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REAL HYPERSURFACES WITH CONSTANT TOTALLY REAL BISECTIONAL CURVATURE IN COMPLEX SPACE FORMS

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Abstract. In this paper we classify real hypersurfaces with constant totally real bisectional curvature in a non flat complex space form $M_m(c)$, $c \neq 0$ as those which have constant holomorphic sectional curvature given in [6] and [13] or constant totally real sectional curvature given in [11].

Keywords: real hypersurfaces, totally real bisectional curvature, sectional curvature, holomorphic sectional curvature

MSC 2000: 53C40, 53C15

Introduction

The sectional curvature offers a lot of information concerning the intrinsic geometry of a Riemannian manifold. For instance, manifolds which have constant sectional curvature have been a great source of study. In complex manifolds, the holomorphic sectional curvature and the totally real sectional curvature arise naturally and it is known that the constancy of the holomorphic sectional curvature is equivalent to the constancy of the totally real sectional curvature (See Goldberg and Kobayashi [4] and Houh [5]).

A complex m-dimensional Kaehler manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_m(c)$. A complete and simply connected complex space form is either a complex projective space $P^m(c)$,

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a complex Euclidean space $\mathbb{C}^m$ or a complex hyperbolic space $H^m(c)$, provided $c > 0$, $c = 0$ or $c < 0$, respectively.

Let M be a connected real hypersurface of a non flat complex space form $M_m(c)$, $c \neq 0$, N a local unit normal vector field to M. If J is the almost complex structure of $M_m(c)$, $c \neq 0$, we will denote $\xi = -JN$. Given a vector field X tangent to M, we will write $JX = \varphi X + \eta(X)N$, where φX and $\eta(X)N$ are the tangential and the normal component of JX, respectively. We recall that M is ruled if the distribution $\mathfrak{D}(p) = \{X \in T_pM\colon\ X \perp \xi\}$, $p \in M$, is integrable and its leaves are totally geodesic $M_{m-1}(c)$.

If π is a 2-plane included in $\mathfrak{D}(p)$, where $p \in M$, we will say that π is totally real if $\varphi\pi$ is orthogonal to π. We denote by $T(\pi) = T(X,Y)$ the sectional curvature of a totally real 2-plane $\pi = \mathrm{Span}\{X,Y\}$ included in $\mathfrak{D}(p)$, $p \in M$, and we will call it the totally real sectional curvature of π. If $T(\pi)$ is constant for any π included in $\mathfrak{D}(p)$ and any $p \in M$, we will say that M has a constant totally real sectional curvature. If the complex dimension of the complex space form is $m = 2$, there are no totally real 2-planes tangent to M. Therefore, the totally real sectional curvature is meaningful when $m \geqslant 3$.

On the other hand, Bishop and Goldberg [2] introduced the notion of totally real bisectional curvature $B(X,Y)$ on a Kaehler manifold M. It is determined by a totally real plane $[X,Y]$ and its image $[JX,JY]$ by the complex structure J, where $[X,Y]$ denotes the plane spanned by linearly independent vector fields X and Y. Moreover, the above two planes $[X,Y]$ and $[JX,JY]$ are orthogonal to each other. And it is known that two orthonormal vectors X and Y span a totally real plane if and only if X, Y and JY are orthonormal.

Houh [5] showed that an $m(\geqslant 3)$-dimensional Kaehler manifold with a constant totally real bisectional curvature is congruent to a complex space form of a constant holomorphic sectional curvature $H(X) = c$, where $H(X)$ is determined by the holomorphic plane $[X,JX]$. Also Barros and Romero[1] asserted that for a connected indefinite Kaehler manifold M with complex dimension $m \geqslant 3$ to be an indefinite complex space form with a constant holomorphic sectional curvature c a necessary and sufficient condition is to have a constant totally real bisectional curvature $\frac{1}{2}c$ at any point.

