subset of $\mathbb{N}$. Then $\psi(\mathbb{N})$ is a locally compact space which is not meta-Lindelöf [2].

Let $\mathscr{P} = \{\mathscr{A}\} \cup \{\mathbb{N}\}$. Then $\mathscr{P}$ is a k-cover for $\psi(\mathbb{N})$ because it is finite. Since $\mathscr{A}$ is a closed discrete subset of $\psi(\mathbb{N})$, $\mathscr{P}$ is a k-cover consisting of metric subspaces. Since a locally compact space with a point-countable k-network has a point-countable base by Corollary 3.6 in [7], $\psi(\mathbb{N})$ has no point-countable k-network. $\square$

Theorem 4. *The following are equivalent for a space X:*

(1) *X has a point-countable mk-system;*

(2) *X is a k-space with a point-countable k-cover consisting of metric compact subspaces of X;*

(3) *X is a k-space with a point-countable compact k-network;*

(4) *X is a k-space with a point-countable closed k-network, and every first countable closed subspace of X is locally compact;*

(5) *X is a (compact-covering and) quotient s-image of a locally compact metric space.*

P r o o f. (1) $\Leftrightarrow$ (2) by Proposition 2.1 in [9], (2) $\Rightarrow$ (3) by Lemma 2, (3) $\Leftrightarrow$ (4) by Lemma 2.1 in [11] and Theorem 4.1 in [7], and (1) $\Leftrightarrow$ (5) by Theorem 1 in [10].

(3) $\Rightarrow$ (1). Suppose that $\mathscr{P}$ is a point-countable compact k-network for X. Each element of $\mathscr{P}$ is metrizable by Corollary 3.7 in [7]. Since every k-network is a k-cover, and X is a k-space, $\mathscr{P}$ is a mk-system by Lemma 1. $\square$

The following examples show that the condition "k-spaces" and "metrizable properties" are essential in Theorem 4.

(1) Let $\beta\mathbb{N}$ be the Stone-Čech compactification of $\mathbb{N}$, $p \in \beta\mathbb{N} \setminus \mathbb{N}$, and $X = \mathbb{N} \cup \{p\}$ with a subspace topology of $\beta\mathbb{N}$. Then every compact set of X is finite, thus X is a non-k-space with a point-countable compact k-network.

(2) M. Sakai [15] or Huaipeng Chen [4] constructed a space Y such that Y has a point-countable closed k-network and every first countable closed subspace of Y is compact, but Y has no point-countable compact k-network.

(3) $\beta\mathbb{N}$ is a k-space with a k-cover $\{\beta\mathbb{N}\}$, which is not metrizable.

By Tanaka's theorem the following corollary holds.

Corollary 5. *The following are equivalent for a space X:*

(1) *X is a closed s-image of a locally compact metric space;*

(2) *X is a Fréchet space with a point-countable mk-system;*

(3) *X is a Fréchet space with a point-countable compact k-network.*

Question 6. Let X be a regular and Fréchet space with a point-countable k-network. Is X a space with a point-countable k-network consisting of separable subsets of X if every first countable closed subspace of X is locally separable?

Next, we discuss some relationships among mk-systems, k-networks and k-covers about HCP-families. The following example states that point-countable families cannot be replaced by σ-closure-preserving families in Lemma 2 or Theorem 4.

Example 7. There is a space X with a closure-preserving mk-system, but X having no σ-closure-preserving network.

P r o o f. The fact can be showed by Example 3.1 in [3]. Let $\mathbb{I}$ be the closed unit interval, and $X = \mathbb{I} \times \mathbb{I}$. The set X is endowed with the following topology: each point in $\mathbb{I} \times (0,1]$ is isolated in X; the local base of point $(s,0) \in X$ consists of the sets of the form $V \times \mathbb{I} \setminus (\{s\} \times (0,1])$ for each $s \in \mathbb{I}$, where V is a neighborhood of s in $\mathbb{I}$. Then X is a regular and first countable space with a closed map $f\colon X \to \mathbb{I}$ with no Lindelöf fibre [3]. Thus X has no σ-closure-preserving network by Theorem 1.1 in [3].

