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A SEQUENTIAL ITERATION ALGORITHM WITH NON-MONOTONEOUS BEHAVIOUR IN THE METHOD OF PROJECTIONS ONTO CONVEX SETS

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Abstract. The method of projections onto convex sets to find a point in the intersection of a finite number of closed convex sets in a Euclidean space, may lead to slow convergence of the constructed sequence when that sequence enters some narrow "corridor" between two or more convex sets. A way to leave such corridor consists in taking a big step at different moments during the iteration, because in that way the monotoneous behaviour that is responsible for the slow convergence may be interrupted. In this paper we present a technique that may introduce interruption of the monotony for a sequential algorithm, but that at the same time guarantees convergence of the constructed sequence to a point in the intersection of the sets. We compare experimentally the behaviour concerning the speed of convergence of the new algorithm with that of an existing monotoneous algorithm.

Keywords: projections onto convex sets, nonlinear operators, slow convergence

MSC 2000: 47N10, 47H09, 40A99

1. Introduction

Suppose that in the m-dimensional Euclidean space $\mathbb{R}^m$ a finite number of closed convex sets $\{C_j\}_{j=1}^n$ with non-empty intersection $C^* \equiv \bigcap_{j=1}^{n} C_j$ are given. A problem that often arises in applied mathematics is: find a point in C^*; this problem is also known as the convex feasibility problem.

When the individual sets C_j are such that for each of them its corresponding metric projection operator P_j is explicitly known (P_j may be either linear or nonlinear), by the method of projections onto convex sets (often abbreviated as POCS) a sequence is constructed that converges to a point of C^*. Depending on the number r ($1 \leqslant r \leqslant n$) of projections that is used at each step in constructing the sequence, one can speak of a sequential method (when $r = 1$), or a block-iterative method (when $1 < r < n$), or a (fully) parallel method (when $r = n$). We refer to [1], [2], [3], [4], [7] and [12] for an overview of general problems and methods, to [6] and [9] for some general applications, and to [14] and [15] for specific applications in image processing. Recently, the convex feasibility problem attracted new attention, as for some *Interior Point Methods* for solving *Semidefinite Programming Problems* a feasible starting point is required [13]. There is, however, a serious drawback concerning the standard methods of projections onto convex sets: the constructed sequence often converges very slowly. Experimentally it may be observed that, for a given problem with a fixed position of the closed convex sets and for a fixed standard algorithm, choosing different starting points may lead to big differences concerning the number of iterations that is needed to reach a point in the intersection. As there is usually no indication about the choice of the starting point in order to avoid slow convergence, it is clear that new algorithms are needed that are less dependent on that choice. Although there are some good rate-of-convergence results connected to classical monotoneous algorithms when closed linear subspaces are used, at this moment we don't have applicable theoretical results about the rate of convergence for the non-monotoneous algorithm that we are going to construct. Hence, the usefulness of a new algorithm should at first be judged on the base of experimental results. Some earlier acceleration schemes for special problems have been presented in [8] and [10].

In [5] we presented a parallel algorithm that, as can be concluded from experimental results, can eliminate to a great extent the influence of the starting point, based on the idea of interrupting at different iteration steps during the procedure the "monotone" way of convergence. The monotoneous behaviour (also called the Fejér monotony [11]) of the iteration sequence $\{\mathbf{x}_k\}_{k=0}^{+\infty}$ in $\mathbb{R}^m$, expressed by the inequality $\|\mathbf{x}_{k+1} - \mathbf{v}\| \leqslant \|\mathbf{x}_k - \mathbf{v}\|$, $\forall\, \mathbf{v} \in C^*$, for all k, is present in all traditional POCS-algorithms and may be responsible for slow convergence connected to bad starting points. By interrupting the monotony, we want to approach the intersection set C^* from a different direction, and this may give us the possibility to leave some small corridor that leads to the intersection but that causes the slow convergence. The experimental results based on the algorithm in [5] were very promising concerning the number of iterations that was needed to obtain an intersection point irrespective of the starting point, but part of the computations had to be done in the space $(\mathbb{R}^m)^n$ instead of $\mathbb{R}^m$, and this was a complicating factor.

In this paper, we present a new algorithm that is based on the idea of interrupting the monotoneous behaviour of the iteration sequence $\{\mathbf{x}_k\}_{k=0}^{+\infty}$, but that at two essential points is different from [5]: the algorithm now is sequential (instead of parallel), and all computations are done in $\mathbb{R}^m$. Essentially, the algorithm to construct $\mathbf{x}_{k+1}$

from $\mathbf{x}_k$ uses suitable relaxed projections for those values of $k+1$ where interruption is allowed, and uses pure projections for those values of $k+1$ where monotoneous behaviour is wanted. We want to stress the fact that the algorithm allows a lot of flexibility that is not described in the underlying version as, e.g., allowing relaxation (instead of using pure projections) for those values of $k+1$ where monotony is wanted, or allowing more freedom of choice for determining the possible interruption points. However, as it was our first aim to explain the essential ideas behind the algorithm and to show its convergence, we didn't endeavour to present the most general form of the algorithm.

In Section 2, we explain how the iteration sequence $\{\mathbf{x}_k\}_{k=0}^{+\infty}$ is constructed. In Section 3, we prove that the sequence really converges to a point of C^*. Finally, in Section 4 we present a few examples to compare the number of iterations needed to obtain convergence for the new algorithm and for a classical one.

2. Construction of the algorithm

2.1. Let $\mathbb{R}^m$ be the m-dimensional Euclidean space with the standard inner product $\langle\ ,\ \rangle$ and the norm $\|\ \|$ derived from $\langle\ ,\ \rangle$; we denote $(\mathbb{R}^m, \langle\ ,\ \rangle, \|\ \|)$ for brevity by H. Elements of H are denoted by boldface letters.

In H, n closed convex sets $\{C_j\}_{j=1}^n$ are given, having non-empty intersection $C^* \equiv \bigcap_{j=1}^n C_j$. Projection onto C_j is denoted either as P_{C_j} or as P_j. We want to obtain a point in C^* by a sequential iterative procedure, i.e., starting from some given point $\mathbf{x}_0$ in H we want to construct a sequence $\{\mathbf{x}_k\}_{k=0}^{+\infty}$ in H that converges to a point in C^*, and such that, in going from $\mathbf{x}_k$ to $\mathbf{x}_{k+1}$, only one projection operator (with or without relaxation) is used; this is the meaning of the word "sequential" in the iterative procedure. On the other hand, contrary to most methods we no longer need the Fejér monotony of the method, i.e., it is no longer necessary that at each step $k+1$ the inequality $\|\mathbf{x}_{k+1} - \mathbf{v}\| \leqslant \|\mathbf{x}_k - \mathbf{v}\|$ for all $\mathbf{v} \in C^*$ is true.