Goldberg and Kobayashi [4] introduced the notion of a holomorphic bisectional curvature $H(X,Y)$, which is determined by two holomorphic planes $[X,JX]$ and $[Y,JY]$, and asserted that a complex projective space $P^m(c)$ is the only compact Kaehler manifold with a positive holomorphic bisectional curvature $H(X,Y)$ and a constant scalar curvature. If we compare the notion of $B(X,Y)$ with $H(X,Y)$ and $H(X)$, the holomorphic bisectional curvature $H(X,Y)$ turns out to be the totally real bisectional curvature $B(X,Y)$ (respectively, the holomorphic sectional curvature

$H(X)$) when two holomorphic planes $[X, JX]$ and $[Y, JY]$ are orthogonal to each other (coincide with each other).

Summing up all of the situations mentioned above, the main goal of this paper is to study whether the relations of constancy among some kinds of curvatures defined on Kaehler manifolds also hold on real hypersurfaces in non flat complex space forms $M_m(c)$, $c \neq 0$. Real hypersurfaces with a constant holomorphic sectional curvature in $M_m(c)$, $c \neq 0$, $m \geqslant 3$, have been classified by Kimura in [6] when $c > 0$, i.e., in the complex projective space $P^m(c)$, and by the authors in [10] and [13] when $c < 0$, i.e., in the complex hyperbolic space $H^m(c)$.

Now in this paper we give our results as follows:

Theorem. *Let M be a real hypersurface of $M_m(c)$, $c \neq 0$, $m \geqslant 3$, on which the totally real bisectional curvature B is constant. Then $B = \frac{1}{2}c$ and we have one of the following cases:*

(a) *M is a ruled real hypersurface,*
(b) *M is a real hypersurface which admits a foliation of codimension two such that each leaf is contained in a totally geodesic $M_{m-1}(c)$, $c \neq 0$, as a ruled real hypersurface,*
(c) *$c > 0$ and M is an open subset of a geodesic hypersphere,*
(d) *$c < 0$ and M is an open subset of either*
 d.1) *a tube over a totally geodesic $H^{m-1}(c)$, or*
 d.2) *a Montiel tube, or*
 d.3) *a geodesic hypersphere.*

On the other hand, a 2-plane π tangent to M is called holomorphic if it admits an orthonormal basis of the form $\{X, \varphi X\}$. The holomorphic sectional curvature is the sectional curvature of any holomorphic 2-plane tangent to M. We will denote it by $H(\pi) = H(X)$. Moreover, by virtue of Theorem A and Theorem B in the next Preliminaries all real hypersurfaces of types (a), (b), (c) and (d) mentioned above have a constant holomorphic sectional curvature $H(X) = \text{const}$. The next corollary gives an affirmative answer to the main question of this paper.

Corollary. *Let M be a real hypersurface of $M_m(c)$, $c \neq 0$, $m \geqslant 3$. Then M has a constant holomorphic sectional curvature if and only if M has a constant totally real bisectional curvature.*

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1. Preliminaries

Let M be a real hypersurface in $M_m(c)$, $c \neq 0$, $m \geqslant 3$. Let ∇ be the Levi-Civita connection of M. In the introduction we wrote $JX = \varphi X + \eta(X)N$ for all $X \in TM$. Thus, φ is a skew-symmetric tensor field of type (1,1) of M and η is a 1-form on M. We denote by g both the metric on $M_m(c)$ and the induced metric on M. Now it is easy to see that $\eta(X) = g(X,\xi)$. The set (φ,ξ,η,g) is called an almost contact metric structure on M and its elementary properties are

$$\begin{aligned}&\varphi^2 X = -X + \eta(X)\xi \quad \text{and} \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0,\\ &g(\varphi X, Y) + g(X, \varphi Y) = 0 \quad \text{and} \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)\end{aligned}$$

for any $X, Y \in TM$, where A is the Weingarten endomorphism associated with N.

Since the ambient space $M_m(c)$ is of a constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are respectively given as

$$\begin{aligned}R(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y + g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y\\ &- 2g(\varphi X,Y)\varphi Z\} + g(AY,Z)AX - g(AX,Z)AY,\end{aligned} \tag{1.1}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X,Y)\xi\}, \tag{1.2}$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X.

