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## PROJECTIVE MODULES AND PRIME SUBMODULES

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*Abstract.* In this paper, we use Zorn's Lemma, multiplicatively closed subsets and saturated closed subsets for the following two topics:

- (i) The existence of prime submodules in some cases,
- (ii) The proof that submodules with a certain property satisfy the radical formula.

We also give a partial characterization of a submodule of a projective module which satisfies the prime property.

*Keywords:* prime submodule, primary submodule,  $\mathcal{S}$ -closed subsets, the radical formula

*MSC 2000:* 113A10, 13A99, 13C10

### 0. INTRODUCTION

Throughout the paper  $R$  will denote a commutative ring with identity. Let  $M$  be a unitary module over  $R$ . Let  $B$  and  $C$  be two submodules of  $M$ . Then it is clear that the set  $\{r \in R: rC \leq B\}$  is an ideal of  $R$ , denoted by  $(B : C)$ . A proper submodule  $N$  of  $M$  with  $\mathfrak{P} = (N : M)$  is said to be  $\mathfrak{P}$ -prime if  $rm \in N$  for  $r \in R$  and  $m \in M$  implies that  $r \in \mathfrak{P}$  or  $m \in N$ . It is well-known that a proper submodule  $N$  of  $M$  is prime if and only if  $\mathfrak{P} = (N : M)$  is a prime ideal in  $R$  and the  $R/\mathfrak{P}$ -module  $M/N$  is torsion free. For any submodule  $N$  of  $M$ , the radical of  $N$  in  $M$  is defined to be the intersection of all prime submodules of  $M$  containing  $N$ , denoted by  $M\text{-rad}_R N$ . Also  $M\text{-rad}_R 0$  is defined to be the intersection of all prime submodules of  $M$ . If there is no prime submodule containing  $N$ , then  $M\text{-rad}_R N = M$ . The radical of submodules has been studied in recent years (see, for example, [6], [8], [8]). In this paper we continue these investigations for a certain case.

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Section 1 is concerned with the existence of prime submodules. We also prove a consequence of the Prime Avoidance Theorem for modules and give its application.

In Section 2, the main aim is to give a necessary and sufficient condition for the equality  $M\text{-rad}_R N = \sqrt{(N : M)}M$  where  $N$  is a submodule of a projective  $R$ -module  $M$ . For a submodule  $N$  of a finitely generated projective module  $M$ , we prove that  $N$  is prime if and only if  $(N : M)$  is prime and  $M/N$  is a projective  $R/P$ -module. Moreover, we show that  $M\text{-rad}_R N = \sqrt{(N : M)}M + N = RE_M(N)$  for a submodule  $N$  of a module  $M$  provided  $M/N$  is projective. In particular, we show that  $M\text{-rad}_R 0 = \sqrt{(0 : M)}M$  for a projective  $R$ -module  $M$ .

## 1. $\mathcal{S}$ -CLOSED SUBSET OF MODULES

In the first half of this section, we give a consequence of the Prime Avoidance Theorem for modules to which Lu extended the Prime Avoidance Theorem for rings in [4] and we give an application of them. Now we start by recalling the Prime Avoidance Theorem for modules.

**Theorem 1.1** (Prime Avoidance Theorem [4]). *Let  $M$  be an  $R$ -module. Let  $N_1, N_2, \dots, N_n$  be a finite number of submodules of  $M$  and let  $N$  be a submodule of  $M$  such that  $N \subseteq N_1 \cup \dots \cup N_n$ . Assume that at most two of the  $N_i$ 's ( $1 \leq i \leq n$ ) are not prime and that  $(N_j : M) \not\subseteq (N_k : M)$  whenever  $j \neq k$ . Then  $N \subseteq N_i$  for some  $i$ .*

Now we extend [11, Theorem 3.64] to the module case by using Theorem 1.1.

