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INDECOMPOSABLE MATRICES OVER
A DISTRIBUTIVE LATTICE

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Abstract. In this paper, the concepts of indecomposable matrices and fully indecomposable matrices over a distributive lattice L are introduced, and some algebraic properties of them are obtained. Also, some characterizations of the set $F_n(L)$ of all $n \times n$ fully indecomposable matrices as a subsemigroup of the semigroup $H_n(L)$ of all $n \times n$ Hall matrices over the lattice L are given.

Keywords: distributive lattice, indecomposable matrix, fully indecomposable matrix, semigroup, characterization

MSC 2000: 15A33, 15A18

1. INTRODUCTION

The concept of indecomposable nonnegative matrices first appeared in 1912 in a paper by Frobenius [1] dealing with the spectral properties of nonnegative matrices, and the concept of fully indecomposable nonnegative matrices was introduced by Marcus and Minc [2]. Their properties and characterizations have been studied by many authors.

In 1973, Š. Schwarz [3] was the first to introduce the concepts of indecomposable Boolean matrices (or indecomposable relations) and fully indecomposable Boolean matrices (or fully indecomposable relations), and obtained some algebraic properties of them. Since then, a number of works in this area were published (see e.g. [4]–[10]).

In this paper we shall develop these concepts, introduce the concepts of indecomposable matrices and fully indecomposable matrices over a distributive lattice L and

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give some algebraic properties and characterizations of them. In Section 3, we obtain some algebraic properties of indecomposable matrices and fully indecomposable matrices over the lattice L . In Section 4, we shall show that the set $F_n(L)$ of all $n \times n$ fully indecomposable matrices over the lattice L forms a nilpotent semigroup having the universal matrix as the zero element and that the index of nilpotency of $F_n(L)$ is equal to the number $n - 1$. Also, we show that $F_n(L)$ is the maximal nilpotent ideal of the semigroup $H_n(L)$ of all $n \times n$ Hall matrices over the lattice L . Some results obtained in this paper generalize former results on Boolean matrices in [3].

2. DEFINITIONS AND PRELIMINARY LEMMAS

Let (L, \leq, \vee, \wedge) be a distributive lattice with the least and the greatest elements 0 and 1, respectively. The join $a \vee b$ and the meet $a \wedge b$ of a, b in L will be denoted by $a + b$ and $a \cdot b$ (or ab), respectively. It is clear that if L is a linear lattice, especially the Boolean algebra $B_0 = \{0, 1\}$ or the fuzzy algebra $F = [0, 1]$, then $a + b = \max\{a, b\}$ and $ab = \min\{a, b\}$ for all a and b in L .

Let $V_n(L)$ ($n \geq 1$) denote the set of all n -tuples (n -vectors) over the lattice L . For $\alpha = (a_1, a_2, \dots, a_n)$, $\beta = (b_1, b_2, \dots, b_n)$ in $V_n(L)$ we define $\alpha + \beta = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$ and $\alpha \leq \beta \iff a_i \leq b_i$ for $i = 1, 2, \dots, n$; we also define $\alpha < \beta \iff \alpha \leq \beta$ and there exists $i \in \{1, 2, \dots, n\}$ such that $a_i < b_i$. The *norm* of a vector α is defined by $\|\alpha\| = \sum_{i=1}^n a_i$. Let $0 = (0, 0, \dots, 0)$ and $e = (1, 1, \dots, 1)$. The vector 0 is called the *zero vector* of $V_n(L)$. Let e_i denote the n -tuple with 1 as its i th coordinate, 0 otherwise.

The multiplication of a vector α by a scalar λ in L is defined by $\lambda\alpha = (\lambda a_1, \dots, \lambda a_n)$. The vector α is called a *constant vector* if $\alpha = \lambda e = (\lambda, \lambda, \dots, \lambda)$ for some λ in L , otherwise, α is called *nonconstant*.

Let $M_n(L)$ ($n \geq 1$) be the set of $n \times n$ matrices over L (*lattice matrices*). We shall denote by A_{ij} or a_{ij} the element of L which is the (i, j) -entry of A in $M_n(L)$. We define:

$A + B = C$ iff $c_{ij} = a_{ij} + b_{ij}$ for $i, j = 1, 2, \dots, n$, $A \leq B$ iff $a_{ij} \leq b_{ij}$ for $i, j = 1, 2, \dots, n$, $A < B$ iff $A \leq B$ and $a_{ij} < b_{ij}$ for some couple $i, j \in \{1, 2, \dots, n\}$, $AB = C$ iff $c_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$ for $i, j = 1, 2, \dots, n$, $A^T = C$ iff $c_{ij} = a_{ji}$ for $i, j = 1, 2, \dots, n$,

$$I_n = (\delta_{ij}), \quad \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \quad i, j = 1, 2, \dots, n,$$

$J_n = (a_{ij})$, where $a_{ij} = 1$ for $i, j = 1, 2, \dots, n$. J_n is called the *universal matrix*.

Further, $A^0 = I_n$, $A^{k+1} = A^k A$, $k = 0, 1, 2, \dots$. We shall denote by a_{ij}^k the element at the (i, j) -entry of A^k .

The following properties are derived immediately from these definitions.

- a) $M_n(L)$ is a monoid with respect to multiplication.
- b) $(M_n(L), +, \cdot)$ is a semiring and for any A, B, C and D in $M_n(L)$, $A + A = A$, and if $A \leq B$ and $C \leq D$ then $AC \leq BD$.

A matrix in $M_n(L)$ is called a *permutation matrix* if one of the elements of its every row and every column is 1 and the others are 0. A matrix A in $M_n(L)$ is called *invertible* if there exists a matrix B in $M_n(L)$ such that $AB = BA = I_n$. The matrix B is called the *inverse* of A and is denoted by A^{-1} .

It is clear that the set $S_n(L)$ of all invertible matrices in $M_n(L)$ is the group of the units of the monoid $M_n(L)$.

Remark 2.1. A square matrix A over the Boolean algebra B_0 is invertible iff A is a permutation matrix.

A matrix A in $M_n(L)$ is called a *Hall matrix* (see [11]) if there exists a matrix P in $S_n(L)$ such that $P \leq A$. The matrix A is called *reflexive* if $I_n \leq A$. It is clear that the set $H_n(L)$ of all $n \times n$ Hall matrices over L forms a subsemigroup of the semigroup $M_n(L)$ and contains the group $S_n(L)$.