Now let us consider a distribution $\mathfrak{D}$ of the tangent space T_pM, $p \in M$ defined in such a way that

$$\mathfrak{D}(p) = \{X \in T_pM\colon\ g(X,\xi) = 0\}.$$

Then its unit distribution of $\mathfrak{D}$ can be given by

$$\mathfrak{U}\mathfrak{D}(p) = \{X \in \mathfrak{D}(p)\colon\ \|X\| = 1\}.$$

By the Gauss equation (1.1) we compute the following expressions for the totally real bisectional curvature and the holomorphic sectional curvature of M, respectively:

$$B(X,Y) = \frac{c}{2} + g(A\varphi X,\varphi Y)g(AX,Y) - g(AX,\varphi Y)g(A\varphi X,Y) \tag{1.3}$$

where $X, Y \in \mathfrak{U}\mathfrak{D}$ and $g(X,Y) = g(\varphi X, Y) = 0$, and

$$H(X) = c + g(AX,X)g(A\varphi X,\varphi X) - g(AX,\varphi X)^2 \tag{1.4}$$

for any $X \in \mathfrak{U}\mathfrak{D}$.

Now in order to prove our main theorem we introduce the following results.

Theorem A ([6]). *Let M be a real hypersurface in $P^m(c)$, $m \geqslant 3$, which has a constant holomorphic sectional curvature H. Then M is one of the following cases:*

(a) *an open subset of a geodesic hypersphere, $H > c$,*
(b) *a ruled real hypersurface, $H = c$,*
(c) *a real hypersurface which admits a foliation of codimension two such that each leaf is contained in a totally geodesic hyperplane $P^{m-1}(c)$ as a ruled real hypersurface $H = c$.*

Theorem B ([10] and [13]). *Let M be a real hypersurface of $H^m(c)$, $c < 0$, $m \geqslant 3$, which has a constant holomorphic sectional curvature H. Then M is one of the following cases:*

(a) *an open subset of a geodesic hypersphere of radius $r > 0$, $\frac{3}{4}c < H = \{1 - \frac{1}{4}\sinh^2(r)\}c$,*
(b) *an open subset of a Montiel tube, $H = \frac{3}{4}c$,*
(c) *an open subset of a tube of radius $r > 0$ over a hyperplane $H^{m-1}(c)$, $c < H = \{1 - \frac{1}{4}\cosh^2(r)\}c < \frac{3}{4}c$,*
(d) *ruled, $H = c$,*
(e) *a real hypersurface which admits a foliation of codimension two such that each leaf is contained in a totally geodesic hyperplane $H^{m-1}(c)$ as a ruled real hypersurface, $H = c$.*

If p is a point of M, the rank of A at p is called the type number of M at p, and will be denoted by $t(p)$.

Theorem C ([10] and [14]). *Let M be a real hypersurface of $M_m(c)$, $c \neq 0$, $m \geqslant 3$, which satisfies $t(p) \leqslant 2$ for all $p \in M$. Then M is a ruled real hypersurface.*

Now let us recall some relations for $B(X, Y)$, $H(X)$ and $T(X, Y)$ on a Kaehler manifold (See Goldberg and Kobayashi [4], Houh [5]). For any totally real two planes $[X, Y]$, $[JX, JY]$ we have

$$B(X, Y) = T(X, Y) + T(JX, Y). \tag{1.5}$$

In fact,

$$\begin{aligned} B(X, Y) &= g(R(X, JX)JY, Y) \\ &= -g(R(JX, JY)X, Y) - g(R(JY, X)JX, Y) \\ &= g(R(X, Y)Y, X) + g(R(JY, X)X, JY) \\ &= K(X, Y) + K(JY, X) \\ &= T(X, Y) + T(JY, X), \end{aligned} \tag{1.6}$$

where $T(X, Y)$ denotes the totally real sectional curvature determined by the totally real plane $[X, Y]$ and we have used the fact that

$$g(R(JX, JY)Y, X) = g(R(X, Y)Y, X). \tag{1.7}$$

From these formulas on a Kaehler manifold we know that it is not trivial to show the constancy of the totally real sectional curvature $T(X, Y)$ from the constancy of the totally real bisectional curvature $B(X, Y)$.

When we consider real hypersurfaces M in a complex space form $M_m(c)$, we can consider a distribution $\mathfrak{D}$ in the tangent space T_pM, $p \in M$. Then the distribution $\mathfrak{D}$ is invariant with respect to the structure tensor φ and can be regarded as a holomorphic distribution in a Kaehler manifold.