Before explaining our new algorithm, we repeat the following well-known facts that we resume as Procedure PP (in which PP may be seen as an abbreviation for pure projection).

Procedure PP. Suppose that, starting from some point $\mathbf{x}_0$ in H, the sequence $\{\mathbf{x}_k\}_{k=0}^{+\infty}$ in H is constructed as follows: when $\mathbf{x}_k$ has been obtained, and when $\mathbf{x}_k \notin C^*$, choose an index $f(k+1) \in \{1, 2, \dots, n\}$ such that $\mathbf{x}_k \notin C_{f(k+1)}$, and construct $\mathbf{x}_{k+1}$ by $\mathbf{x}_{k+1} = P_{f(k+1)}\mathbf{x}_k$; otherwise said, in going from $\mathbf{x}_k$ to $\mathbf{x}_{k+1}$ a pure projection onto a single closed convex set to which $\mathbf{x}_k$ doesn't yet belong is used, and $\mathbf{x}_{k+1} \in C_{f(k+1)}$. *This ends procedure PP.*

Then the following inequalities are true for all $\mathbf{v} \in C^*$ (see [15]):

$$\langle \mathbf{x}_k - \mathbf{x}_{k+1}, \mathbf{v} - \mathbf{x}_{k+1} \rangle \leqslant 0 \tag{1}$$

$$\|\mathbf{x}_{k+1} - \mathbf{v}\|^2 \leqslant \|\mathbf{x}_k - \mathbf{v}\|^2 - \|\mathbf{x}_k - \mathbf{x}_{k+1}\|^2. \tag{2}$$

In particular, in this manner the resulting sequence $\{\mathbf{x}_k\}_{k=0}^{+\infty}$ has a monotoneous behaviour with respect to the intersection of all sets $\{C_j\}_{j=1}^n$. The function f may be seen as a choice function from the set of positive integers to the set $\{1, 2, \ldots, n\}$.

2.2. We now want to modify the above method such that at regular steps during the iteration the monotoneous behaviour may be interrupted, but such that nevertheless the resulting sequence is convergent to a point of C^*. Let N and J be positive integers, $N > 2$ and $J > N$, but otherwise free to choose. Suppose that, starting from some point $\mathbf{x}_0$ in H, the points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_J$ have been constructed according to procedure PP. For construction of the points $\mathbf{x}_{k+1}$ with index $k+1 > J$ (assuming that $\mathbf{x}_k$ doesn't yet belong to C^*), we will make a distinction depending on the index $k+1$, as follows:

—when $\mathbf{x}_k$ $(k \geqslant J)$ has been obtained, and $k + 1 \notin \{J + 1 + pN\}_{p=0}^{+\infty}$, then $\mathbf{x}_{k+1}$ is constructed from $\mathbf{x}_k$ by using a pure projection according to procedure PP; in particular, the inequalities (1) and (2) are true in this situation. In the last subsection of Section 2 we will put some restriction (to be called: control strategy) on the choice of the function f in procedure PP as part of our algorithm, in order to guarantee convergence of the resulting sequence to a point of C^*.

—When $\mathbf{x}_k$ $(k \geqslant J)$ has been obtained and $k + 1 \in \{J + 1 + pN\}_{p=0}^{+\infty}$, then $\mathbf{x}_{k+1}$ is obtained from $\mathbf{x}_k$ by using a suitable relaxed projection. We first choose an index $f(k+1)$ for a suitable set $C_{f(k+1)}$ such that $\mathbf{x}_k \notin C_{f(k+1)}$, leading to the associated projection operator $P_{f(k+1)}$. In Procedure Relax we now explain how to choose the (positive) relaxation parameter λ_{k+1} in order to construct $\mathbf{x}_{k+1}$.

Procedure Relax. Put

$$\mathbf{w}_{k+1} \equiv P_{f(k+1)}\mathbf{x}_k, \tag{3}$$

and determine the next iteration point $\mathbf{x}_{k+1}$ by

$$\mathbf{x}_{k+1} = \mathbf{w}_{k+1} + \lambda_{k+1}(\mathbf{w}_{k+1} - \mathbf{x}_k). \tag{4}$$

A routine calculation leads to the following equality, for each $\mathbf{v} \in C^*$:

$$\|\mathbf{x}_{k+1} - \mathbf{v}\|^2 = \|\mathbf{w}_{k+1} - \mathbf{v}\|^2 + \lambda_{k+1}^2\|\mathbf{w}_{k+1} - \mathbf{x}_k\|^2 + 2\lambda_{k+1}\langle \mathbf{w}_{k+1} - \mathbf{v}, \mathbf{w}_{k+1} - \mathbf{x}_k \rangle, \tag{5}$$

from which the following inequality results in view of (3) and of (1) (in which $\mathbf{x}_{k+1}$ has to be replaced by $\mathbf{w}_{k+1}$):

$$\|\mathbf{x}_{k+1} - \mathbf{v}\|^2 \leqslant \|\mathbf{w}_{k+1} - \mathbf{v}\|^2 + \lambda_{k+1}^2 \|\mathbf{w}_{k+1} - \mathbf{x}_k\|^2. \tag{6}$$

For the first term on the right hand side of (6), the inequality (2) (with $\mathbf{x}_{k+1}$ replaced by $\mathbf{w}_{k+1}$) is valid; hence,

$$\|\mathbf{x}_{k+1} - \mathbf{v}\|^2 \leqslant \|\mathbf{x}_k - \mathbf{v}\|^2 - \|\mathbf{x}_k - \mathbf{w}_{k+1}\|^2 + \lambda_{k+1}^2 \|\mathbf{w}_{k+1} - \mathbf{x}_k\|^2. \tag{7}$$