A set $S = \{a_1, a_2, \dots, a_m\}$ of elements in L is called a *decomposition* of 1 in L if $\sum_{i=1}^m a_i = 1$; S is called *orthogonal* if $a_i a_j = 0$ holds for all i and j provided that $i \neq j$; S is called an *orthogonal decomposition* of 1 in L if it is orthogonal and a decomposition of 1 in L .

A semigroup S with zero element z_0 is called *nilpotent with the index of nilpotency* l if $S^l = \{z_0\}$ while $S^{l-1} \neq \{z_0\}$. A two-sided ideal (or ideal) Q of a semigroup S is called a *prime ideal* if $V \cdot W \subseteq Q$ implies either $V \subseteq Q$ or $W \subseteq Q$ for all two-sided ideals V, W of S , where $V \cdot W = \{vw : v \in V, w \in W\}$ and $S^l = \{s_1 s_2 \dots s_l : s_i \in S, i = 1, 2, \dots, l\}$.

The following lemmas will be used:

Lemma 2.1. Let L be a distributive lattice. Then

- (1) for $a, a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ in L , we have

$$\sum_{i=1}^n a_i b_i = \prod_{U \subseteq N} \left(\sum_{i \in U} a_i + \sum_{j \in N-U} b_j \right) \quad \text{and} \quad a + \prod_{i=1}^n a_i = \prod_{i=1}^n (a + a_i)$$

where $N = \{1, 2, \dots, n\}$;

(2) for $a_{ij} \in L$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, we have

$$\prod_{i \in N} \left(\sum_{j \in M} a_{ij} \right) = \sum_{\sigma \in m(N, M)} \prod_{i \in N} a_{i\sigma(i)}$$

where $M = \{1, 2, \dots, m\}$ and $m(N, M)$ is the set of all maps from N to M .

Proof. (1) can be obtained from Lemma 2.1 in [12]; (2) is the dual of Lemma 2.1 (2) in [13]. \square

Lemma 2.2. Let $A \in M_n(L)$. Then the following statements are all equivalent.

- (1) A is invertible;
- (2) $AA^T = A^T A = I_n$;
- (3) there exists a positive integer l such that $A^l = I_n$;
- (4) each row and each column of A is an orthogonal decomposition of 1 in L .

Proof. The proof of Lemma 2.2 can be found in [14]. \square

Remark 2.2. Note that if A is invertible in $M_n(L)$ then $A^{-1} = A^T$.

Lemma 2.3. Let $P = (p_{ij}) \in S_n(L)$. Then

- (1) for any $\alpha, \beta \in V_n(L)$, $\alpha < \beta \Rightarrow \alpha P < \beta P$;
- (2) for any α in $V_n(L)$, α is a constant vector iff αP is a constant vector;
- (3) $\sum_{i \in U, j \in V} p_{ij} = 1$ for any $U, V \subseteq N$ with $|U| + |V| \geq n + 1$.

Proof. (1) First, it is clear that $\alpha < \beta$ implies $\alpha P \leq \beta P$. Suppose that $\alpha P = \beta P$. Then $\alpha P P^{-1} = \beta P P^{-1}$, and so $\alpha = \beta$, which contradicts our hypothesis. This proves (1).

(2) Let $\alpha = \lambda e$ for some λ in L . Then $\alpha P = (\lambda e)P = \lambda e$ (by Lemma 2.2 (4)). Conversely, if $\alpha P = \lambda e$ for some $\lambda \in L$ then $\alpha = (\lambda e)P^{-1} = \lambda e$. This proves (2).

(3) Since $p_{i1} + p_{i2} + \dots + p_{in} = 1$ for any $i \in U$, we have

$$\begin{aligned} 1 &= \prod_{i \in U} \left(\sum_{j \in N-V} p_{ij} + \sum_{j \in V} p_{ij} \right) \\ &\leq \prod_{i \in U} \left(\sum_{k \in N-V} p_{ik} + \sum_{i \in U, j \in V} p_{ij} \right) \\ &= \prod_{i \in U} \left(\sum_{k \in N-V} p_{ik} \right) + \sum_{i \in U, j \in V} p_{ij} \quad (\text{by Lemma 2.1 (1)}) \\ &= \sum_{\sigma \in m(U, N-V)} \prod_{i \in U} p_{i\sigma(i)} + \sum_{i \in U, j \in V} p_{ij} \quad (\text{by Lemma 2.1 (2)}), \end{aligned}$$

where $m(U, N - V)$ is the set of all maps of U to $N - V$.

Since $|U| \geq (n - |V|) + 1 = |N - V| + 1$, for any $\sigma \in m(U, N - V)$ there must be a couple $s, t \in U$ such that $\sigma(s) = \sigma(t)$. Therefore $\prod_{i \in U} p_{i\sigma(i)} = 0$ for all σ in $m(U, N - V)$ (by Lemma 2.2 (4)). Hence $\sum_{i \in U, j \in V} p_{ij} = 1$. This proves (3). \square

Lemma 2.4. *Let $A \in H_n(L)$. Then*

- (1) *there exists a matrix P in $S_n(L)$ such that $\alpha \leq \alpha(PA)$ for any α in $V_n(L)$;*
- (2) *if $I_n \leq A$, then $A^k = A^{n-1}$ holds for $k \geq n$.*

Proof. (1) Let $A \in H_n(L)$. Then there exists a matrix Q in $S_n(L)$ such that $Q \leq A$. Let $Q^{-1} = P$. Then $P \in S_n(L)$ and $I_n \leq PA$. Clearly, $\alpha \leq \alpha(PA)$ for any α in $V_n(L)$.

(2) can be obtained from Theorem 4 in [14]. \square

3. INDECOMPOSABLE LATTICE MATRICES AND FULLY INDECOMPOSABLE LATTICE MATRICES

In this section, we shall introduce the concepts of indecomposable matrices and fully indecomposable matrices over a lattice L , and discuss some of their properties.

To do this, we first recall the notions of indecomposable Boolean matrices and fully indecomposable Boolean matrices and give some of their characterizations.