**Theorem 1.2.** *Let  $M$  be an  $R$ -module. Suppose that  $N_1, \dots, N_r$  are prime submodules of  $M$  such that  $(N_i : M) \not\subseteq (N_j : M)$  for  $i \neq j$  where  $r \geq 1$ , let  $N$  be a submodule of  $M$  and let  $m \in M$  be such that  $mR + N \not\subseteq \bigcup_{i=1}^r N_i$ . Then there exists*

*$n \in N$  such that  $m + n \notin \bigcup_{i=1}^r N_i$ .*

*Proof.* Suppose that  $m$  lies in each of  $N_1, \dots, N_k$  but in none of  $N_{k+1}, \dots, N_r$ . If  $k = 0$  then  $m = m + 0 \notin \bigcup_{i=1}^r N_i$  and so there is nothing to prove. Now we assume that our claim is true for  $k \geq 1$ .

Now  $N \not\subseteq \bigcup_{i=1}^k N_i$ , for otherwise by the Prime Avoidance Theorem we would have a contradiction. Thus there exists  $d \in N \setminus (N_1 \cup \dots \cup N_k)$ . Hence we have  $N_{k+1} \cap \dots \cap N_r \not\subseteq N_1 \cup \dots \cup N_k$ . Otherwise, since  $N_j$  is a prime submodule, by the Prime Avoidance Theorem we get a contradiction. Thus there exists  $b \in (N_{k+1} : M) \cap \dots \cap (N_r : M) \setminus ((N_1 : M) \cup \dots \cup (N_k : M))$ . Let  $n = bd \in N$ . On the other hand,

$n \in \bigcap_{j=k+1}^r N_j$ . Then  $n = bd \notin N_1 \cup \dots \cup N_k$ . Otherwise,  $bd \in N_i$  for  $1 \leq i \leq k$ . Since  $N_i$  is prime, either  $d \in N_i$  or  $b \in (N_i : M)$ . Then  $n \in (N_{k+1} \cap \dots \cap N_r) \setminus (N_1 \cup \dots \cup N_k)$ . Therefore, since  $m \in (N_1 \cup \dots \cup N_k)$ , it follows that  $m + n \notin \bigcup_{i=1}^r N_i$ .  $\square$

Now our main aim is to use Zorn's Lemma for the existence of prime submodules under a certain condition. It is concerned with a subset which is closed relative to a multiplicatively closed subset in a commutative ring. Throughout this section, we assume that every multiplicatively closed subset of  $R$  contains 1, but does not contain 0. Let  $\mathcal{S}$  be a multiplicatively closed subset of a ring  $R$  and let  $M$  be an  $R$ -module. Then following [4], a non-empty subset  $S^*$  of  $M$  is said to be  $\mathcal{S}$ -closed if  $sm \in S^*$  for every  $s \in \mathcal{S}$  and  $m \in S^*$ . Further, an  $\mathcal{S}$ -closed subset  $S^*$  is *saturated* if the following condition is satisfied: whenever  $rm \in S^*$  for  $r \in R$  and  $m \in M$ , then  $r \in \mathcal{S}$  and  $m \in S^*$ .

Let  $N$  be a prime submodule of an  $R$ -module  $M$ . Evidently, if  $S^* = M \setminus N$  and  $\mathcal{S} = R \setminus (N : M)$ , then  $S^*$  is a saturated  $\mathcal{S}$ -closed subset of  $M$ . Now we give the main theorem of this section.