But, if we apply the equation of Gauss to the above formulas, the situation is not the same as in a Kaehler manifold. Even the formulas (1.6) and (1.7) need not hold for such real hypersurfaces in a complex space form $M_m(c)$. So it is not trivial to show the constancy mentioned above. Due to such a situation the purpose of this paper is to classsify all real hypersurfaces in $M_n(c)$ with a constant totally real bisectional curvature and to assert that such a hereditary property also can be hold for real hypersurfaces.

2. Proof of the Theorem

Let p be a point of M. Let $X, Y, Z \in \mathfrak{U}\mathfrak{D}(p)$ be such that $\mathrm{Span}\{Y, X\}$ is totally real and $g(X, Z) = 0$. Then there is a curve $X(t)$, $t \in (-\delta, \delta)$, such that $X(t) \in \mathfrak{U}\mathfrak{D}(p)$, $\mathrm{Span}\{Y, X(t)\}$ is totally real, $X(0) = X$ and $X'(0) = Z$.

Now we assume that M is a real hypersurface in $M_n(c)$ on which the totally real bisectional curvature $B(X, Y)$ determined by two totally real planes $[X, Y]$ and $[JX, JY]$ is constant. Then by differentiating (1.3) we obtain

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0} g(A\varphi X(t), \varphi Y)g(AX(t), Y) - g(AX(t), \varphi Y)g(A\varphi X(t), Y) = 0,$$

thus

$$\begin{aligned} g(A\varphi Z, \varphi Y)g(AX, Y) + g(A\varphi X, \varphi Y)g(AZ, Y) \\ - g(AZ, \varphi Y)g(A\varphi X, Y) - g(AX, \varphi Y)g(A\varphi Z, Y) = 0 \end{aligned} \tag{2.1}$$

for any X, Y and Z in $\mathfrak{U}\mathfrak{D}(p)$ such that $\mathrm{Span}\{Y, X\}$ is totally real and $g(X, Z) = 0$.

Let us take an orthonormal basis $\{\xi_1, E_1, \ldots, E_{2m-2}\}$ of T_xM such that

$$AE_i|_{\mathfrak{D}} = a_iE_i, i = 1, 2, \ldots, 2m - 2. \tag{$*$}$$

Now let us choose $i \in \{1, 2, \ldots, 2m-2\}$ such that $[X, Z] \perp [E_i, \varphi E_i]$. Differentiating (2.1) one more time and using $X'(0) = Z$, we arrive at

$$g(A\varphi Z, \varphi Y)g(AZ, Y) - g(AZ, \varphi Y)g(A\varphi Z, Y) = 0. \tag{2.2}$$

Now let us replace Z by φY in (2.1), because $[X, Y]$ and $[Y, \varphi X]$ are also totally real sections such that $g(X, \varphi Y) = 0$. Then it follows that

$$\begin{aligned} g(AY, \varphi Y)g(AX, Y) + g(A\varphi X, \varphi Y)g(A\varphi Y, Y) \\ - g(A\varphi Y, \varphi Y)g(A\varphi X, Y) + g(AX, \varphi Y)g(AY, Y) = 0. \end{aligned} \tag{2.3}$$

If we substitute $Y = E_i$ into the above equation and use the formula $(*)$, we have

$$a_i g(AX, \varphi E_i) - g(A\varphi E_i, \varphi E_i)g(A\varphi X, E_i) = 0. \tag{2.4}$$

For a totally real plane $[\varphi E_i, X]$ we know that $a_i g(AX, \varphi E_i) = 0$. Thus for any X orthogonal to E_i and φE_i we know that $a_i g(AX, \varphi E_i) = 0$.