Definition 3.1. Let $A \in M_n(B_0)$. A is said to be *decomposable* if there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} B & O \\ C & D \end{bmatrix}$$

where B and D are square. Otherwise, A is called *indecomposable*; A is said to be *partly decomposable* if there exist permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} B & O \\ C & D \end{bmatrix}$$

where B and D are square. Otherwise, A is called *fully indecomposable*.

Remark 3.1. Note that a matrix $A \in M_n(B_0)$ is indecomposable if and only if there is no proper nonempty subset U of the set $N = \{1, 2, \dots, n\}$ such that $a_{ij} = 0$ for all $i \in U$ and $j \in N - U$.

Remark 3.2. Note that any fully indecomposable Boolean matrix is indecomposable.

Proposition 3.1. *Let $A \in M_n(B_0)$. Then*

(1) *A is indecomposable if and only if*

$$(I_n + A)^{n-1} = J_n,$$

(2) *A is fully indecomposable if and only if for any k in $\{1, 2, \dots, n-1\}$, every $k \times n$ ($n \times k$) submatrix of A has at least $k+1$ columns ($k+1$ rows) which are not zero vectors,*

(3) *A is fully indecomposable if and only if there exists a permutation matrix P such that $I_n \leq PA$ and PA is indecomposable.*

Proof. (1) *Sufficiency:* Suppose that A is decomposable. Then, by Remark 3.1, there exists a proper nonempty subset U of N such that $a_{ij} = 0$ for all $i \in U$ and $j \in N - U$. Now let $u \in U$ and $v \in N - U$. Since $J_n = (I_n + A)^{n-1} = I_n + A + \dots + A^{n-1}$, we have $(I_n + A + \dots + A^{n-1})_{uv} = 1$, and so there exists a k in $\{1, 2, \dots, n-1\}$ such that $(A^k)_{uv} = 1$. But

$$(A^k)_{uv} = \sum_{1 \leq i_1, \dots, i_{k-1} \leq n} a_{ui_1} a_{i_1 i_2} \dots a_{i_{k-1} v},$$

hence there exists a sequence i_1, \dots, i_{k-1} such that $a_{ui_1} = a_{i_1 i_2} = \dots = a_{i_{k-1} v} = 1$. Let i_t be the last member in the sequence $i_0, i_1, \dots, i_{k-1}, i_k$ which is in U (taking $i_0 = u$ and $i_k = v$). Then $i_t \in U$ and $i_{t+1} \in N - U$. But $a_{i_t i_{t+1}} = 1$, a contradiction.

Necessity: Suppose that A is indecomposable. Then by Proposition 5.2.3 in [15] we have that for any $i, j \in N$, there exists a sequence $\gamma_1, \dots, r_{k(i,j)-1}$ such that $a_{i\gamma_1} = a_{\gamma_1\gamma_2} = \dots = a_{\gamma_{k(i,j)-1}j} = 1$ (including the empty sequence with $a_{ij} = 1$). Therefore $(A^{k(i,j)})_{ij} = 1$. Let $k = \max_{i,j \in N} \{k(i,j)\}$. Then $((I_n + A)^k)_{ij} = (I_n + A + \dots + A^k)_{ij} = 1$ for all i, j in N , and so $(I_n + A)^k = J_n$. Since $I_n \leq I_n + A$, we have $(I_n + A)^m = (I_n + A)^{n-1}$ for all $m \geq n$ (by Lemma 2.4(2)). If $k \geq n$, then $(I_n + A)^{n-1} = (I_n + A)^k = J_n$; if $k \leq n-1$, then $J_n = (I_n + A)^k \leq (I_n + A)^{n-1}$, and so $(I_n + A)^{n-1} = J_n$. This proves (1).

(2) By Definition 3.1, A is partly decomposable if and only if A contains an $s \times (n-s)$ zero submatrix with $1 \leq s \leq n-1$. That is to say, A is fully indecomposable if and only if for any $s \times t$ zero submatrix of A we have $s+t \leq n-1$. Therefore, A is fully indecomposable if and only if for any k in $\{1, 2, \dots, n-1\}$, every $k \times n$ ($n \times k$) submatrix of A has at least $k+1$ columns ($k+1$ rows) which are not zero vectors. This proves (2).

(3) *Sufficiency:* Let $B = PA$. Then $I_n \leq B$ and B is indecomposable. Let $B[U|V]$ denote the $|U| \times |V|$ submatrix of B consisting precisely of those elements b_{ij} of B for which $i \in U$ and $j \in V$, where U and V are nonempty subsets of the set N . Then for

any proper nonempty subset U of N , the matrix $B[U|N - U]$ is not the zero matrix (by Remark 3.1) and $I_k \leq B[U|U]$, where $k = |U|$, and so the matrix $B[U|N]$ has at least $k + 1$ columns which are not zero vectors. By (2), B is fully indecomposable and so is A .

Necessity: Suppose that A is fully indecomposable. Then the first row of A has at least two elements which are 1, say $a_{1j_1} = a_{1j'_1} = 1$, where $j_1 \neq j'_1$. By (2), the j_1 th column of A has at least two elements which are 1. Assume that $a_{1j_1} = a_{2j_1} = 1$ without loss of generality. By (2), the second row of A has at least two elements a_{2j_2} and $a_{2j'_2}$ such that $a_{2j_2} = a_{2j'_2} = 1$ and $j_2 \neq j_1$. Similarly, the k th row ($3 \leq k \leq n$) of A has at least two elements a_{kj_k} and $a_{kj'_k}$ such that $a_{kj_k} = a_{kj'_k} = 1$ and $j_k \notin \{j_1, j_2, \dots, j_{k-1}\}$. Therefore, we have that $a_{1j_1} = a_{2j_2} = \dots = a_{nj_n} = 1$ and that j_1, j_2, \dots, j_n are distinct. Now put $\bar{A} = (\bar{a}_{il})_{n \times n}$ such that

$$\bar{a}_{il} = \begin{cases} a_{il} & \text{if } l = j_i, \\ 0 & \text{if } l \neq j_i. \end{cases}$$

It is clear that \bar{A} is a permutation matrix and $\bar{A} \leq A$. Let $P = (\bar{A})^{-1}$. Then P is a permutation matrix and $I_n \leq PA$. Clearly, PA is indecomposable. This proves (3). \square

By Proposition 3.1, the indecomposable Boolean matrices and the fully indecomposable Boolean matrices can be described as follows:

Definition 3.1'. Let $A \in M_n(B_0)$. A is called *indecomposable* if $(I_n + A)^{n-1} = J_n$; A is called *fully indecomposable* if there exists a permutation matrix P such that $I_n \leq PA$ and PA is indecomposable.