**Theorem 1.3.** *Let  $N$  be a submodule of an  $R$ -module  $M$  and let  $\mathcal{S}$  be a multiplicatively closed subset of  $R$ . Also suppose that  $S^*$  is an  $\mathcal{S}$ -closed subset of  $M$  with  $N \cap S^* = \emptyset$  and  $\mathfrak{P} = (N : M)$  is a maximal ideal in  $R \setminus \mathcal{S}$  such that  $M/\mathfrak{P}M$  is a finitely generated  $R$ -module. Then the set*

$$\Psi = \{K \leq M : N \leq K, K \cap S^* = \emptyset \text{ and } (K : M) = (N : M)\}$$

*of submodules of  $M$  has at least one maximal element, and any such maximal element of  $\Psi$  is a prime submodule of  $M$ . Moreover, it is a maximal submodule in  $M \setminus S^*$ .*

**Proof.** Clearly the set  $\Psi$  is non-empty. Let  $\Delta$  be a non-empty totally ordered subset of  $\Psi$ . Then  $Q = \bigcup_{K_i \in \Delta} K_i$  is a submodule of  $M$  such that  $N \subseteq Q$  and  $Q \cap S^* = \emptyset$ . Since  $M/\mathfrak{P}M$  is finitely generated, we have  $(Q : M) = (N : M)$ . Thus  $Q$  is an upper bound for  $\Delta$  in  $\Psi$  and so it follows from Zorn's Lemma that  $\Psi$  has at least one maximal element.

Let  $U$  be an arbitrary maximal element of  $\Psi$ . Then  $U$  is a proper submodule of  $M$ . Take  $a \in M \setminus U$ . Then there exist  $s \in S^*$ ,  $r \in R$  and  $u \in U$  such that  $s = u + ra$ . On the other hand,  $\mathcal{S} \cap (N : M) = \emptyset$ . Take  $b \in R \setminus (N : M)$ . Then  $\mathcal{S} \cap ((N : M) + Rb) \neq \emptyset$  and so there exist  $s' \in \mathcal{S}$ ,  $q \in \mathfrak{P}$  and  $r' \in R$  such that  $s' = q + r'b$ . Hence we have  $ss' = uq + ur'b + raq + rr'ab$  and so  $ab \notin U$ . Thus  $U$  is prime.

For the second claim, let  $T$  be a submodule in  $M \setminus S^*$  such that  $U \subset T$ . Then  $(U : M)$  is strictly contained in  $(T : M)$ . Thus there exists an element  $x$  in  $(T : M) \cap \mathcal{S}$ . But this yields that  $xs \in S^* \cap T = \emptyset$  for any  $s \in S^*$ , a contradiction.  $\square$

Let  $M$  be an  $R$ -module and let  $\mathfrak{P}$  be a prime ideal of  $R$ . Then we recall  $M(\mathfrak{P})$ , the following subset of  $M$  from [8]:  $M(\mathfrak{P}) = \{m \in M : Am \subseteq \mathfrak{P}M \text{ for some ideal } A \not\subseteq \mathfrak{P}\}$ . It is clear that  $M(\mathfrak{P})$  is a submodule of  $M$  and  $\mathfrak{P}M \subseteq M(\mathfrak{P})$ .

**Corollary 1.4.** *Let  $N$  be a submodule of an  $R$ -module  $M$  and let  $S$  be a multiplicatively closed subset of  $R$ . Also suppose that  $S^*$  is an  $S$ -closed subset of  $M$  with  $N \cap S^* = \emptyset$  and  $\mathfrak{P} = (N : M)$  is a maximal ideal in  $R \setminus S$  such that  $M/\mathfrak{P}M$  is a finitely generated  $R$ -module. Then*

- (1) *there exists a prime submodule  $P$  of  $M$  such that  $\mathfrak{P} = (P : M)$ ;*
- (2)  *$\mathfrak{P} = (\mathfrak{P}M : M)$ ;*
- (3)  *$M(\mathfrak{P})$  is a  $\mathfrak{P}$ -prime submodule of  $M$ .*

The following corollary is clear by Proposition 1.8 in [8] but we give it here as an illustration of Corollary 1.4.

