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SIGNED AND MINUS DOMINATION IN BIPARTITE GRAPHS

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Abstract. The paper studies the signed domination number and the minus domination number of the complete bipartite graph $K_{p,q}$.

Keywords: signed domination number, minus domination number, complete bipartite graph

MSC 2000: 05C69

Here we shall study two numerical invariants of graphs concerning domination, namely the signed domination number and the minus domination number [1].

If f is a function which maps the vertex set V of a graph G into some set of numbers and $S \subseteq V$, then $f(S) = \sum_{x \in S} f(x)$.

Let $f: V \rightarrow \{-1, 1\}$. If for the closed neighbourhood $N[v]$ of any vertex $v \in V$ we have $f(N[v]) \geq 1$, then f is called a signed dominating function (SDF) of G . The value $f(V)$ is called the weight $w(f)$ of f . The minimum of $w(f)$ taken over all SDF's is called the signed domination number $\sigma_{\text{sg}}(G)$ of G .

If in this definition we replace the set $\{-1, 1\}$ by $\{-1, 0, 1\}$ we obtain the definition of the minus dominating function (MDF) and of the minus domination number $\sigma^-(G)$ of G .

We shall study $\sigma_{\text{sg}}(K_{p,q})$ and $\sigma^-(K_{p,q})$ for the complete bipartite graph $K_{p,q}$. We suppose always that $q \leq p$.

We start with the signed domination number. If a SDF f on $K_{p,q}$ is given, we use the following notation:

The bipartition classes of $K_{p,q}$ are P, Q with $|P| = p, |Q| = q$. We define $V^+ = \{v \in V: f(v) = 1\}$, $V^- = \{v \in V: f(v) = -1\}$. Further $P^+ = V^+ \cap P$,

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$P^- = V^- \cap P$, $Q^+ = V^+ \cap Q$, $Q^- = V^- \cap Q$ and $p^+ = |P^+|$, $p^- = |P^-|$, $q^+ = |Q^+|$, $q^- = |Q^-|$. Therefore $w(f) = p^+ + q^+ - p^- - q^-$.

Now we express a theorem.

Theorem 1. *Let $K_{p,q}$ be a complete bipartite graph with the bipartition classes P , Q such that $|P| = p$, $|Q| = q$, $q \leq p$. Let $\sigma_{\text{sg}}(K_{p,q})$ be the signed domination number of $K_{p,q}$. Then*

- (i) *for $q = 1$ there is $\sigma_{\text{sg}}(K_{p,q}) = p + 1$;*
- (ii) *for $2 \leq q \leq 3$ there is $\sigma_{\text{sg}}(K_{p,q}) = q$ for p even and $\sigma_{\text{sg}}(K_{p,q}) = q + 1$ for p odd;*
- (iii) *for $q \geq 4$ there is $\sigma_{\text{sg}}(K_{p,q}) = 4$ for both p and q even, $\sigma_{\text{sg}}(K_{p,q}) = 6$ at both p , q odd and $\sigma_{\text{sg}}(K_{p,q}) = 5$ for one of the numbers p , q even and the other odd.*

Proof. First we prove (i). Let $q = 1$. Then $K_{p,q}$ is either K_2 , or a star with p edges. For the first case the assertion is evident. Thus let $K_{p,q}$ be a star. Then $Q = \{c\}$, where c is the central vertex and P is the set of vertices of degree 1. Let $x \in P$. Then $N[x] = \{x, c\}$ and $f(N[x]) = f(x) + f(c) \geq 2$ for any SDF f . This implies $f(x) = f(c) = 1$. As x was chosen arbitrarily, $K_{p,q}$ has the unique SDF f which has the value 1 in all vertices. Thus $w(f) = p + 1$ and also $\sigma_{\text{sg}}(K_{p,q}) = p + 1$.

The continuation of the proof will consist from a series of claims.

Claim 1. *Let $Q^- = \emptyset$. Then if f is a SDF, then $w(f) \geq q$ for p even and $w(f) \geq q + 1$ for p odd.*

Proof. Let f be a SDF and $Q^- = \emptyset$. Then $Q = Q^+$ and $f(Q) = q$. Let $x \in Q$. Then $N[x] = \{x\} \cup P$ and $f(N[x]) = f(x) + f(P) = 1 + f(P)$. The inequality $f(N[x]) \geq 1$ holds only if $f(P) \geq 0$. We have $f(P) = p^+ - p^-$, $p = p^+ + p^-$ and this implies $f(P) = 2p^+ - p$. If $f(P) \geq 0$ and p is even, then $p^+ \geq \frac{1}{2}p$, $p^- \leq \frac{1}{2}p$, $f(P) \geq 0$. If p is odd, then $p^+ \geq \frac{1}{2}(p + 1)$, $p^- \leq \frac{1}{2}(p - 1)$ and $f(P) \geq 1$. This implies the assertion. □

Claim 2. *Let $P^- = \emptyset$. Then if f is a SDF, then $w(f) \geq p$ for q even and $w(f) \geq p + 1$ for q odd.*

Proof. The proof of this claim is analogous to that of Claim 1. Note that $q \leq p$ and thus such a lower bound is greater than or equal to the bound from Claim 1. □

Claim 3. Let $Q \neq \emptyset$. Then $f(P) \geq 2$ for p even and $f(P) \geq 3$ for p odd.