But inequality (2) is, with the necessary replacements, also valid for the following couples of points: $(\mathbf{x}_k, \mathbf{x}_{k-1}), (\mathbf{x}_{k-1}, \mathbf{x}_{k-2}), \ldots, (\mathbf{x}_{k+1-(N-1)}, \mathbf{x}_{k+1-N})$, due to the fact that in each couple the point with the larger index is obtained from the other point by procedure PP. Hence, repeatedly using inequality (2) as we did in going from (6) to (7), we obtain from (7), for each $\mathbf{v} \in C^*$:

$$\begin{aligned} \|\mathbf{x}_{k+1} - \mathbf{v}\|^2 &\leqslant \|\mathbf{x}_{k-1} - \mathbf{v}\|^2 - \|\mathbf{x}_{k-1} - \mathbf{x}_k\|^2 - \|\mathbf{w}_{k+1} - \mathbf{x}_k\|^2 \\ &\quad + \lambda_{k+1}^2 \|\mathbf{w}_{k+1} - \mathbf{x}_k\|^2 \\ &\leqslant \|\mathbf{x}_{k-2} - \mathbf{v}\|^2 - \|\mathbf{x}_{k-2} - \mathbf{x}_{k-1}\|^2 - \|\mathbf{x}_{k-1} - \mathbf{x}_k\|^2 \\ &\quad - \|\mathbf{w}_{k+1} - \mathbf{x}_k\|^2 + \lambda_{k+1}^2 \|\mathbf{w}_{k+1} - \mathbf{x}_k\|^2 \\ &\leqslant \ldots \\ &\leqslant \|\mathbf{x}_{k+1-N} - \mathbf{v}\|^2 - \|\mathbf{x}_{k+1-N} - \mathbf{x}_{k+1-(N-1)}\|^2 - \ldots \\ &\quad - \|\mathbf{x}_{k-2} - \mathbf{x}_{k-1}\|^2 - \|\mathbf{x}_{k-1} - \mathbf{x}_k\|^2 - \|\mathbf{w}_{k+1} - \mathbf{x}_k\|^2 \\ &\quad + \lambda_{k+1}^2 \|\mathbf{w}_{k+1} - \mathbf{x}_k\|^2. \end{aligned} \tag{8}$$

Putting

$$\begin{aligned} R_{k+1} = \|\mathbf{x}_{k+1-N} - \mathbf{x}_{k+1-(N-1)}\|^2 + \ldots + \|\mathbf{x}_{k-2} - \mathbf{x}_{k-1}\|^2 + \|\mathbf{x}_{k-1} - \mathbf{x}_k\|^2 \\ + \|\mathbf{w}_{k+1} - \mathbf{x}_k\|^2, \end{aligned} \tag{9}$$

inequality (8) may be rewritten as

$$\|\mathbf{x}_{k+1} - \mathbf{v}\|^2 \leqslant \|\mathbf{x}_{k+1-N} - \mathbf{v}\|^2 - R_{k+1} + \lambda_{k+1}^2 \|\mathbf{w}_{k+1} - \mathbf{x}_k\|^2. \tag{10}$$

Now we want to determine the relaxation coefficient λ_{k+1} such that, although it may no longer be true that the Fejér monotony property $\|\mathbf{x}_{k+1} - \mathbf{v}\| \leqslant \|\mathbf{x}_k - \mathbf{v}\|$ for all $\mathbf{v} \in C^*$ is valid in going from $\mathbf{x}_k$ to $\mathbf{x}_{k+1}$, the monotoneous behaviour is nevertheless repaired with respect to the "former" possible interruption point

$\mathbf{x}_{k+1-N}$ (of course, when $k = J$, then at the point $\mathbf{x}_{J+1-N}$ the monotony is not yet interrupted); otherwise said, we want that:

$$\|\mathbf{x}_{k+1} - \mathbf{v}\| \leqslant \|\mathbf{x}_{k+1-N} - \mathbf{v}\|, \tag{11}$$

for all $\mathbf{v} \in C^*$ and for all $k+1 \in \{J+1+pN\}_{p=0}^{+\infty}$.

Let γ be a given positive real number, $0 < \gamma < 1$. In view of inequality (10), we see that (11) may be obtained when λ_{k+1} is chosen such that $\lambda_{k+1}^2 \|\mathbf{w}_{k+1} - \mathbf{x}_k\|^2 \leqslant \gamma R_{k+1}$. Then we derive from (10) that:

$$\|\mathbf{x}_{k+1} - \mathbf{v}\|^2 \leqslant \|\mathbf{x}_{k+1-N} - \mathbf{v}\|^2 - (1-\gamma) R_{k+1}. \tag{12}$$

In particular, it is sufficient to choose λ_{k+1} as follows:

$$0 < \lambda_{k+1} \leqslant \sqrt{\frac{\gamma R_{k+1}}{\|\mathbf{w}_{k+1} - \mathbf{x}_k\|^2}}. \tag{13}$$

This ends Procedure Relax.

A supplementary condition on the choice of λ_{k+1} to guarantee convergence of the constructed sequence will be given in the last subsection of Section 2.

We finally remark that expression (4) to determine $\mathbf{x}_{k+1}$ from $\mathbf{x}_k$ in the Procedure Relax may be written as follows:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + (1 + \lambda_{k+1})(P_{f(k+1)}\mathbf{x}_k - \mathbf{x}_k),$$

which means that we also can write $\mathbf{x}_{k+1} = T_{k+1}\mathbf{x}_k$ with $T_{k+1} = \mathbf{1} + (1 + \lambda_{k+1})(P_{f(k+1)} - \mathbf{1})$, in which $\mathbf{1}$ is the identity operator on H. When $0 < 1 + \lambda_{k+1} < 2$, it is well-known that T_{k+1} is a non-expansive operator, leading to Fejér monotony. Hence, creation of a possible interruption of the monotony at $\mathbf{x}_{k+1}$ will depend on the value of λ_{k+1} that has to be chosen according to (13).

2.3. In this last subsection of Section 2, we give the necessary refinements concerning the choice of the function f and of the relaxation parameter λ_{k+1}, in order to obtain convergence of the constructed sequence $\{\mathbf{x}_k\}_{k=0}^{+\infty}$ to a point of C^*.