**Corollary 1.5.** *Let  $M$  be a finitely generated faithful module and  $\mathfrak{P}$  a prime ideal of  $R$ . Then there is a prime submodule  $P$  of  $M$  such that  $(P : M) = \mathfrak{P}$ .*

*Proof.* By using the determinant argument, we get that  $\mathfrak{P} = (\mathfrak{P}M : M)$ . Also we can get a maximal submodule  $N$  of  $M$  containing  $\mathfrak{P}M$ . Let  $S = R \setminus \mathfrak{P}$  and  $S^* = M \setminus N$ . Since  $N$  is a prime submodule of  $M$ ,  $S^*$  is an  $S$ -closed subset of  $M$ . Now the result follows from Corollary 1.4.  $\square$

We now turn our attention to the characterization of submodules which satisfy the radical formula by using a saturated closed subset of  $M$ . First we recall the following elementary definitions.

Let  $N$  be a submodule of an  $R$ -module  $M$  with  $N \neq M$ . The *envelope* of  $N$  in  $M$  is defined by  $\{rm : r \in R \text{ and } m \in M \text{ such that } r^n m \in N \text{ for } n \in \mathbb{N}\}$  and is denoted by  $E_M(N)$ . We use  $RE_M(N)$  to denote the submodule of  $M$  generated by  $E_M(N)$ . Following [7], we say that  $N$  *satisfies the radical formula (s.t.r.f.)* in  $M$  provided  $M\text{-rad}_R N = RE_M(N)$ , and  $M$  is said to *s.t.r.f.* if every submodule of  $M$  s.t.r.f. in  $M$  and analogously a ring  $R$  *s.t.r.f.* whenever every  $R$ -module s.t.r.f.

Let  $N$  be a submodule of an  $R$ -module  $M$ . Also suppose that  $M\text{-rad}_R N$  is generated by the set  $U$ . We say that  $N$  *satisfies (\*)* if  $Rm \cap N = 0$  whenever  $m \in U \setminus N$ . Clearly  $N$  satisfies (\*) provided that  $N$  is a summand submodule of  $M\text{-rad}_R N$ . Further if  $M$  is a  $\mathbb{Z}$ -module  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$  and  $N = \mathbb{Z}/2\mathbb{Z} \oplus 0$  then  $M\text{-rad}_R N$  is generated by the set  $\{(1 + 2\mathbb{Z}, 0 + 8\mathbb{Z}), (0 + 2\mathbb{Z}, 2 + 8\mathbb{Z})\}$  and so  $N$  satisfies (\*).

Let  $N$  be a submodule of an  $R$ -module  $M$  with (\*) and let  $\mathcal{Q} = (N : M)$  be a non-zero ideal of  $R$ . Then  $M\text{-rad}_R N$  is equal to  $N$  provided  $N$  contains the torsion subset  $T(M) = \{m \in M : \text{there exists } 0 \neq r \in R \text{ such that } rm = 0\}$ .

**Theorem 1.6.** *Let  $N$  be a submodule of an  $R$ -module  $M$  with  $(*)$ . Also suppose that  $\mathcal{Q} = (N : M)$  is a non zero prime ideal of  $R$ . If  $\mathcal{Q}$  contains the set of all zero divisors on  $M$  then  $M\text{-rad}_R N = RE_M(N)$ . In particular, whenever  $N$  is summand  $M\text{-rad}_R N = N$ .*

*Proof.* Let  $M\text{-rad}_R N$  be generated by the set  $U$ . It is enough to show that  $M\text{-rad}_R N \subseteq RE_M(N)$ . Take  $b \in M\text{-rad}_R N$  with  $b \notin RE_M(N)$  and look for a contradiction. Write  $b = r_1 m_1 + \dots + r_n m_n$  for some  $r_i \in R$  and  $m_i \in U$  ( $1 \leq i \leq n$ ). Without loss of generality, we can assume that  $r_1 m_1 \notin RE_M(N)$ . Hence  $m_1 \notin N$  and for all  $t \in \mathbb{N}$ ,  $r_1^t m_1 \notin N$ . Let  $\mathcal{S} = R \setminus \mathcal{Q}$  and  $S^* = Sm_1$ . Then  $S^*$  is an  $\mathcal{S}$ -closed subset of  $M$  and clearly,  $r_1 m_1 \in S^*$  and  $S^* \cap N = \emptyset$ . By Theorem 1.3, there exists a prime submodule  $P$  containing  $N$  such that  $P \cap S^* = \emptyset$  and  $(P : M) = \mathcal{Q}$ . It follows that  $r_1 m_1 \notin P$  and so  $r_1 m_1 \notin M\text{-rad}_R N$ . So we get a contradiction and this completes the proof.  $\square$