Let $\mathscr{S} = \{\{(x_n, y_n)\colon n \in \mathbb{N}\}\colon \{x_n\}$ is a convergent sequence in $\mathbb{I}$ with all x_n's distinct and $y_n \in (0,1]\}$, $Y = \mathbb{I} \times \{0\}$, and $\mathscr{P} = \{Y\} \cup \{Y \cup S\colon S \in \mathscr{S}\}$.

For each $S \in \mathscr{S}$, then $\overline{S}$ is metric and compact in X, thus $Y \cup S$ is a compact and metric subspace of X, hence $\mathscr{P}$ is a compact and metric cover of X. If $\mathscr{P}'$ is a subset of $\mathscr{P}$, then $Y \subset \bigcup \mathscr{P}'$, so $\bigcup \mathscr{P}'$ is closed in X, hence $\mathscr{P}$ is closure-preserving in X. Suppose a subset A of X is such that $P \cap A$ is closed in P for each $P \in \mathscr{P}$, we shall show that A is closed in X. Let $z \in X \setminus A$. If $z \notin Y$, then $\{z\}$ is open and $\{z\} \cap A = \emptyset$. If $z = (s,0) \in Y$, put $Z = A \cap Y$, then Z is closed, and $z \notin Z$, thus there exists an open neighborhood V of s in $\mathbb{I}$ with $\overline{V \times \{0\}} \cap Z = \emptyset$. Let $D = \{x \in \mathbb{I}\colon$ there is $y \in \mathbb{I}$ such that $(x,y) \in A \cap (V \times \mathbb{I})\}$, then D is finite. If not, there is a sequence $\{(x_n, y_n)\}$ in A such that each $x_n \in V$, all x_n's are distinct and $y_n \in (0,1]$ because $(V \times \{0\}) \cap Z = \emptyset$. We can assume that the sequence $\{x_n\}$ is convergent to $x_0 \in \mathbb{I}$, then $x_0 \in \overline{V}$, thus the sequence $\{(x_n, y_n)\}$ converges to $(x_0, 0)$ in X. Take $S = \{(x_n, y_n)\colon n \in \mathbb{N}\}$, then $S \in \mathscr{S}$ and $(Y \cup S) \cap A = Z \cup S$. Since $(x_0, 0) \notin Z$, $(Y \cup S) \cap A$ is not closed, a contradiction. This shows that D is finite, so there exists an open neighborhood V' of s in $\mathbb{I}$ with $V' \subset V$ and $(V' \times \mathbb{I} \setminus (\{s\} \times (0,1])) \cap A = \emptyset$, hence A is closed in X. Therefore, X is determined by $\mathscr{P}$, and X has a closure-preserving mk-system. $\square$

Lemma 8. *If X has a σ-*HCP *k-cover consisting of metric closed subspaces, then it has a σ-*HCP *closed k-network consisting of metric subspaces.*

P r o o f. Suppose $\mathscr{P} = \bigcup\limits_{n \in \mathbb{N}} \mathscr{P}_n$ is a σ-HCP k-cover consisting of metric closed subspaces of X, where each $\mathscr{P}_n$ is HCP. We can assume that each $\mathscr{P}_n \subset \mathscr{P}_{n+1}$, and put $X_n = \bigcup \mathscr{P}_n$, $Z_n = \bigoplus \mathscr{P}_n$, and let $f_n\colon Z_n \to X_n$ be the natural map. Then Z_n is a metric space, and f_n is a closed map because $\mathscr{P}_n$ is HCP. By the Nagata-Smirnov metrization theorem, Z_n has a σ-locally finite closed k-network $\mathscr{Q}_n$. Put $\mathscr{R} = \bigcup\limits_{n \in \mathbb{N}} f_n(\mathscr{Q}_n)$. Then $\mathscr{R}$ is a σ-HCP cover consisting of closed subsets of X by the closeness of the map f_n. If K is compact in X, then $K \subset X_m$ for some $m \in \mathbb{N}$. In fact, suppose not, then $K \setminus X_n \neq \emptyset$ for each $n \in \mathbb{N}$, and so there exists a sequence $\{x_i\}$ in K such that each $x_i \in X_{n_{i+1}} \setminus X_{n_i}$ and $n_i < n_{i+1}$. If D is a subset of $\{x_i\colon i \in \mathbb{N}\}$ and $P \in \mathscr{P}$, then $P \in \mathscr{P}_{n_k}$ for some $k \in \mathbb{N}$, thus $D \cap P \subset \{x_i\colon i < k\}$ is finite.