Now we introduce the concepts of indecomposable matrices and fully indecomposable matrices over a lattice L .

Definition 3.2. Let $A \in M_n(L)$. A is said to be *indecomposable* if $(I_n + A)^{n-1} = J_n$; A is said to be *fully indecomposable* if there exists a P in $S_n(L)$ such that $I_n \leq PA$ and PA is indecomposable.

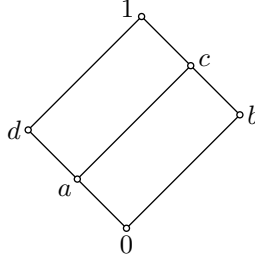
The sets of indecomposable matrices and fully indecomposable matrices in $M_n(L)$ are denoted by $I_n(L)$ and $F_n(L)$, respectively.

Example 3.1. Consider the lattice $L = \{0, a, b, c, d, 1\}$ whose diagram is shown below:

It is easy to see that L is a distributive lattice.

Now let

$$A = \begin{bmatrix} 0 & d & b \\ c & 0 & d \\ d & 1 & 0 \end{bmatrix}$$



and

$$B = \begin{bmatrix} b & 1 & d \\ d & b & 1 \\ 1 & d & b \end{bmatrix}.$$

Then

$$I_3 + A = \begin{bmatrix} 1 & d & b \\ c & 1 & d \\ d & 1 & 1 \end{bmatrix} \quad \text{and} \quad (I_3 + A)^2 = J_3,$$

and so A is indecomposable.

Let $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. It is clear that $P \in S_n(L)$ and $PB = \begin{bmatrix} 1 & d & b \\ b & 1 & d \\ d & b & 1 \end{bmatrix}$.

Therefore $I_n \leq PB$. Since $(PB)^2 = J_3$, we have that A is fully indecomposable.

Proposition 3.2. *Let $A \in I_n(L)$. Then*

- (1) *for any P in $S_n(L)$, we have $PAP^T \in I_n(L)$;*
- (2) *if $\sum_{i=1}^n a_{ii} = 1$, then $A^{2n-1} = J_n$.*

Proof. (1) Let $A \in I_n(L)$. Then $(I_n + A)^{n-1} = J_n$. Therefore

$$(I_n + PAP^T)^{n-1} = (P(I_n + A)P^T)^{n-1} = P(I_n + A)^{n-1}P^T = PJ_nP^T = J_n,$$

and so

$$PAP^T \in I_n(L).$$

This proves (1).

(2) Since $A = (a_{ij}) \in I_n(L)$ we have

$$I_n + A + A^2 + \dots + A^{n-1} = J_n,$$

and so

$$(3.1) \quad a_{ij} + a_{ij}^2 + \dots + a_{ij}^{n-1} = 1 \quad \text{for all } i \neq j.$$

For any i and j in $N = \{1, 2, \dots, n\}$, we have

$$\begin{aligned} a_{ij}^{2n-1} &= \sum_{1 \leq i_1, \dots, i_{2n-2} \leq n} a_{ii_1} a_{i_1 i_2} \dots a_{i_{2n-2} j} \geq \sum_{k=1}^n \sum_{p+d+r=2n-1} a_{ik}^p a_{kk}^d a_{kj}^r \\ &\geq \sum_{k=1}^n a_{kk} \sum_{p+r \leq 2n-2} a_{ik}^p a_{kj}^r \quad (\text{because } a_{kk}^d \geq a_{kk}) \\ &\geq \sum_{k=1}^n a_{kk} \left(\sum_{p=1}^{n-1} a_{ik}^p \right) \left(\sum_{r=1}^{n-1} a_{kj}^r \right). \end{aligned}$$

Case I: $i \neq j$. In this case

$$\begin{aligned} a_{ij}^{2n-1} &\geq \sum_{k \neq i, j} a_{kk} + a_{ii} \left(\sum_{p=1}^{n-1} a_{ii}^p \right) + a_{jj} \left(\sum_{r=1}^{n-1} a_{jj}^r \right) \quad (\text{by (3.1)}) \\ &= \sum_{k=1}^n a_{kk} = 1. \end{aligned}$$

Case II: $i = j$. In this case

$$a_{ii}^{2n-1} \geq \sum_{k \neq i} a_{kk} + a_{ii} \left(\sum_{p=1}^{n-1} a_{ii}^p \right) = \sum_{k=1}^n a_{kk} = 1.$$

Therefore $A^{2n-1} = J_n$. This proves (2). □

Proposition 3.3. Let $A = (a_{ij}) \in F_n(L)$. Then

- (1) $A \in H_n(L) \cap I_n(F)$;
- (2) for any P_1, P_2 in $S_n(L)$, $P_1 A P_2 \in F_n(L)$;
- (3) for any nonempty subsets U, V of $N = \{1, 2, \dots, n\}$ with $|U| + |V| \geq n$, we have

$$\sum_{i \in U, j \in V} a_{ij} = 1.$$

Proof. (1) Clearly, $A \in H_n(L)$. Now we shall show that $A \in I_n(L)$. Since $A \in F_n(L)$, there exists a matrix P in $S_n(L)$ such that $I_n \leq PA$ and PA is indecomposable. Therefore

$$\begin{aligned} I_n + A &= I_n + P^{-1}(PA) = I_n + P^{-1}(I_n + PA) \quad (\text{because } I_n \leq PA) \\ &= (I_n + A) + (I_n + P^{-1}). \end{aligned}$$

By Lemma 2.2 (3), there exists a positive integer l such that $P^l = I_n$. Thus $P^{l-1} = P^{-1}$. Since the integers l and $l-1$ are relatively prime, there exists a positive integer u such that $u(l-1) \equiv 1 \pmod{l}$, and so $P^{u(l-1)} = P$. Now

$$\begin{aligned} (I_n + A)^u &= ((I_n + A) + (I_n + P^{l-1}))^u \geq (I_n + A)^u + (I_n + P^{l-1})^u \\ &\geq A + P^{u(l-1)} = A + P. \end{aligned}$$

Thus

$$(I_n + A)^{2u} \geq (A + P)^2 \geq PA = I_n + PA,$$

and so

$$\begin{aligned} (I_n + A)^{n-1} &= (I_n + A)^{2u(n-1)} \quad (\text{by Lemma 2.4 (2)}) \\ &\geq (I_n + PA)^{n-1} = J_n \quad (\text{because } PA \text{ is indecomposable}). \end{aligned}$$

Then $(I_n + A)^{n-1} = J_n$, i.e., A is indecomposable. This proves (1).