Let  $M$  be an  $R$ -module. Note that  $M$  is said to be a *multiplication module* provided for each submodule  $N$  of  $M$  there exists an ideal  $\mathcal{I}$  of  $R$  such that  $N = \mathcal{I}M$ . In particular, invertible and more generally projective ideals of  $R$  are multiplication  $R$ -modules. On the other hand, cyclic modules are multiplication modules. (For more details, see for example [1]).

For the rest of this section, we assume  $R$  to be a ring in which every ideal is cyclic,  $M$  to be a multiplication  $R$ -module and  $S^*$  to be an  $\mathcal{S}$ -closed subset of  $M$  relative to a multiplicatively closed subset  $\mathcal{S}$  of  $R$ . Our aim is to prove that every subset of  $M$  is contained in a minimal saturated closed subset. In [4, Theorem 4.3] Lu assumes  $M$  to be a cyclic  $R$ -module. Now we take one more step and assume  $M$  to be a multiplication module.

**Lemma 1.7.** *Let  $R, M, S^*$  and  $\mathcal{S}$  be as above. Let  $N$  be a maximal submodule in  $M \setminus S^*$ . If  $S^*$  is saturated, then the ideal  $(N : M)$  is maximal in  $R \setminus \mathcal{S}$  so that  $(N : M)$  is a prime ideal of  $R$ .*

Thus due to Lemma 1.7, [4, Theorem 4.8] can be improved. Hence we have

**Lemma 1.8.** *Let  $R, M, S^*$  and  $\mathcal{S}$  be as above. Then  $S^*$  is a saturated  $\mathcal{S}$ -closed subset of  $M$  if and only if  $S^* = M \setminus \bigcup_{i \in I} P_i$  and  $\mathcal{S} = R \setminus \bigcup_{i \in I} \mathfrak{P}_i$  where  $P_i$  is a  $\mathfrak{P}_i$ -prime submodule of  $M$  such that  $P_i \cap S^* = \emptyset$  for all  $i$ .*

Assume that  $M$  is a multiplication  $R$ -module. For any subset  $T$  of  $M$ , define  $\overline{T} = M \setminus \bigcup_{i \in I} P_i$  and  $\mathcal{S} = R \setminus \bigcup_{i \in I} \mathfrak{P}_i$  where  $P_i$  is a  $\mathfrak{P}_i$ -prime submodule of  $M$  such that  $P_i \cap T = \emptyset$  for all  $i$ . Now we have

**Theorem 1.9.** *Let  $R, M, S, T$  and  $\overline{T}$  be as above. Then  $\overline{T}$  is a minimal saturated  $\mathcal{S}$ -closed subset of  $M$  containing  $T$ .*

*Proof.* Clearly  $\overline{T}$  is a saturated  $\mathcal{S}$ -closed subset of  $M$ . Assume that  $K$  is a saturated  $\mathcal{S}_0$ -closed subset of  $M$  such that  $T \subseteq K \subseteq \overline{T}$ . Then by Lemma 1.8,  $K = M \setminus \bigcup Q_i$  and  $\mathcal{S}_0 = R \setminus \bigcup Q_i$  where  $Q_i$  is a  $\mathcal{Q}_i$ -prime submodule of  $M$  such that  $Q_i \cap K = \emptyset$  for all  $i$ . Let  $x \in \overline{T} = M \setminus \bigcup P_i$ . Hence,  $x \notin \bigcup P_i$  and so  $x \notin \bigcup Q_i$ . Therefore,  $K = \overline{T}$  and  $\mathcal{S} = \mathcal{S}_0$ . This completes the proof.  $\square$