First of all, as we want to obtain a point in the intersection of the n sets C_j ($j = 1, \ldots, n$), it is clear that during the construction of the sequence $\{\mathbf{x}_k\}_{k=0}^{+\infty}$ each index $j \in \{1, \ldots, n\}$ should be involved an infinite number of times. We express the control strategy for the choice of the indices $j \in \{1, \ldots, n\}$ by the following form of what is often called

Almost cyclic control: there exists some positive integer $M > n$ such that, for each $k \in \mathbb{Z}^+$, each of the sets $C_1, \ldots, C_n$ contains at least one point from the set $\{\mathbf{x}_{k+1}, \ldots, \mathbf{x}_{k+M}\}$ of the M points following $\mathbf{x}_k$.

A practical application follows after the algorithm. In particular, we derive from this control strategy that the following result is true, that we formulate as Lemma 2.1, to be used further on:

Lemma 2.1. *When* $k+1 \in \{J+1+pN\}_{p=0}^{+\infty}$ *and* $\mathbf{w}_{k+1}$ *belongs to the set* C_I *for some index* $I \in \{1,\dots,n\}$*, then at least one of the* M *points* $\mathbf{x}_{k+1},\dots,\mathbf{x}_{k+M}$ *also belongs to* C_I*.*

In the second place, there seems to be a need to put a bound on the values of the relaxation parameters $\{\lambda_{k+1}\}_{k+1\in\{J+1+pN\}_{p=0}^{+\infty}}$. To explain that need, we prove the following preparatory result:

Lemma 2.2. *For the sequence* $\{\mathbf{x}_k\}_{k=0}^{+\infty}$ *constructed by the algorithm described above by combining Procedure PP and Procedure Relax, we have that:*

$$\lim_{\substack{k+1\to+\infty\\ k+1\notin\{J+1+pN\}_{p=0}^{+\infty}}} \|\mathbf{x}_k-\mathbf{x}_{k+1}\| = 0$$

and

$$\lim_{\substack{k+1\to+\infty\\ k+1\in\{J+1+pN\}_{p=0}^{+\infty}}} \|\mathbf{x}_k-\mathbf{w}_{k+1}\| = 0.$$

P r o o f. Assume that $\mathbf{x}_k$ has been obtained, and that $k+1 \in \{J+1+pN\}_{p=0}^{+\infty}$. Then $\mathbf{x}_{k+1}$ is determined from $\mathbf{x}_k$ by Procedure Relax, and inequality (12) is valid with R_{k+1} as given by (9). In particular, it follows from (12) that for each point $\mathbf{v}\in C^*$ the sequence of numbers $\{\|\mathbf{x}_{k+1}-\mathbf{v}\|\}_{k+1\in\{J+1+pN\}_{p=0}^{+\infty}}$ is non-negative and non-increasing, and hence there exists a non-negative number $d(\mathbf{v})$ such that:

$$\lim_{\substack{k+1\to+\infty\\ k+1\in\{J+1+pN\}_{p=-1}^{+\infty}}} \|\mathbf{x}_{k+1-N}-\mathbf{v}\| = d(\mathbf{v}) = \lim_{\substack{k+1\to+\infty\\ k+1\in\{J+1+pN\}_{p=0}^{+\infty}}} \|\mathbf{x}_{k+1}-\mathbf{v}\|.$$

This, combined with (12), leads to:

$$\lim_{\substack{k+1\to+\infty\\ k+1\in\{J+1+pN\}_{p=0}^{+\infty}}} R_{k+1} = 0,$$

from which in turn we derive the following separate limit results, all of them for $k+1 \to +\infty$ and $k+1 \in \{J+1+pN\}_{p=0}^{+\infty}$:

$$\begin{aligned} 0 &= \lim \|\mathbf{x}_{k+1-N} - \mathbf{x}_{k+1-(N-1)}\| \\ &= \lim \|\mathbf{x}_{k+1-(N-1)} - \mathbf{x}_{k+1-(N-2)}\| \\ &= \dots \\ &= \lim \|\mathbf{x}_{k-2} - \mathbf{x}_{k-1}\| \\ &= \lim \|\mathbf{x}_{k-1} - \mathbf{x}_{k}\| \\ &= \lim \|\mathbf{x}_{k} - \mathbf{w}_{k+1}\|. \end{aligned}$$

As we assumed during the derivation of these results that $k+1 \in \{J+1+pN\}_{p=0}^{+\infty}$, we see that the two indices that appear together in each one of the obtained expressions of limits in which two $\mathbf{x}$-points are figuring (i.e., without the very last expression) are consecutive, that the smallest one of the complete set of them is $k+1-N$ and as such belongs to the index set $\{J+1+pN\}_{p=-1}^{+\infty}$, and that the largest one of the complete set of them is k, and as such does not belong to the index set $\{J+1+pN\}_{p=0}^{+\infty}$. This is precisely the statement of the first part of Lemma 2.2. The last statement of Lemma 2.2 is obvious from the proof. □

In verbal-algebraic form, the contents of the first part of Lemma 2.2 may be stated as follows:

$$\lim_{k+1\to+\infty} \|\mathbf{x}_k - \mathbf{x}_{k+1}\| = 0,$$

on condition that the largest index $k+1$, appearing in the difference of two consecutive elements of the sequence $\{\mathbf{x}_k\}_{k=0}^{+\infty}$, does not belong to the set $\{J+1+pN\}_{p=0}^{+\infty}$. To prove convergence of the sequence $\{\mathbf{x}_k\}_{k=0}^{+\infty}$, we will need that also $\lim_{\substack{k+1\to+\infty \\ k+1\in\{J+1+pN\}_{p=0}^{+\infty}}} \|\mathbf{x}_k - \mathbf{x}_{k+1}\| = 0$. For such index $k+1$ we know that $\mathbf{x}_{k+1} = \mathbf{w}_{k+1} + \lambda_{k+1}(\mathbf{w}_{k+1} - \mathbf{x}_k)$ where $\mathbf{w}_{k+1} = P_{f(k+1)}\mathbf{x}_k$, and hence:

$$\|\mathbf{x}_{k+1} - \mathbf{x}_k\| = (1+\lambda_{k+1})\|\mathbf{w}_{k+1} - \mathbf{x}_k\|. \tag{14}$$

Now, in view of the last part in the statement of Lemma 2.2, we know that

$$\lim_{\substack{k+1\to+\infty \\ k+1\in\{J+1+pN\}_{p=0}^{+\infty}}} \|\mathbf{w}_{k+1} - \mathbf{x}_k\| = 0.$$

Hence, to obtain the result that also

$$\lim_{\substack{k+1\to+\infty \\ k+1\in\{J+1+pN\}_{p=0}^{+\infty}}} \|\mathbf{x}_k - \mathbf{x}_{k+1}\| = 0,$$

it is sufficient to put an upper bound on the set of numbers $\{1 + \lambda_{k+1}\}$ where $k+1 \in \{J+1+pN\}_{p=0}^{+\infty}$. We do it as follows: let B be a given (big) positive real number, and choose λ_{k+1} by:

$$\lambda_{k+1} = \min\left(B, \sqrt{\frac{\gamma R_{k+1}}{\|\mathbf{w}_{k+1} - \mathbf{x}_k\|^2}}\right). \tag{15}$$

We formulate our result as Lemma 2.3.