(2) Let $A \in F_n(L)$. Then there exists a P in $S_n(L)$ such that $I_n \leq PA$ and $PA \in I_n(L)$. Let $Q = P_2^{-1}PP_1^{-1}$. Then $Q \in S_n(L)$ and

$$Q(P_1AP_2) = (P_2^{-1}PP_1^{-1})(P_1AP_2) = P_2^{-1}(PA)P_2 \geq P_2^{-1}I_nP_2 = I_n.$$

Furthermore, $Q(P_1AP_2) = P_2^{-1}(PA)P_2$ is indecomposable since PA is indecomposable. Therefore, P_1AP_2 is fully indecomposable. This proves (2).

(3) By the definition of A , there exists P in $S_n(L)$ such that $I_n \leq PA$ and PA is indecomposable. Thus, we have $I_n < PA$. Let $PA = B = (b_{ij})$. Then

$$b_{ii} = 1 \quad \text{for } i = 1, 2, \dots, n,$$

and so

$$\sum_{i \in U, j \in V} b_{ij} = 1 \quad \text{for any } U \text{ and } V \text{ with } U \cap V \neq \emptyset.$$

If $U \cap V = \emptyset$, then $U \cup V = N$. Let now $\alpha = \sum_{i \in U} e_i + \lambda \sum_{i \in V} e_i$, where $\lambda \in L$ and $\lambda \neq 1$. Then

$$\alpha B = \left(\sum_{i \in U} b_{i1} + \lambda \sum_{i \in V} b_{i1}, \dots, \sum_{i \in U} b_{in} + \lambda \sum_{i \in V} b_{in} \right).$$

For any $j \in N$, if $j \in U$, then $\sum_{i \in U} b_{ij} + \lambda \sum_{i \in V} b_{ij} = 1$; if $j \in V$, then $\sum_{i \in U} b_{ij} + \lambda \sum_{i \in V} b_{ij} = \sum_{i \in U} b_{ij} + \lambda$. Since $\alpha < \alpha B$, there is $j \in V$ such that $\sum_{i \in U} b_{ij} + \lambda > \lambda$, and so $\sum_{i \in U, j \in V} b_{ij} + \lambda > \lambda$ for all $\lambda \in L$ with $\lambda \neq 1$. Thus $\sum_{i \in U, j \in V} b_{ij} = 1$.

Now $A = P^{-1}B$. Let $P^{-1} = (d_{ij})$. Then

$$\begin{aligned} \sum_{i \in U, j \in V} a_{ij} &= \sum_{i \in U} \sum_{j \in V} \sum_{t=1}^n d_{it} b_{tj} = \sum_{t=1}^n \left(\sum_{i \in U} d_{it} \right) \left(\sum_{j \in V} b_{tj} \right) \\ &= \prod_{W \subseteq N} \left(\sum_{i \in U, s \in W} d_{is} + \sum_{j \in V, t \in N-W} b_{tj} \right) \quad (\text{by Lemma 2.1 (1)}). \end{aligned}$$

$$\text{Let } \Delta(W) = \sum_{i \in U, s \in W} d_{is} + \sum_{j \in V, t \in N-W} b_{tj}.$$

If $(N - W) \cap V \neq \emptyset$ or $|V| + |N - W| \geq n$, then $\sum_{j \in V, t \in N-W} b_{tj} = 1$, and so $\Delta(W) = 1$.

If $(N - W) \cap V = \emptyset$ and $|V| + |N - W| \leq n - 1$, then $V \not\subseteq W$, and so $|W| \geq |V| + 1$. Thus $|U| + |W| \geq |U| + |V| + 1 \geq n + 1$, and so $\sum_{i \in U, s \in W} d_{is} = 1$ (by Lemma 2.3 (3)). Therefore $\Delta(W) = 1$ for any $W \subseteq N$. Hence $\sum_{i \in U, j \in V} a_{ij} = 1$. This proves (3). \square

Proposition 3.4. *Let $A \in M_n(L)$. Then $A \in F_n(L)$ iff there exists a P in $S_n(L)$ such that*

$$\alpha < \alpha PA$$

for any nonconstant vector α in $V_n(L)$.

Proof. Suppose that there exists a matrix P in $S_n(L)$ such that $\alpha < \alpha PA$ for any nonconstant vector α in $V_n(L)$. Take $\alpha = e_1, e_2, \dots, e_n$. Then

$$\begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} < \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} PA.$$

Therefore, we have $I_n < PA$, and so $\alpha \leq \alpha PA \leq \dots \leq \alpha (PA)^{n-1} \leq \alpha (PA)^n$ for any α in $V_n(L)$. By Lemma 2.4 (2), we have $(PA)^{n-1} = (PA)^n$. Therefore $\alpha (PA)^{n-1} = \alpha (PA)^{n-1} (PA)$ for any α in $V_n(L)$, and so $\alpha (PA)^{n-1} = \lambda_\alpha e$ for some λ_α in L . If we take $\alpha = e_1, e_2, \dots, e_n$, then $e_i (PA)^{n-1} = e$, and so

$$\begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} (PA)^{n-1} = J_n.$$

Thus $(I_n + PA)^{n-1} = (PA)^{n-1} = J_n$. Hence $A \in F_n(L)$.

Conversely, suppose that $A \in F_n(L)$. Then there exists a matrix P in $S_n(L)$ such that $I_n \leq PA$ and $PA \in I_n(L)$, and so $\alpha \leq \alpha PA$ for all vectors α in $V_n(L)$. If $\alpha = \alpha PA$, then

$$\alpha = \alpha PA = \alpha(PA)^2 = \dots = \alpha(PA)^{n-1} = \alpha J_n = \|\alpha\|e.$$

Therefore $\alpha < \alpha PA$ for any nonconstant vector α in $V_n(L)$. This proves the proposition. \square

At the end of this section, we will introduce the concepts of weakly indecomposable matrices and weakly fully indecomposable matrices over the lattice L .