## 2. PROJECTIVE MODULES

In this section we deal with the radicals of a submodule. In [6], McCasland and Moore proved that  $M\text{-rad}_R N = \sqrt{(N : M)}M$  for a finitely generated multiplication  $R$ -module  $M$ . And in [1], El-Bast and Smith proved the same result for any multiplication  $R$ -module. In this section the main aim is to give a necessary and sufficient condition for the equality  $M\text{-rad}_R N = \sqrt{(N : M)}M$  for a submodule  $N$  of a projective  $R$ -module  $M$ .

Let  $\mathfrak{P}$  be a prime ideal of  $R$ . Recall that a submodule  $N$  of an  $R$ -module  $M$  is said to be  $\mathfrak{P}$ -primary if  $rm \in N$  for  $r \in R$  and  $m \in M$  implies that either  $m \in N$  or  $r \in \sqrt{(N : M)} = \mathfrak{P}$ . It is well known that  $\mathfrak{P}F$  is a prime submodule of  $F$  such that  $(\mathfrak{P}F : F) = \mathfrak{P}$  for a free  $R$ -module  $F$ . Hence we have the following known lemma

**Lemma 2.1.** *Let  $F$  be a free  $R$ -module and  $\mathcal{P}$  a  $\mathfrak{P}$ -primary ideal of  $R$ . Then  $\mathcal{P}F$  is a  $\mathfrak{P}$ -primary submodule of  $F$ .*

**Theorem 2.2.** *Let  $M$  be a projective  $R$ -module. Then either  $\mathcal{P}M = M$  or  $\mathcal{P}M$  is a  $\mathfrak{P}$ -primary submodule of  $M$  for every  $\mathfrak{P}$ -primary ideal  $\mathcal{P}$  of  $R$ .*

*Proof.* Let  $M$  be a projective  $R$ -module. Thus  $F = M \oplus A$  where  $F$  is a free module and  $A$  is an  $R$ -module. Let  $\{f_i = m_i + a_i\}_{i \in I}$  be a basis for  $F$  where  $m_i \in M$  and  $a_i \in A$ . Assume that  $\mathcal{P}M \neq M$  for a  $\mathfrak{P}$ -primary ideal  $\mathcal{P}$  of  $R$ . First, we show that  $\sqrt{(\mathcal{P}M : M)} = \mathfrak{P}$ . Take a non-zero element  $r \in \sqrt{(\mathcal{P}M : M)}$  but not in  $\mathfrak{P}$ . Then for some integer  $n$  we have  $r^n M \subseteq \mathcal{P}M \subseteq \mathcal{P}F$ . Then by Lemma 2.1 we get  $M \subseteq \mathcal{P}F$ . Let  $x \in M$  and so  $x = \sum r_i f_i$  where  $r_i \in \mathcal{P}$ . Then  $x - \sum r_i m_i = \sum r_i a_i \in M \cap A = 0$  and hence  $x \in \mathcal{P}M$ . It follows that  $\mathcal{P}M = M$ , a contradiction. Therefore, we get  $\sqrt{(\mathcal{P}M : M)} = \mathfrak{P}$ .

Let  $r \in R$  and  $m \in M$  be such that  $rm \in \mathcal{P}M$  with  $r \notin \mathfrak{P}$ . Then  $m \in \mathcal{P}F$  and so  $m \in \mathcal{P}M$ . This completes the proof.  $\square$

As corollaries to Theorem 2.2 we have

**Corollary 2.3.** *Let  $M$  be a projective  $R$ -module and let  $\mathfrak{P}$  be a prime ideal of  $R$ . Then the following statements are equivalent.*

- (1)  $\mathfrak{P}M$  is a prime submodule of  $M$ .
- (2) There exists a prime submodule  $U$  of  $M$  such that  $\mathfrak{P} = (U : M)$ .
- (3)  $(\mathfrak{P}M : M) = \mathfrak{P}$ .