Lemma 2.3. *When λ_{k+1} is chosen as in* (15), *then:*

$$\lim_{\substack{k+1 \to +\infty \\ k+1 \in \{J+1+pN\}_{p=0}^{+\infty}}} \|\mathbf{x}_k - \mathbf{x}_{k+1}\| = 0.$$

We now have all the ingredients to state our algorithm.

Algorithm. In $H \equiv (\mathbb{R}^m, \langle\ ,\ \rangle, \|\ \|)$, let $C_1, \ldots, C_n$ be n given closed convex non-empty sets with corresponding metric projection operators $P_1, \ldots, P_n$ and with non-empty intersection $C^* \equiv \bigcap_{i=1}^{n} C_i$. Let N, J and M be positive integer numbers, $N > 2$, $J > N$, $M > n$; let γ be a real number, $0 < \gamma < 1$; let B be a big positive real number. Starting from some point $\mathbf{x_0}$ in $\mathbb{R}^m$, construct the sequence $\{\mathbf{x}_k\}_{k=0}^{+\infty}$ in $\mathbb{R}^m$ as follows: when $\mathbf{x}_k$ has been obtained, and when $\mathbf{x}_k \notin C^*$, then in constructing the next point $\mathbf{x}_{k+1}$ the following control strategy has to be used:

Control Strategy: For each $k \in \mathbb{Z}^+$ it has to be true that each of the sets $C_1, \ldots, C_n$ contains at least one point from the set of M points $\{\mathbf{x}_{k+1}, \ldots, \mathbf{x}_{k+M}\}$ that follow $\mathbf{x}_k$.

Taking into account this control strategy in what follows, the next point $\mathbf{x}_{k+1}$ is constructed as follows:

a) When $k+1 \notin \{J+1+pN\}_{p=0}^{+\infty}$, choose an index $f(k+1)$ in the set $\{1, 2, \ldots, n\}$ such that $\mathbf{x}_k \notin C_{f(k+1)}$, and put $\mathbf{x}_{k+1} = P_{f(k+1)}\mathbf{x}_k$.

b) When $k+1 \in \{J+1+pN\}_{p=0}^{+\infty}$, choose an index $f(k+1)$ in the set $\{1, 2, \ldots, n\}$ such that $\mathbf{x}_k \notin C_{f(k+1)}$, put: $\mathbf{w}_{k+1} = P_{f(k+1)}\mathbf{x}_k$, $R_{k+1} = \|\mathbf{x}_{k+1-N} - \mathbf{x}_{k+1-(N-1)}\|^2 + \ldots + \|\mathbf{x}_{k-2} - \mathbf{x}_{k-1}\|^2 + \|\mathbf{x}_{k-1} - \mathbf{x}_k\|^2 + \|\mathbf{x}_k - \mathbf{w}_{k+1}\|^2$, $\lambda_{k+1} = \min(B, \sqrt{\gamma R_{k+1}/\|\mathbf{w}_{k+1} - \mathbf{x}_k\|^2})$, and construct $\mathbf{x}_{k+1}$ by:

$$\mathbf{x}_{k+1} = \mathbf{w}_{k+1} + \lambda_{k+1}(\mathbf{w}_{k+1} - \mathbf{x}_k).$$

Remark. When looking at the practical application of the algorithm under the strategy of almost cyclic control, there is the fact that you do not see that this control strategy has been applied. The reason is that the number M in the control strategy may be choosen as a very big number. For instance, suppose that there are 15 convex sets C_j $(j = 1, 2, \ldots, 15)$ (e.g., 15 strips in $\mathbb{R}^2$), and that you take $M = 5000$. Then, for almost cyclic control, it is sufficient, for instance, that from the constructed sequence of points x_k, each of the sets $C_1, C_2, \ldots, C_{15}$ contains at least one point from the set of points with indices $4984, 4985, \ldots, 4999$ (a condition that in practice would be very easy to realize, by using pure projections for those points); that the same is true for the set of points with indices $9984, 9985, \ldots, 9999$; and so on. But in practice, it usually does not take 4900 iterations before convergence has appeared. On the other hand, what is well done of course in the practical algorithm (and this is the new thing), is the possible creation of interruption of monotony, by applying a suitable N as in the algorithm. As a consequence, in the practical application the index $f(k+1)$ of the next convex set is determined at random; the only thing you have to check, is that the current iteration point x_k is not already an element of the set $C_{f(k+1)}$; if it is, then take another index at random and do the same checking; as long as the current iteration point is not yet a point of the intersection, a suitable set $C_{f(k+1)}$ can be found.

In the next section we prove that the sequence $\{\mathbf{x}_k\}_{k=0}^{+\infty}$, constructed according to this algorithm, converges to a point of C^*.

3. Proof of convergence of the constructed sequence

In this Section, we prove that the sequence $\{\mathbf{x}_k\}_{k=0}^{+\infty}$ that has been constructed according to the algorithm, is convergent to a point $\mathbf{a} \in C^*$. The constructed sequence $\{\mathbf{x}_k\}_{k=0}^{+\infty}$ contains in particular the subsequence $\{\mathbf{x}_{J+1+pN}\}_{p=0}^{+\infty}$ consisting of points where the monotoneous behaviour possibly may be interrupted; for brevity, we will often denote this subsequence as $\{\mathbf{x}_{n_p}\}_{p=0}^{+\infty}$. The proof of convergence of the complete sequence $\{\mathbf{x}_k\}_{k=0}^{+\infty}$ will result from the following separate parts:

(i) The subsequence $\{\mathbf{x}_{n_p}\}_{p=0}^{+\infty}$ contains a subsequence that converges to a point $\mathbf{a} \in C^*$.