Now we have to discuss the following three cases:

Case 1. At least two of the a_i's are not zero. We can suppose without losing any generality that a_1, a_2 are not zero. Let $\Omega = \{p \in M \colon a_1(p) \neq 0, a_2(p) \neq 0\}$. Throughout this case, $p \in \Omega$ unless otherwise stated. From this and (2.4) we know that $a_1 g(AX, \varphi E_1) = 0$ for any $X \perp E_1, \varphi E_1$. This means

$$A\varphi E_1 \in [\xi, \varphi E_1].$$

Similarly, for any $X \perp E_2, \varphi E_2$ we have

$$A\varphi E_2 \in [\xi, \varphi E_2].$$

By virtue of this fact, even if we consider the case that all of a_i, $i = 1, \ldots, m-1$ are different from zero, all of its holomorphic sectional curvatures are constant:

$$H(E_i) = c + g(AE_i, E_i)g(A\varphi E_i, \varphi E_i) - g(AE_i, \varphi E_i)^2 = c.$$

Now let us consider the case when $a_3 = \ldots = a_{m-1} = 0$. Then it follows that

$$g(A\varphi E_k, E_j) = g(\varphi E_k, AE_j) = a_j g(\varphi E_k, E_j) = 0 \quad \text{for } j \geqslant 3.$$

This means

$$A\varphi E_k \in \operatorname{Span}\{\xi, \varphi E_1, \ldots, \varphi E_{m-1}\}.$$

Now putting $Z = E_i$ in (2.1) and using $AE_i|\mathfrak{D} = 0$ for $i = 3, \ldots, m-1$, for any totally real section $[X, Y]$ such that $X, Y \in \mathfrak{U}\mathfrak{D}$ we know

$$g(A\varphi E_i, \varphi Y)g(AX, Y) - g(AX, \varphi Y)g(A\varphi E_i, Y) = 0. \tag{2.5}$$

We take another totally real section $[X', Y']$ defined by

$$X' = \frac{1}{\sqrt{2}}(X+Y), \quad Y' = \frac{1}{\sqrt{2}}(X-Y).$$

Then it follows that

$$\begin{aligned}\{g(A\varphi E_i, \varphi X) - g(A\varphi E_i, \varphi Y)\}\{g(AX, X) - g(AY, Y)\} \\ - \{g(AX, \varphi X) + g(AY, \varphi X) - g(AX, \varphi Y) \\ - g(AY, \varphi Y)\}\{g(A\varphi E_i, X) - g(A\varphi E_i, Y)\} = 0.\end{aligned}$$

Taking $X = E_j$ and $Y = E_l$ such that $AE_j|_{\mathfrak{D}} = 0$ and $AE_l|_{\mathfrak{D}} = a_l E_l$, $a_l \neq 0$, $l = 1, 2$ we have

$$\{g(A\varphi E_i, \varphi E_j) - g(A\varphi E_i, \varphi E_l)\}g(AE_l, E_l) = 0.$$

Since $AE_l|_{\mathfrak{D}} = a_l E_l$, $a_l \neq 0$, it follows that

$$g(A\varphi E_i, \varphi E_j) = g(A\varphi E_i, \varphi E_1) = g(A\varphi E_i, \varphi E_2) = g(\varphi E_i, A\varphi E_2) = 0$$

for any distinct $i \neq j \in \{3, \ldots, m-1\}$. This yields

$$AE_i|\mathfrak{D} = d_i E_i \quad \text{and} \quad A\varphi E_i|\mathfrak{D} = b_i \varphi E_i.$$

Now let us put $\alpha = \frac{1}{\sqrt{3}}$, $\beta = \sqrt{2/3}$. Let us consider the vectors

$$X = \alpha E_i + \beta E_j, Y = \beta E_i - \alpha E_j \quad \text{and} \quad Z = \beta\varphi E_i - \alpha\varphi E_j.$$

Then it can be easily seen that the planes $[X, Y]$ and $[X, Z]$ are totally real and their bisectional curvature is given by

$$B(X, Y) = \frac{c}{2} + g(A\varphi X, \varphi Y)g(AX, Y) - g(AX, \varphi Y)g(A\varphi X, Y). \tag{2.6}$$

So it follows that

$$\begin{aligned}B(X, Y) - \frac{c}{2} \\ = \alpha^2\beta^2 g(A\varphi E_i, \varphi E_i)g(AE_i, E_i) - \alpha^2\beta^2 g(A\varphi E_i, \varphi E_i)g(AE_j, E_j) \\ - \alpha^2\beta^2 g(A\varphi E_j, \varphi E_j)g(AE_i, E_i) + \alpha^2\beta^2 g(A\varphi E_j, \varphi E_j)g(AE_j, E_j) \\ = \alpha^2\beta^2(d_i - d_j)(b_i - b_j).\end{aligned} \tag{2.7}$$