Definition 3.3. Let $A \in M_n(L)$. A is called *weakly decomposable* if there exists a matrix P in $S_n(L)$ such that

$$PAP^T = \begin{bmatrix} B & O \\ C & D \end{bmatrix}$$

where B and D are square. Otherwise, A is called *weakly indecomposable*; A is called *weakly partly decomposable* if there exist matrices P and Q in $S_n(L)$ such that

$$PAQ = \begin{bmatrix} B & O \\ C & D \end{bmatrix}$$

where B and D are square. Otherwise, A is called *weakly fully indecomposable*.

Remark 3.3. Note that any indecomposable matrix is weakly indecomposable and any fully indecomposable matrix is weakly fully indecomposable over the lattice L . However, the converse is not true.

Example 3.2. Consider the lattice L from Example 3.1. Let $A = \begin{bmatrix} a & c \\ d & b \end{bmatrix}$, $B = \begin{bmatrix} a & c \\ d & a \end{bmatrix} \in M_2(L)$. For any $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ and $D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$ in $S_n(L)$, using Lemma 2.2 (4), we have

$$PAP^T = \begin{pmatrix} p_{11}a + p_{12}b & p_{12}p_{21}d + p_{11}p_{22}c \\ p_{11}p_{22}d + p_{12}p_{21}c & p_{21}a + p_{22}b \end{pmatrix}.$$

Since

$$\begin{aligned} p_{12}p_{21}d + p_{11}p_{22}c &\geq (p_{11}p_{22} + p_{12}p_{21})a \\ &= (p_{12} + p_{11}p_{22})(p_{21} + p_{11}p_{22})a \quad (\text{by Lemma 2.1 (1)}) \\ &= (p_{12} + p_{11})(p_{12} + p_{22})(p_{21} + p_{11})(p_{21} + p_{22})a \\ &\hspace{15em} (\text{by Lemma 2.1 (1)}) \\ &= a \quad (\text{by Lemma 2.2 (4)}) > 0, \end{aligned}$$

A is weakly indecomposable. But $(I_2 + A)^{2-1} = \begin{bmatrix} 1 & c \\ d & 1 \end{bmatrix} \neq J_2$, hence A is not indecomposable.

Also,

$$\begin{aligned} PBD &\geq \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \\ &= \begin{bmatrix} (p_{11} + p_{12})a & (p_{11} + p_{12})a \\ (p_{21} + p_{22})a & (p_{21} + p_{22})a \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \\ &= \begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \quad (\text{by Lemma 2.2 (4)}) \\ &= \begin{bmatrix} (d_{11} + d_{21})a & (d_{12} + d_{22})a \\ (d_{11} + d_{21})a & (d_{12} + d_{22})a \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix} \quad (\text{by Lemma 2.2 (4)}). \end{aligned}$$

Therefore, B is weakly fully indecomposable.

For any $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \in S_n(L)$, we have

$$PB = \begin{bmatrix} p_{11}a + p_{12}d & p_{11}c + p_{12}a \\ p_{21}a + p_{22}d & p_{21}c + p_{22}a \end{bmatrix}.$$

Since $p_{11}a + p_{12}d \leq a + d = d < 1$ and $p_{21}c + p_{22}a \leq c + a = c < 1$, we have that $I_2 \not\leq PB$. Thus B is not fully indecomposable.

Remark 3.4. If L is the Boolean algebra B_0 , then the concept of indecomposable matrices coincides with that of weakly indecomposable matrices and the concept of fully indecomposable matrices coincides with that of weakly fully indecomposable matrices over L .

4. THE SEMIGROUP OF FULLY INDECOMPOSABLE LATTICE MATRICES

In this section, we shall give some characterizations of $F_n(L)$ as a semigroup.

Theorem 4.1.

- (1) $F_n(L)$ is a nilpotent semigroup having J_n as the zero element.
- (2) The index of nilpotency of $F_n(L)$ is equal to the number $n - 1$.

Proof. (1) J_n is clearly the zero element of $F_n(L)$ since $J_n \in F_n(L)$ and for any $A \in F_n(L)$ we have $AJ_n = J_nA = J_n$. Suppose that $A, B \in F_n(L)$. Then

there exist P_1, P_2 in $S_n(L)$ such that $I_n \leq P_1A$, $I_n \leq P_2B$, $(P_1A)^{n-1} = J_n$ and $(P_2B)^{n-1} = J_n$. Therefore

$$\begin{aligned} I_n &\leq P_2B = P_2I_nB \leq P_2(P_1A)B = (P_2P_1)(AB), \\ J_n &= (P_2B)^{n-1} = (P_2I_nB)^{n-1} \leq (P_2(P_1A)B)^{n-1} = ((P_2P_1)(AB))^{n-1}, \end{aligned}$$

and so

$$J_n = ((P_2P_1)(AB))^{n-1}.$$

Let now $P = P_2P_1$. Then $P \in S_n(L)$, $I_n \leq P(AB)$ and $J_n = (P(AB))^{n-1}$, and so $AB \in F_n(L)$. Hence, $F_n(L)$ is a semigroup.

Suppose that $A_1, A_2, \dots, A_{n-1} \in F_n(L)$. Let $T = A_1A_2 \dots A_{n-1}$, $A_l = (a_{ij}^{(l)})$, $l = 1, 2, \dots, n-1$. Then

$$t_{ij} = \sum_{1 \leq i_1, \dots, i_{n-2} \leq n} a_{ii_1}^{(1)} a_{i_1i_2}^{(2)} \dots a_{i_{n-2}j}^{(n-1)}.$$

Let $\Delta_{ij}^{(0)} = t_{ij}$ and $\Delta_{ij}^{(l)} = \sum_{1 \leq i_{l+1}, \dots, i_{n-2} \leq n} a_{ii_{l+1}}^{(l+1)} \dots a_{i_{n-2}j}^{(n-1)}$, $l = 1, 2, \dots, n-2$. It is clear that

$$\Delta_{ij}^{(l)} = \sum_{i_{l+1}=1}^n a_{ii_{l+1}}^{(l+1)} \Delta_{i_{l+1}j}^{(l+1)}.$$