By Lemma 1, K is determined by $\mathscr{P}_{|K} = \{P \cap K \colon P \in \mathscr{P}\}$, D is closed in K, thus $\{x_i \colon i \in \mathbb{N}\}$ is an infinite discrete subset of K, a contradiction to the compactness of K. We shall show that $\mathscr{R}$ is a k-network for X. For each $K \subset V$ with K compact and V open in X, then $K \subset X_m$ for some $m \in \mathbb{N}$. Since f_m is a closed map, f_m is compact-covering [13], i.e., there exists a compact subset L in Z_m such that $f_m(L) = K$. Because $\mathscr{Q}_m$ is a k-network for Z_m, so $L \subset \bigcup \mathscr{Q}'_m \subset f_m^{-1}(X_m \cap V)$ for some finite subset $\mathscr{Q}'_m$ of $\mathscr{Q}_m$. Thus $K \subset \bigcup f_m(\mathscr{Q}'_m) \subset V$. Hence $\mathscr{R}$ is a σ-HCP closed k-network consisting of metric subspaces. □

The Gillman-Jerison space $\psi(\mathbb{N})$ in Example 3 shows that the closedness of subsets is essential in Lemma 8 because $\psi(\mathbb{N})$ has not any σ-HCP k-network by Corollary 6 in [5].

Theorem 9. *The following are equivalent for a space X:*

(1) *X has a σ-HCP mk-system;*

(2) *X is a k-space with a σ-HCP k-cover consisting of metric compact subspaces of X;*

(3) *X is a k-space with a σ-HCP compact k-network;*

(4) *X is a k-space with a σ-HCP closed k-network, and every first countable closed subspace of X is locally compact.*

P r o o f. (3) $\Rightarrow$ (1). Suppose $\mathscr{P}$ is a σ-HCP compact k-network for a k-space X. By Lemma 1, $\mathscr{P}$ is a k-system for X. Since X has a σ-HCP k-network, X is a σ-space (i.e., a regular space with a σ-locally finite network), and so each compact subset of X is metrizable [6]. Thus $\mathscr{P}$ is a σ-HCP mk-system for X.

(1) $\Rightarrow$ (2). Suppose $\mathscr{P}$ is a σ-HCP mk-system for X, then X is a k-space. $\mathscr{P}$ is a σ-HCP k-cover consisting of metric compact subspaces of X by Proposition 2.1 in [8].

(2) $\Rightarrow$ (3) by Lemma 8, and (3) $\Leftrightarrow$ (4) by Theorem 3.1 in [11]. □

Corollary 10. *The following are equivalent for a space X:*

(1) *X is a closed image of a locally compact metric space;*

(2) *X is a Fréchet space with a σ-HCP mk-system;*

(3) *X has a HCP mk-system;*

(4) *X is a Fréchet space with a σ-HCP compact k-network.*

P r o o f. (2) $\Leftrightarrow$ (4) by Theorem 9, (1) $\Leftrightarrow$ (4) by Corollary 3.2 in [11], and (2) $\Leftrightarrow$ (3) by the proof of Theorem 2.5 in [8]. □

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