**Corollary 2.4.** *Let  $M$  be a projective  $R$ -module. Then either  $M(\mathfrak{P}) = \mathfrak{P}M$  or  $M(\mathfrak{P}) = M$  for every prime ideal  $\mathfrak{P}$  of  $R$ .*

**Lemma 2.5.** *Let  $N$  be a submodule of a projective  $R$ -module  $M$ . Then*

$$M\text{-rad}_R[(N : M)M] = \sqrt{(N : M)M}.$$

*Proof.* Let  $U$  be a prime submodule of  $M$  containing  $(N : M)M$ . Then  $(N : M) \subseteq (U : M)$  and so  $\sqrt{(N : M)M} \subseteq (U : M)M \subseteq U$ . This means that  $\sqrt{(N : M)M} \subseteq M\text{-rad}_R[(N : M)M]$ . For the converse, let  $\mathfrak{P}$  be a prime ideal of  $R$  such that  $(N : M) \subseteq \mathfrak{P}$ . Then  $\mathfrak{P}M$  is a prime submodule of  $M$  or  $\mathfrak{P}M = M$ . So we have  $M\text{-rad}_R[(N : M)M] \subseteq \bigcap(\mathfrak{P}M)$ . Since  $M$  is a projective  $R$ -module,  $M\text{-rad}_R[(N : M)M] \subseteq \bigcap(\mathfrak{P}M) = (\bigcap \mathfrak{P})M = \sqrt{(N : M)M}$ . This completes the proof.  $\square$

Let  $N$  be a proper submodule of a module  $M$ . Now we give the following definition to prove our main aim in this paper: We say that  $N$  *satisfy the prime property (s.t.p.p.)* in  $M$  provided  $(N : M) \subseteq \mathfrak{P}$  for a prime ideal  $\mathfrak{P}$  of  $R$ ,  $N \subseteq \mathfrak{P}M$ .

**Example 2.6.**

- (i) Let  $M$  be an  $R$ -module and let  $I$  be an ideal of a ring  $R$ . Then it is easy to check that the submodule  $IM$  s.t.p.p. in  $M$ .
- (ii) Let  $M$  be the  $\mathbb{Z}$ -module  $\mathbb{Z} \oplus 36\mathbb{Z}$  and  $N = 6\mathbb{Z} \oplus 36\mathbb{Z}$ . Then  $(N : M) = 12\mathbb{Z}$ . It can be seen that  $N$  s.t.p.p. in  $M$ .

By using the prime property and projective modules, we obtain a characterization for a prime submodule. Now we recall the fact from [3] that if  $R$  is a domain and  $M$  is a torsion-free  $R$ -module then  $M$  is flat if and only if  $(I \cap J)M = IM \cap JM$  for all ideals  $I$  and  $J$  of  $R$ . It is also known that a finitely generated flat module over a domain is projective. Hence we have



**Proposition 2.7.** *Let  $M$  be a finitely generated projective  $R$ -module and let  $N$  be a submodule which s.t.p.p. in  $M$ . Then  $N$  is a prime submodule if and only if  $P = (N : M)$  is a prime ideal of  $R$  and  $M/N$  is a projective  $R/P$ -module.*

*Proof.* Sufficiency is evident. Let  $N$  be a prime submodule of  $M$  which s.t.p.p. in  $M$ . Then  $\mathfrak{P} = (N : M)$  is a prime ideal of  $R$  and  $(\mathcal{I} \cap \mathcal{J})(M/N) = \mathcal{I}(M/N) \cap \mathcal{J}(M/N)$  for all ideals  $\mathcal{I}$  and  $\mathcal{J}$  in  $R/\mathfrak{P}$ . Then the result follows from [3, Theorem 1 and Corollary 1].  $\square$

**Corollary 2.8.** *Let  $M$  be a finitely generated projective  $R$ -module. Then  $M/\mathfrak{P}M$  is a projective  $R/\mathfrak{P}$ -module for every prime ideal  $\mathfrak{P}$  of  $R$ .*

The prime property gives also another characterization for radical submodules.