(ii) Each converging subsequence of $\{\mathbf{x}_{n_p}\}_{p=0}^{+\infty}$ converges to the same point $\mathbf{a}$.

(iii) The sequence $\{\mathbf{x}_k\}_{k=0}^{+\infty}$ converges to $\mathbf{a}$.

Proof of (i). The "interrupting" subsequence $\{\mathbf{x}_{n_p}\}_{p=0}^{+\infty}$ is bounded, and hence it contains a subsequence that converges to some point $\mathbf{a} \in H$. As the number of sets $\{C_i\}_{i=1}^{n}$ is finite, this last subsequence contains itself a subsequence that we denote as $\{\mathbf{x}_{n_{p_s}}\}_{s=0}^{+\infty}$, such that also $\{\mathbf{x}_{n_{p_s}}\}_{s=0}^{+\infty}$ converges to $\mathbf{a}$, and such that in the original sequence $\{\mathbf{x}_k\}_{k=0}^{+\infty}$ each element that follows after each $\mathbf{x}_{n_{p_s}}$ has been obtained by pure projection onto a fixed closed convex set, say C_α for some $\alpha \in \{1, \dots, n\}$; i.e., $\mathbf{x}_{n_{p_s}+1} = P_{C_\alpha}\mathbf{x}_{n_{p_s}}$. Hence, $\|\mathbf{a} - P_{C_\alpha}\mathbf{x}_{n_{p_s}}\| \leqslant \|\mathbf{a} - \mathbf{x}_{n_{p_s}}\| + \|\mathbf{x}_{n_{p_s}} - P_{C_\alpha}\mathbf{x}_{n_{p_s}}\|$, and both terms on the right hand side of this inequality tend to zero, the first one by convergence of $\{\mathbf{x}_{n_{p_s}}\}_{s=0}^{+\infty}$ to $\mathbf{a}$, the second one by Lemma 2.2. This means that the sequence $\{P_{C_\alpha}\mathbf{x}_{n_{p_s}}\}_{s=0}^{+\infty}$ converges to $\mathbf{a}$; hence, $\mathbf{a} \in C_\alpha$.

Let now β be any index from $\{1, \dots, n\}$, $\beta \neq \alpha$; we prove that $\mathbf{a}$ also belongs to C_β.

If it is the case that an infinite number of elements of the converging subsequence $\{\mathbf{x}_{n_{p_s}}\}_{s=0}^{+\infty}$ also belongs to C_β, then it is clear that also $\mathbf{a}$ belongs to C_β.

On the other hand, if the former assumption is not true, then due to the control strategy the following is certainly true: within each M iterations in the original sequence at least one point belonging to C_β will be appearing. Somewhat more concretely, when $\mathbf{x}_{n_{p_s}+1}$ belongs to C_α, then among the M elements that precede $\mathbf{x}_{n_{p_s}+1}$ in the original sequence there is at least one element that belongs to C_β; we denote that element by $\mathbf{x}_{q_s}$. Then of course the following inequality (INEQ) is true, for each positive index s:

$$\text{(INEQ)} \qquad \|\mathbf{x}_{q_s} - \mathbf{a}\| \leqslant \|\mathbf{x}_{q_s} - \mathbf{x}_{n_{p_s}+1}\| + \|\mathbf{x}_{n_{p_s}+1} - \mathbf{a}\|.$$

For the first term on the right hand side of (INEQ) we have that, applying the triangle inequality no more than M times, an upper bound of that term is obtained consisting of no more than M terms, in which each term tends to zero when s tends to infinity due to Lemma 2.2 and Lemma 2.3. Hence, that first term tends to zero. For the second term on the right hand side of (INEQ) it has already been proved that it tends to zero when s tends to infinity. Hence, the subsequence $\{\mathbf{x}_{q_s}\}_{s=0}^{+\infty}$ converges to $\mathbf{a}$, and we know that each element $\mathbf{x}_{q_s}$ belongs to the closed set C_β. We conclude that $\mathbf{a} \in C_\beta$. As β was an arbitrary index, it results that $\mathbf{a} \in C^*$. This concludes the proof of (i). □

Proof of (ii). Let us suppose that the subsequence $\{\mathbf{x}_{n_p}\}_{p=0}^{+\infty}$, which is a short-hand notation for $\{\mathbf{x}_{J+1+pN}\}_{p=0}^{+\infty}$, contains besides a subsequence, say $\{\mathbf{x}_{n_{p_r}}\}_{r=0}^{+\infty}$, that converges to the point $\mathbf{a} \in C^*$, also the subsequence $\{\mathbf{x}_{n_{p_t}}\}_{t=0}^{+\infty}$ that converges to a point $\mathbf{a}'$. With a proof as in "Proof of (i)" above, it will follow that also $\mathbf{a}'$ belongs to C^*. We now prove that $\mathbf{a}' = \mathbf{a}$.

From the fact that the points $\mathbf{a}$ and $\mathbf{a}'$ belong to C^* and from inequality (12) in which $k+1 \in \{J+1+pN\}_{p=0}^{+\infty}$, we derive that the number sequences $\{\|\mathbf{x}_{n_p} - \mathbf{a}\|\}_{p=0}^{+\infty}$ and $\{\|\mathbf{x}_{n_p} - \mathbf{a}'\|\}_{p=0}^{+\infty}$ are convergent, with respective limits that we denote as $d(\mathbf{a})$ and $d(\mathbf{a}')$; of course the same is true for their respective subsequences $\{\|\mathbf{x}_{n_{p_r}} - \mathbf{a}\|\}_{r=0}^{+\infty}$, $\{\|\mathbf{x}_{n_{p_t}} - \mathbf{a}\|\}_{t=0}^{+\infty}$, $\{\|\mathbf{x}_{n_{p_r}} - \mathbf{a}'\|\}_{r=0}^{+\infty}$ and $\{\|\mathbf{x}_{n_{p_t}} - \mathbf{a}'\|\}_{t=0}^{+\infty}$.