Now let us take another totally real section $[X, Y]$ spanned by two vectors $X = \frac{1}{\sqrt{2}}(E_i + E_j)$ and $Y = \frac{1}{\sqrt{2}}(\varphi E_i - \varphi E_j)$. Then we know $g(X, \varphi Y) = 0$, which together with (2.6) implies that

$$\begin{aligned}
(2.8)\quad 4\Big\{B(X,Y) - \frac{c}{2}\Big\} \\
&= g(\alpha A\varphi E_i + \beta A\varphi E_j, -\beta E_i + \alpha E_j)g(\alpha AE_i + \beta AE_j, \beta\varphi E_i - \alpha\varphi E_j) \\
&\quad - g(\alpha AE_i + \beta AE_j, -\beta E_i + \alpha E_j)g(\alpha A\varphi E_i + \beta A\varphi E_j, \beta\varphi E_i - \alpha\varphi E_j) \\
&= (d_i - d_j)(b_i - b_j).
\end{aligned}$$

Then (2.7) and (2.8) yield that

$$(2.9)\qquad (d_i - d_j)(b_i - b_j) = 0$$

for any $i \neq j$. From this together with the assumption we know that all of the totally real bisectional curvatures are constant, that is $B(X, Y) = \frac{1}{2}c$ for any totally real section $[X, Y]$. Then for another totally real plane $[X, Y]$ spanned by the vectors $X = \frac{1}{\sqrt{2}}(E_i + \varphi E_j)$ and $Y = \frac{1}{\sqrt{2}}(\varphi E_i + E_j)$, from (2.6) we have

$$\begin{aligned}
(2.10)\qquad 0 &= g(A\varphi E_i - AE_j, -E_i + \varphi E_j)g(AE_i + A\varphi E_j, \varphi E_i + E_j) \\
&\quad - g(AE_i + A\varphi E_j, -E_i + \varphi E_j)g(A\varphi E_i - AE_j, \varphi E_i + E_j) \\
&= -(b_i - d_j)(b_j - d_i)
\end{aligned}$$

for any distinct $i \neq j$. From (2.9) we know that $b_i = b_j$ or $d_i = d_j$. Hence taking into account (2.10) we can consider the cases

$$b_i = b_j = d_j \quad \text{or} \quad b_i = b_j = d_i$$

for any distinct $i \neq j$. Similarly we can also consider the cases

$$d_i = d_j = b_i \quad \text{or} \quad d_i = d_j = b_j.$$

At any case we assert that all b_k and d_k coincide with each other. So it follows that

$$AX\big|_{\mathfrak{D}} = bX$$

for any $X \in \mathfrak{U}\mathfrak{D}$. This implies that the holomorphic sectional curvature $H(X) = c + b^2$ is constant.

Now in order to complete our proof we consider two cases. To do this we suppose that the interior of the set $\Gamma = \{p \in M :\ a_1(p) = 0 \text{ or } a_2(p) = 0\}$ is not empty.

Case 2. Let us suppose $a_1 = \ldots = a_{2m-2} = 0$ on an open subset Ω of M, which can be supposed to be included in Γ. By virtue of $(*)$, $t(p) \leqslant 2$ on Ω. By Theorem C, Ω is a ruled real hypersurface. On such an open subset Ω we can write

$$A\xi = \alpha\xi + \beta U,\ AU = \beta\xi,\ A\varphi U = 0 \text{ and } AX = 0$$

for any $X \in \mathfrak{D}$ orthogonal to U and φU. If the function β is equal to 0, then the set Ω has at most two distinct pincipal curvatures. So the set Ω is a geodesic hypersphere for $c > 0$ (See Cecil and Ryan [3]) and a horosphere or a geodesic hypersphere for $c < 0$ (See Montiel [8]). This makes a contradiction. From this we know that the function β is non-vanishing on Ω, that is, $\Omega = \{p \in M\colon\ \beta(p) \neq 0\}$.