Hence

$$\begin{aligned} t_{ij} &= \sum_{i_1=1}^n a_{ii_1}^{(1)} \Delta_{i_1j}^{(1)} \\ &= \prod_{U_1 \subseteq N} \left(\sum_{i_1 \in U_1} a_{ii_1}^{(1)} + \sum_{j_1 \in N-U_1} \Delta_{j_1j}^{(1)} \right) \quad (\text{by Lemma 2.1 (1)}) \\ &= \prod_{U_1 \subseteq N} \left(\sum_{i_1 \in U_1} a_{ii_1}^{(1)} + \sum_{j_1 \in N-U_1} \left(\sum_{i_2=1}^n a_{j_1i_2}^{(2)} \Delta_{i_2j}^{(2)} \right) \right) \\ &= \prod_{U_1 \subseteq N} \left(\sum_{i_1 \in U_1} a_{ii_1}^{(1)} + \sum_{i_2=1}^n \left(\sum_{j_1 \in N-U_1} a_{j_1i_2}^{(2)} \right) \Delta_{i_2j}^{(2)} \right) \\ &= \prod_{U_1 \subseteq N} \left(\sum_{i_1 \in U_1} a_{ii_1}^{(1)} + \prod_{U_2 \subseteq N} \left(\sum_{i_2 \in U_2} \left(\sum_{j_1 \in N-U_1} a_{j_1i_2}^{(2)} \right) + \sum_{j_2 \in N-U_2} \Delta_{j_2j}^{(2)} \right) \right) \\ &= \prod_{U_1, U_2 \subseteq N} \left(\sum_{i_1 \in U_1} a_{ii_1}^{(1)} + \sum_{i_2 \in U_2} \sum_{j_1 \in N-U_1} a_{j_1i_2}^{(2)} + \sum_{j_2 \in N-U_2} \Delta_{j_2j}^{(2)} \right) \\ &\hspace{15em} (\text{by Lemma 2.1 (1)}). \end{aligned}$$

Repeating this process we can obtain that

$$t_{ij} = \prod_{U_1, \dots, U_{n-2} \subseteq N} \left(\sum_{i_1 \in U_1} a_{ii_1}^{(1)} + \sum_{i_2 \in U_2, j_1 \in N-U_1} a_{j_1 i_2}^{(2)} + \dots \right. \\ \left. + \sum_{i_{n-2} \in U_{n-2}, j_{n-3} \in N-U_{n-3}} a_{j_{n-3} i_{n-2}}^{(n-2)} + \sum_{j_{n-2} \in N-U_{n-2}} a_{j_{n-2} j}^{(n-1)} \right).$$

For any $U_1, U_2, \dots, U_{n-2} \subseteq N$, let

$$\Delta(U_1, \dots, U_{n-2}) = \sum_{i_1 \in U_1} a_{ii_1}^{(1)} + \sum_{i_2 \in U_2, j_1 \in N-U_1} a_{j_1 i_2}^{(2)} + \dots \\ + \sum_{i_{n-2} \in U_{n-2}, j_{n-3} \in N-U_{n-3}} a_{j_{n-3} i_{n-2}}^{(n-2)} + \sum_{j_{n-2} \in N-U_{n-2}} a_{j_{n-2} j}^{(n-1)}.$$

If $|U_1| \geq n-1$, then $\sum_{i_1 \in U_1} a_{ii_1}^{(1)} = 1$ (by Proposition 3.3 (3)), and so $\Delta(U_1, \dots, U_{n-2}) =$

1. Similarly, if $|N - U_{n-2}| \geq n-1$, we have that $\sum_{j_{n-2} \in N-U_{n-2}} a_{j_{n-2} j}^{(n-1)} = 1$, and so $\Delta(U_1, \dots, U_{n-2}) = 1$. This means that $\Delta(U_1, \dots, U_{n-2}) = 1$ if $|U_1| \geq n-1$ or $|N - U_{n-2}| \geq n-1$. Let now $|U_1| \leq n-2$ and $|N - U_{n-2}| \leq n-2$. Since

$$|U_1| + (|U_2| + |N - U_1|) + \dots + (|U_{n-2}| + |N - U_{n-3}|) + |N - U_{n-2}| = (n-2)n,$$

we have

$$(|U_2| + |N - U_1|) + \dots + (|U_{n-2}| + |N - U_{n-3}|) \geq (n-2)n - 2(n-2) = (n-2)^2.$$

Hence there must be an l in $\{1, 2, \dots, n-3\}$ such that $|U_{l+1}| + |N - U_l| \geq n$. Since $A_{l+1} \in F_n(L)$, we have

$$\sum_{i_{l+1} \in U_{l+1}, j_l \in N-U_l} a_{j_l i_{l+1}}^{(l+1)} = 1 \quad (\text{by Proposition 3.3 (3)}),$$

and so $\Delta(U_1, \dots, U_{n-2}) = 1$.

Therefore, we have $\Delta(U_1, \dots, U_{n-2}) = 1$ for all $U_1, \dots, U_{n-2} \subseteq N$, and so

$$t_{ij} = \prod_{U_1, U_2, \dots, U_{n-2} \subseteq N} \Delta(U_1, \dots, U_{n-2}) = 1, \quad \text{i.e., } T = J_n.$$

Hence $F_n(L)$ is a nilpotent semigroup having J_n as the zero element and $(F_n(L))^{n-1} = \{J_n\}$. This proves (1).

(2) By (1), the index of nilpotency of $F_n(L) \leq n - 1$. To show that the index of nilpotency is exactly $n - 1$, it is sufficient to show that for any $n > 1$ there is an $A \in F_n(L)$ such that $A^{n-2} \neq J_n$. It is easy to prove that the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

has this property. This proves (2). □

Remark 4.1. Theorem 4.1 is a generalization of Theorem 1.2 in [3].

Corollary 4.1. For any $A \in F_n(L)$ we have $A^{n-1} = J_n$.

We now give some characterizations of $F_n(L)$ as a subsemigroup of $H_n(L)$.

Theorem 4.2. The set $F_n(L)$ is a two-sided ideal of $H_n(L)$.