The sequence $\{\mathbf{x}_{n_{p_r}}\}_{r=0}^{+\infty}$ is convergent to the point $\mathbf{a}$. Writing $\mathbf{x}_{n_{p_r}} - \mathbf{a}'$ as $\mathbf{x}_{n_{p_r}} - \mathbf{a} + \mathbf{a} - \mathbf{a}'$, and developing, leads to:

$$\|\mathbf{x}_{n_{p_r}} - \mathbf{a}'\|^2 - \|\mathbf{x}_{n_{p_r}} - \mathbf{a}\|^2 = 2\langle \mathbf{x}_{n_{p_r}} - \mathbf{a}, \mathbf{a} - \mathbf{a}'\rangle + \|\mathbf{a} - \mathbf{a}'\|^2, \tag{16}$$

and analogously we obtain:

$$\|\mathbf{x}_{n_{p_t}} - \mathbf{a}\|^2 - \|\mathbf{x}_{n_{p_t}} - \mathbf{a}'\|^2 = 2\langle \mathbf{x}_{n_{p_t}} - \mathbf{a}', \mathbf{a}' - \mathbf{a}\rangle + \|\mathbf{a}' - \mathbf{a}\|^2. \tag{17}$$

Taking in (16) and (17) the limit, respectively for $r \to +\infty$ and for $t \to +\infty$, we obtain:

$$d(\mathbf{a}')^2 - d(\mathbf{a})^2 = 0 + \|\mathbf{a} - \mathbf{a}'\|^2$$

and

$$d(\mathbf{a})^2 - d(\mathbf{a}')^2 = 0 + \|\mathbf{a}' - \mathbf{a}\|^2.$$

Hence, $d(\mathbf{a}) = d(\mathbf{a}')$, and from this we easily deduce that $\mathbf{a}' = \mathbf{a}$. □

Proof of (iii). We finally have to prove that the originally constructed sequence $\{\mathbf{x}_k\}_{k=0}^{+\infty}$ converges to $\mathbf{a}$, when we know that this sequence contains the specific subsequence $\{\mathbf{x}_{n_p}\}_{p=0}^{+\infty}$ that converges to $\mathbf{a} \in C^*$. Let j be any index in $\mathbb{Z}^+$; then there exist successive indices n_p and n_q of the subsequence $\{\mathbf{x}_{n_p}\}_{p=0}^{+\infty}$ such that $n_p < j \leqslant n_q$, with $n_p \equiv J + 1 + (p-1)N$ and $n_q \equiv J + 1 + pN$. When $n_p < j < n_q$, then $\mathbf{x}_j$ has been obtained by pure projection, and from (2) we know that $\|\mathbf{x}_j - \mathbf{a}\| \leqslant \|\mathbf{x}_{j-1} - \mathbf{a}\| \leqslant \ldots \leqslant \|\mathbf{x}_{n_p} - \mathbf{a}\|$, and $\|\mathbf{x}_{n_p} - \mathbf{a}\| \to 0$ when $n_p \to +\infty$. Hence, also $\mathbf{x}_j \to \mathbf{a}$ when $j \to +\infty$. When $j = n_q$ there is nothing to prove. □

4. Examples

In this final section, we consider two examples to illustrate some aspects concerning the number of iterations that is needed to reach the intersection set, once by using the sequential algorithm in which each iteration corresponds to the use of a single pure projection on a randomly chosen set (this algorithm is in fact procedure PP, and hence is denoted as PP) and once by using the newly introduced non-monotoneous algorithm with the following fixed set of parameters: $N = 5$, $J = 10$, $\gamma = 0.9$ (we denote this algorithm as NONMONOT). The only difference between the two algorithms consists in the fact that, when $\mathbf{x}_k$ has been obtained, then in PP we have that $\mathbf{x}_{k+1} = P_{f(k+1)}\mathbf{x}_k$ for **all** $k+1$, while in NONMONOT we have that $\mathbf{x}_{k+1}$ is given by an analogous expression as in PP when $k+1 \notin \{J+1+pN\}_{p=0}^{+\infty}$ but that $\mathbf{x}_{k+1}$ is obtained by a suitable relaxed projection when $k+1 \in \{J+1+pN\}_{p=0}^{+\infty}$.

The explicit form of the projection operator onto a strip in a real Hilbert space $H = (H, \langle\ ,\ \rangle, \|\ \|)$ with inner product $\langle\ ,\ \rangle$ and norm $\|\ \|$ derived from it, can be found in [12, pp. 98–99]. The result is as follows:

Let b_1 and b_2 be two real numbers, $b_1 < b_2$. Let $\mathbf{a}$ be a non-zero vector in $\mathbf{H}$, and let $\mathbf{y}$ be a generic element of $\mathbf{H}$. Put $C = \{\mathbf{y}\colon\ b_1 \leqslant < \mathbf{y}, \mathbf{a} > \leqslant b_2\}$. Then the projection of a point $\mathbf{x}$ of $\mathbf{H}$ onto the strip C is as follows (formula (3.3–10) in [12]):

$$P\mathbf{x} = \mathbf{x} \quad \textbf{if x belongs to } C;$$

$$P\mathbf{x} = \mathbf{x} + \frac{b_1 - \langle \mathbf{x}, \mathbf{a} \rangle}{\|\mathbf{a}\|^2}\mathbf{a}, \quad \text{if } \langle \mathbf{x}, \mathbf{a} \rangle < b_1;$$

$$P\mathbf{x} = \mathbf{x} + \frac{b_2 - \langle \mathbf{x}, \mathbf{a} \rangle}{\|\mathbf{a}\|^2}\mathbf{a}, \quad \text{if } \langle \mathbf{x}, \mathbf{a} \rangle > b_2.$$

As our first example, we take as closed convex sets $\{C_i\}_{i=1}^{15}$ 15 "strips" in $\mathbb{R}^2$, i.e., when $\{a_i^1\}_{i=1}^{15}$, $\{a_i^2\}_{i=1}^{15}$, $\{b_i^1\}_{i=1}^{15}$ and $\{b_i^2\}_{i=1}^{15}$ are sets of real numbers, then each C_i is given by

$$C_i = \{(x, y) \in \mathbb{R}^2 :\ b_i^1 \leqslant a_i^1 x + a_i^2 y \leqslant b_i^2\}$$

The numbers are chosen such that $\bigcap_{i=1}^{15} C_i$ is non-empty.

Let me first describe the 15 strips $C_1, \ldots, C_{15}$ in $\mathbb{R}^2$. Denote by (x, y) a generic point of $\mathbb{R}^2$; the equations of the strips are with respect to some orthonormal reference system.