Proof. Let $x \in Q^-$. Then $f(N[x]) = f(P) - f(x) = f(P) - 1$. Further considerations are analogous to those from the proof of Claim 1. We obtain here $2p^+ - p \geq 2$ and $p^+ \geq \frac{1}{2}p + 1$, $p^- \leq \frac{1}{2}p - 1$ for p even and $p^+ \geq \frac{1}{2}(p + 3)$, $p^- \leq \frac{1}{2}(p - 3)$ for p odd. In the case of p even we have $f(P) = p^+ - p^- \geq 2$, in the case of p odd we have $f(P) \geq 3$. \square

Claim 4. Let $P \neq \emptyset$. Then $f(Q) \geq 2$ for q even and $f(Q) \geq 3$ for q odd.

Proof. The proof of this claim is quite analogous to that of Claim 3. \square

Claim 5. If $P^- \neq \emptyset$ and $Q \neq \emptyset$, then for every SDF f we have $w(f) \geq 4$ for both p, q even, $w(f) \geq 6$ for both p, q odd and $w(f) \geq 5$ for one of the numbers p, q even and the other odd.

Proof. This follows from Claim 3 and Claim 4, noting that $w(f) = f(P) + f(Q)$. \square

Conclusion of the proof of Theorem 1. For $q = 1$ the proof is ready. For $q \geq 6$ evidently the lower bound for $w(f)$ from Claim 5 is less than that from Claim 1 and Claim 2. Evidently also for $2 \leq q \leq 3$ the converse is true. By considering particular cases we see that for $4 \leq q \leq 5$ both bounds coincide. Therefore it remains to construct a SDF f for which the equality occurs. For $2 \leq q \leq 3$ we put $f(x) = 1$ for each $x \in Q$ and for $\frac{1}{2}p$ vertices of P for p even or $\frac{1}{2}(p + 1)$ vertices x of P for p odd. For $q \geq 4$ we assign the value 1 to $\frac{1}{2}p + 1$ vertices of P for p even or $\frac{1}{2}(p + 3)$ vertices of P for p odd and analogously to $\frac{1}{2}q + 1$ vertices of Q for q even or $\frac{1}{2}(q + 3)$ vertices of Q for q odd. This implies the assertion. \square

In the sequel we shall study the minus domination number. We still use the notation F, Q, p, q and a MDF will be denoted by g .

Theorem 2. Let $K_{p,q}$ be a complete bipartite graph with the bipartition classes P, Q such that $|P| = p, |Q| = q, q \leq p$. Let $\sigma^-(K_{p,q})$ be the minus domination number of $K_{p,q}$. Then

- (i) for $q = 1$ there is $\sigma^-(K_{p,q}) = 1$;
- (ii) for $2 \leq q \leq p$ there is $\sigma^-(K_{p,q}) = 2$.

Proof. First we prove (i). Let $q = 1$. Then $K_{p,q}$ is either K_2 , or a star with p edges. For the first case the assertion is evident. Thus let $K_{p,q}$ be a star. Then $Q = \{c\}$, where c is the central vertex and P is the set of vertices of degree 1. Let $x \in P$, and let g be a MDF of $K_{p,q}$. Then $N[x] = \{x, c\}$ and $g(N[x]) = g(x) + g(c)$.

This is possible only if one of the vertices x, c has the value 1 and the other 0 or 1. Therefore $w(g) \geq 1$. We construct MDF g with $w(g) = 1$. It suffices to put $f(c) = 1$ and $f(x) = 0$ for each $x \in P$. This implies the assertion.

Now we prove (ii). Let $2 \leq q \leq p$. Suppose that there exists a MDF g with $w(g) \leq 1$. We have $w(g) = g(P) + g(Q)$; this implies that at least one of these values, say $g(Q) \leq 0$. Let $x \in P$. We have $g(N[x]) = g(x) + g(Q) \leq 1 + 0 = 1$. This is possible only if $g(x) = 1$ and $g(Q) = 0$. As x was chosen arbitrarily, we have $g(x) = 1$ for each $x \in P$ and $g(P) = p$. Then $w(g) = p \geq 2$, which is a contradiction. Therefore $w(g) \geq 2$ for each MDF g . A MDF g with $w(g) = 2$ can be obtained by choosing $u \in P, v \in Q$ and putting $g(u) = g(v) = 1, f(x) = 0$ for any $x \in V - \{u, v\}$. This implies the assertion. \square

References

- [1] *W. T. Haynes, S. T. Hedetniemi, P. J. Slater: Fundamentals of Domination in Graphs.* Marcel Dekker, New York-Basel-Hong Kong, 1998. [Zbl 0890.05002](#)

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