Proof. Suppose that $A \in F_n(L)$ and $B \in H_n(L)$. Then there exist P_1 and P_2 in $S_n(L)$ such that $I_n \leq P_1A$, $I_n \leq P_2B$ and $(P_1A)^{n-1} = J_n$. Therefore $I_n \leq P_2B \leq P_2(P_1A)B = (P_2P_1)(AB)$, $I_n \leq P_1A \leq P_1(P_2B)A = (P_1P_2)(BA)$, $J_n = (P_1A)^{n-1} \leq (P_1(P_2B)A)^{n-1} = ((P_1P_2)(BA))^{n-1}$, and so $((P_1P_2)(BA))^{n-1} = J_n$. Also,

$$J_n = (P_1A)^{n-1} \leq ((P_1A)(BP_2))^{n-1} = (P_1(AB)P_2)^{n-1} = P_2^{-1}((P_2P_1)(AB))^{n-1}P_2.$$

This implies $J_n \leq ((P_2P_1)(AB))^{n-1}$. Thus $((P_2P_1)(AB))^{n-1} = J_n$. Since $P_1P_2, P_2P_1 \in S_n(L)$, we have $AB, BA \in F_n(L)$. This proves that $F_n(L)$ is a two-sided ideal of $H_n(L)$. □

Remark 4.2. Theorem 4.2 is a generalization of Theorem 2.3 in [3].

Definition 4.1. A matrix A in $H_n(L)$ is called *strongly nilpotent* if P_1AP_2 is nilpotent for any P_1 and P_2 in $S_n(L)$, i.e., $(P_1AP_2)^k = J_n$ for some positive integer k .

Theorem 4.3. The semigroup $F_n(L)$ is exactly the set of all strongly nilpotent elements in $H_n(L)$.

Proof. Let $A \in F_n(L)$. Then by Proposition 3.3 (2), $P_1AP_2 \in F_n(L)$ for any P_1, P_2 in $S_n(L)$, and so P_1AP_2 is nilpotent for any P_1, P_2 in $S_n(L)$ by Corollary 4.1.

Conversely, let $A \in H_n(L)$ and let P_1AP_2 be nilpotent for any P_1, P_2 in $S_n(L)$. Since $A \in H_n(L)$, there exists a P in $S_n(L)$ such that $I_n \leq PA$, and so $\alpha \leq \alpha PA$

for any α in $V_n(L)$. If $\alpha = \alpha PA$, then $\alpha = \alpha(PA) = \alpha(PA)^2 = \dots = \alpha(PA)^k = \dots$. But PA is nilpotent, hence there exists an integer k such that $(PA)^k = J_n$ and so $\alpha = \alpha(PA)^k = \alpha J_n = \|\alpha\|e$. Therefore $\alpha < \alpha(PA)$ if α is nonconstant, and so $A \in F_n(L)$ by Proposition 3.4. \square

Theorem 4.4.

- (1) $F_n(L)$ is the maximal nilpotent ideal of $H_n(L)$.
- (2) The semigroup $F_n(L)$ is precisely the intersection of all prime ideals of $H_n(L)$.

P r o o f. (1) Suppose that U is a nilpotent ideal of $H_n(L)$ and $F_n(L) \subsetneq U$. Then there is a nilpotent element $A \in U - F_n(L)$. Since $A \in U$, we have also $P_1AP_2 \in U$ for any P_1, P_2 in $S_n(L) \subseteq H_n(L)$. On the other hand, since $A \notin F_n(L)$, A is not strongly nilpotent, and so there is a couple P_3, P_4 in $S_n(L)$ such that $(P_3AP_4)^k < J_n$ for all k . That is, P_3AP_4 is not nilpotent, a contradiction with the supposition that U is nilpotent.

(2) We first prove that $F_n(L)$ is contained in any prime ideal of $H_n(L)$. Let Q be a prime ideal of $H_n(L)$. Since $F_n(L)^{n-1} = \{J_n\}$ and $J_n \in Q$, $F_n(L) \cdot F_n(L)^{n-2} \subseteq Q$ implies either $F_n(L) \subseteq Q$, in which case our statement is proved, or $F_n(L)^{n-2} \subseteq Q$. This implies $F_n(L) \cdot F_n(L)^{n-3} \subseteq Q$, hence again either $F_n(L) \subseteq Q$ or $F_n(L)^{n-3} \subseteq Q$. Repeating this argument we find $F_n(L) \subseteq Q$.

Our assertion will be proved if we are able to prove that for any $B \in H_n(L) - F_n(L)$ there is a prime ideal Q_B such that $B \notin Q_B$.

Note first that if $B \in H_n(L) - F_n(L)$, then $P_1BP_2 \in H_n(L) - F_n(L)$ for any P_1, P_2 in $S_n(L)$. For, if there were $P_3BP_4 \in F_n(L)$ for some P_3, P_4 in $S_n(L)$, this would imply $P_3^{-1}(P_3BP_4)P_4^{-1} = B \in F_n(L)$, contrary to the choice of B .

Now since $B \notin F_n(L)$, there are P_5, P_6 in $S_n(L)$ such that the matrix $C = P_5BP_6$ is not nilpotent. Hence no member of the sequence

$$(4.1) \quad C, C^2, \dots, C^k, \dots$$

is contained in $F_n(L)$.

Let Q_B be the largest ideal of $H_n(L)$ which does not meet any element of the sequence (4.1). Then Q_B is not empty since it contains $F_n(L)$. We state that Q_B is a prime ideal of $H_n(L)$. Suppose for an indirect proof that there are two ideals V and W of $H_n(L)$ such that $V \not\subseteq Q_B, W \not\subseteq Q_B$ and $V \cdot W \subseteq Q_B$. Since $Q_B \subsetneq Q_B \cup V$ and $Q_B \subsetneq Q_B \cup W$, there are some powers C^u and C^v such that $C^u \in Q_B \cup V, C^v \in Q_B \cup W$, and so $C^u \in V, C^v \in W$. Therefore $C^{u+v} \in V \cdot W \subseteq Q_B$, contrary to the construction of Q_B . Now B is not contained in the ideal Q_B , since otherwise $P_5BP_6 = C$ would be contained in Q_B , contrary to the choice of C . This completes the proof of our statement. \square

Remark 4.3. Theorem 4.4 generalizes Theorems 2.7 and 2.8 in [3].

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