Now let us consider a subdistribution $\mathfrak{D}_1$ of the distribution $\mathfrak{D}$ defined by

$$\mathfrak{D}_1(p) = \{X \in T_pM\colon\ \eta(X) = g(X, U) = g(X, \varphi U) = 0\}.$$

Then for any $X \in \mathfrak{D}_1$ we have $AX = 0$, so it follows that $\nabla_X\xi = 0$. From this, by virtue of the equation of Codazzi (1.2) we obtain that

$$(\nabla_X A)\xi - (\nabla_\xi A)X = -\frac{c}{4}\varphi X$$

and

$$(\nabla_X A)\xi - (\nabla_\xi A)X = (X\alpha)\xi + \beta\nabla_X U + (X\beta)U + A(\nabla_\xi X).$$

This gives finally $\nabla_X U = -\frac{1}{4}\beta^{-1}c\varphi X$ on Ω. Since M is connected, this shows that $M = \Omega$, because if the sequence $\{x_j\}$ is a sequence in Ω which converges to some boundary point of Ω, then $\lim_{j\to\infty}\beta(x_j) = 0$, which implies that the sequence $\{\|\nabla_X U\|(x_j)\}$ diverges. Therefore M is ruled. By definition, the distribution $\mathfrak{D}$ is integrable and totally geodesic in M, so that if $X, Y \in \mathfrak{D}$, then $\nabla_X\varphi Y \in \mathfrak{D}$, that is to say, $0 = g(\xi, \nabla_X\varphi Y)$. By (1.1) we conclude

$$0 = g(\nabla_X\xi, \varphi Y) = g(\varphi AX, \varphi Y) = g(AX, Y)$$

for any $X, Y \in \mathfrak{U}\mathfrak{D}$. Then $H(X) = c$, which shows that M has a constant holomorphic sectional curvature. Moreover, it is clear that $B(X, Y) = \frac{1}{2}c$ for any $X, Y \in \mathfrak{U}\mathfrak{D}$.

Case 3. Let us suppose that exactly one of the a_i's is different from zero and all the other a_k's are zero. Without losing any generality we can suppose $i = 1$. Let Ω be an open subset of M where $a_1 \neq 0$, which can be supposed to be included in Γ. Throughout this case, $p \in \Omega$ unless otherwise stated. From $(*)$ we obtain

$$AX|_{\mathfrak{D}} = a_1 g(X, E_1)E_1$$

for any $X \in \mathfrak{D}$ in a neigbourhood of each point $p \in \Omega$. By (1.2) and (1.3), $H(X) = c$ for any $X \in \mathfrak{U}\mathfrak{D}(p)$ and any $p \in \Omega$. By Theorem A and Theorem B, either Ω is ruled or Ω admits a foliation of codimension two such that each leaf is contained in a totally geodesic hyperplane $M_{m-1}(c)$ as a ruled real hypersurface. As $a_1 \neq 0$, $t(p) \geqslant 3$ on Ω, and by Theorem C, Ω cannot be ruled. Then on such an open subset $\Omega = \{p \in M\colon\ a_1(p) \neq 0\}$ we have the following expression for the shape operator:

$$\begin{cases} A\xi = \alpha\xi + \beta_1 e_1 + \overline{\beta}_1 \varphi e_1 + \beta_2 e_2, \\ Ae_1 = \beta_1 \xi + a_1 e_1, \\ A\varphi e_1 = \overline{\beta}_1 \xi, \\ Ae_2 = \beta_2 \xi, \\ A\varphi e_j = Ae_k = 0 \quad (2 \leqslant j \leqslant n-1,\ 3 \leqslant k \leqslant n-1). \end{cases}$$