$C_1 = \{(x, y)\colon\ 1.42038 \leqslant 7.00000x + 1.00000y \leqslant 1.66452\}$;
$C_2 = \{(x, y)\colon\ 0.80972 \leqslant 7.94594x + 0.29497y \leqslant 1.86200\}$;
$C_3 = \{(x, y)\colon\ 0.87437 \leqslant 6.44009x + 1.15324y \leqslant 2.18200\}$;
$C_4 = \{(x, y)\colon\ 0.81815 \leqslant 6.44233x + 1.02001y \leqslant 1.80509\}$;
$C_5 = \{(x, y)\colon\ 1.21392 \leqslant 7.85262x + 1.15318y \leqslant 2.10985\}$;
$C_6 = \{(x, y)\colon\ 0.77656 \leqslant 6.16255x + 1.04938y \leqslant 1.90699\}$;
$C_7 = \{(x, y)\colon\ 0.68006 \leqslant 6.09820x + 0.32307y \leqslant 1.71803\}$;
$C_8 = \{(x, y)\colon\ 1.64581 \leqslant 6.69859x + 1.88168y \leqslant 2.37475\}$;
$C_9 = \{(x, y)\colon\ 2.03135 \leqslant 6.86590x + 1.93341y \leqslant 2.41595\}$;
$C_{10} = \{(x, y)\colon\ 0.76846 \leqslant 6.92938x + 1.18142y \leqslant 2.53657\}$;
$C_{11} = \{(x, y)\colon\ 1.73134 \leqslant 7.87282x + 1.83773y \leqslant 2.87812\}$;
$C_{12} = \{(x, y)\colon\ 1.37252 \leqslant 6.26066x + 1.93449y \leqslant 2.47537\}$;
$C_{13} = \{(x, y)\colon\ 0.99053 \leqslant 6.38268x + 0.77575y \leqslant 1.75816\}$;
$C_{14} = \{(x, y)\colon\ 1.50190 \leqslant 6.07839x + 1.39896y \leqslant 2.25000\}$;
$C_{15} = \{(x, y)\colon\ 0.27526 \leqslant 6.04528x + 0.70341y \leqslant 1.78783\}$.

It was the intention to obtain strips with a nonempty intersection and whose border lines make with each other rather acute angles, in order to create a narrow corridor for the iteration proces. The construction of them went as follows.

A point $\mathbf{w} = (u, v)$ in $\mathbb{R}^2$ with random coordinates $0 < u < 1$ and $0 < v < 1$, was created. Next, the straight line with equation $7.00000x + 1.00000y = b$, was created. The scalar b was choosen such that the point (u, v) was a point on that straight line; writing in short notation $\mathbf{a} = (7.00000, 1.00000)$, b was given by the value $b = \langle a, w\rangle$, with $\langle\ ,\ \rangle$ the usual inner product in $\mathbb{R}^2$. To obtain the first strip C_1 that contains the point $\mathbf{w}$, two random numbers r_1 and r_2 with value between 0 and 1 were generated, and we computed $b_1 = b - r_1$, $b_2 = b + r_2$. This led to $C_1 = \{\mathbf{x}\colon\ b_1 \leqslant \langle \mathbf{a}, \mathbf{x}\rangle \leqslant b_2\}$.

Each new strip was created as follows. The two numbers in the coefficient $\mathbf{a}(\mathbf{new})$ were obtained from the coefficient $\mathbf{a}$ in the first strip by at random adding or subtracting a random number between 0 and 1 to or from the coefficients of the first $\mathbf{a}$. Then, a new scalar b, $b(\text{new})$, was computed such that $\langle \mathbf{a}(\mathbf{new}), \mathbf{w}\rangle = b(\text{new})$. From this scalar $b(\text{new})$, the scalars $b(\text{new})_1$ and $b(\text{new})_2$ were determined at random by the same procedure as for the first strip. And so on. The result is that each of the 15 strips contains certainly the point $\mathbf{w}$, and hence there is a nonempty intersection.

For a fixed chosen starting point and taking into account the random choice of a new set in both algorithms, the algorithms were run 30 times. The numbers in Table 1 give the mean number of iterations (for those 30 runs) in order to obtain an intersection point, starting from 6 different starting points.

Starting point	$(0,0)$	$(-10,-10)$	$(9,2)$	$(-3,6)$	$(5,-1)$	$(7,8)$
PP	1873	2402	760	1035	2183	1034
NONMONOT	195	234	110	194	304	198

Table 1

For our second example, we construct in $\mathbb{R}^5$ 50 sets of the form $\langle A_i, X\rangle \leqslant b_i$ $(i = 1, \ldots, 50)$, with $b_i \in \mathbb{R}$, $A_i \equiv (a_i^1, a_i^2, a_i^3, a_i^4, a_i^5)$ a given 5-tuple of real numbers, and $X \equiv (x_1, x_2, x_3, x_4, x_5)$ a generic point of $\mathbb{R}^5$. Again, the numbers are chosen such that the 50 sets have a non-empty intersection. Just like in the first example, for a given starting point both algorithms were run 30 times with random choice of the next set to project on, and the mean number (over these 30 runs) of iterations in order to obtain an intersection point was noted. This leads to Table 2.

These two examples reflect what has been observed experimentally in several analogous experiments (but what is not stated here in order to save space): although using a random order for projection is often judged favourable in contrast to a fixed order, the classical sequential monotoneous algorithm PP may lead to a very slow convergence dependent on the starting point; in contrast, the possibility of interrupting the monotoneous convergence behaviour as in NONMONOT may considerably improve the speed of convergence. Hence, based on several experimental results, the

Starting point	$(8,0,7,0,9)$	$(6,0,0,2,8)$	$(100,80,1,200,9)$
PP	184	171	101
NONMONOT	27	29	25

Starting point	$(20,20,1,20,20)$	$(-2000,300,-1000,-100,-10)$
PP	99	60
NONMONOT	22	14

Starting point	$(-1000,800,-500,-1000,100)$
PP	13
NONMONOT	11

Table 2

newly introduced algorithm could be a good alternative for existing monotoneous algorithms.

One of the good things in algorithms that are created by the method of Projections onto convex sets, is that the constructed sequence converges for whatever starting point. However, experimentally (as also follows from the examples), the choice of the starting point has a real influence on the number of iterations in order to obtain convergence. As far as I know, until now there is no theoretical study about this dependence. For that reason, the algorithm in the present paper seems to be very good, since it seems that the influence of the starting point on the number of iterations has been reduced when compared to more traditional algorithms.

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