Thus by using the equation of Codazzi (1.2) repeatedly, bearing in mind the above formulas and using a similar method as in Kimura [6], Sohn and the third author [13], we can finally assert that

$$\nabla_{e_1} e_1 = \operatorname{grad} \log |a_1| - \beta_1 \varphi e_1.$$

Then by the connectedness of M we know that $M = \Omega$. In fact, if $M \neq \Omega$, then by the definition of Ω, there is a sequence of points $\{y_j\}$ in Ω which converges to a boundary point of Ω, so that $\lim\limits_{j \to \infty} a_1(y_j) = 0$. Hence the above equation implies that $\{\|\nabla_{e_1} e_1 + \beta_1 \varphi e_1\|(y_j)\}$ diverges. Then $\Omega = M$ and therefore M is either case c) of Theorem A or case e) of Theorem B. Moreover, from the expression of the shape operator mentioned above and (1.3) we can show that

$$B(e_1, e_2) = B(\varphi e_1, e_2) = \frac{c}{2}$$

and also $\frac{1}{2}c$ for other totally real sections. So in this case we conclude that $B(X,Y) = \frac{1}{2}c$ for any $X, Y \in \mathfrak{U}\mathfrak{D}$.

Conversely, in [11] the present authors have already proved that all model spaces of Theorem A and Theorem B have constant totally real sectional curvatures, that is $T(X,Y) = \text{const}$ for any totally real section $[X,Y]$. This, together with the formula (1.5), yields that all the model spaces mentioned in Theorem A and Theorem B have constant totally real bisectional curvatures. This completes the proof of our theorem.

References

[1] *M. Barros and A. Romero*: Indefinite Kaehler manifolds. Math. Ann. *281* (1982), 55–62. Zbl 0476.53013

[2] *R. L. Bishop and S. I. Goldberg*: Some implications of the generalized Gauss-Bonnet Theorem. Trans. Amer. Math. Soc. *112* (1964), 508–535. Zbl 0133.15101

[3] *T. E. Cecil and P. J. Ryan*: Focal sets and real hypersurfaces in complex projective space. Trans. A.M.S. *269* (1982), 481–499. Zbl 0492.53039

[4] *S. I. Goldberg and S. Kobayashi*: Holomorphic bisectional curvature. J. Diff. Geometry *1* (1967), 225–234. Zbl 0169.53202

[5] *C. S. Houh*: On totally real bisectional curvature. Proc. Amer. Math. Soc. 56 (1976), 261–263. Zbl 0328.53012

[6] *M. Kimura*: Sectional curvatures of holomorphic planes on a real hypersurface in P_nC. Math. Ann. *276* (1987), 487–499. Zbl 0605.53023

[7] *Y. Maeda*: On real hypersurfaces of a complex projective space. J. Math. Soc. Japan *28* (1976), 529–540. Zbl 0324.53039

[8] *S. Montiel*: Real hypersurfaces of complex hyperbolic space. J. Math. Soc. Japan. *37* (1985), 515–535. Zbl 0554.53021

[9] *R. Niebergall and P. J. Ryan*: Real hypersurfaces in complex space forms. Tight and taut Submanifolds, edited by T. E. Cecil and S. S. Chern. Cambridge U. Press, 1997, pp. 233–305. Zbl 0904.53005

[10] *M. Ortega and J. D. Pérez*: Constant holomorphic sectional curvature and type number of real hypersurfaces of complex hyperbolic space. Proc. 4th Intnal. Congress of Geometry, Thessaloniki (1996). Zbl 0889.53038

[11] *M. Ortega, J. D. Pérez and Y. J. Suh*: Real hypersurfaces with constant totally real sectional curvature in a complex space form. Czechoslovak Math. J. *50* (2000), 531–537. Zbl 1079.53507

[12] *J. D. Pérez and Y. J. Suh*: Real hypersurfaces of quaternionic projective space satisfying $\nabla_{U_i} R = 0$. Diff. Geom. and Its Appl. *7* (1997), 211–217. Zbl 0901.53011

[13] *D. J. Sohn and Y. J. Suh*: Classification of real hypersurfaces in complex hyperbolic space in terms of constant φ-holomorphic sectional curvature. Kyungpook Math. J. *35* (1996), 801–819. Zbl 1004.53015

[14] *Y. J. Suh*: A characterization of ruled real hypersurfaces in P_nC. J. Korean Math. Soc. *29* (1992), 351–359. Zbl 0762.53005

[15] *M. H. Vernon*: Contact hypersurfaces of complex hyperbolic space. Tôhoku Math. J. *39* (1987), 215–222. Zbl 0636.53060

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