**Theorem 2.9.** *Let  $N$  be a submodule of a projective  $R$ -module  $M$ . Then  $N$  s.t.p.p. in  $M$  if and only if  $M\text{-rad}_R N = \sqrt{(N : M)M}$ . In particular, if  $N$  s.t.p.p. in  $M$  then  $N$  s.t.r.f. in  $M$ .*

*Proof.* Sufficiency is evident. Assume that  $N$  s.t.p.p. in  $M$ . Let  $P$  be a prime submodule of  $M$  containing  $[(N : M)M]$ . Thus we have  $(N : M) \subseteq \mathfrak{P} = (P : M)$ . Then by the prime property, we get  $N \subseteq \mathfrak{P}M \subseteq P$ . Therefore,  $M\text{-rad}_R N \subseteq M\text{-rad}_R[(N : M)M]$ . Now the result follows from Lemma 2.5.  $\square$

**Corollary 2.10.** *Let  $N$  be a submodule of an  $R$ -module  $M$  such that  $M/N$  is a projective  $R$ -module. Then  $M\text{-rad}_R N = \sqrt{(N : M)M} + N = RE_M(N)$ .*

*Proof.* Clearly the zero submodule of  $M/N$  s.t.p.p. in  $M/N$ . Since  $M/N$  is a projective  $R$ -module we have  $M/N\text{-rad}_R 0 = \sqrt{0 : (M/N)}(M/N)$ . On the other hand,  $M/N\text{-rad}_R 0 = M\text{-rad}_R N/N$  and  $\sqrt{0 : (M/N)}(M/N) = (\sqrt{N : MM} + N)/N$ . Therefore,  $M\text{-rad}_R N = \sqrt{N : MM} + N = RE_M(N)$ .  $\square$

Using Corollary 2.10 we can improve the result [2, Corollary 8].

**Corollary 2.11.** *If  $M$  is a projective  $R$ -module then  $M\text{-rad}_R(0) = RE_M(0) = \sqrt{0 : MM}$ .*

Compare the next corollary with [10, Corollary 1.5].

**Corollary 2.12.** *Let  $M$  be a primary projective  $R$ -module. Then the radical of  $M$  is a prime submodule of  $M$ .*

*Proof.* Since  $M$  is a projective  $R$ -module  $M$  contains a prime submodule and so  $M\text{-rad}_R 0$  is not equal to  $M$ . Therefore we can prove that  $M\text{-rad}_R 0 = \sqrt{0 : MM}$  is a prime submodule of  $M$  by using the same argument as in Theorem 2.2.  $\square$

If  $N$  is a primary submodule of a projective  $R$ -module  $M$  which s.t.p.p. in  $M$  then by Theorem 2.2 and Theorem 2.9, the radical of  $N$  in  $M$  is a prime submodule of  $M$  or  $M\text{-rad}_R N = M$ . On the other hand, the following example shows that a partial converse of Theorem 2.9 is not true in general.

**Example 2.13.** Let  $R$  be a principal ideal domain and  $M = R \oplus R$ . Let  $N$  be a non-zero cyclic submodule of  $M$ . It can be easily seen that  $M\text{-rad}_R N \neq \sqrt{(N : M)}M$ . Hence  $N$  s.t.r.f. but not s.t.p.p. in  $M$ .

Let  $N = \bigoplus N_i$  be a submodule of an  $R$ -module  $M$ . Then provided  $N_i$  s.t.p.p. in  $M$  for all  $i = 1, \dots, n$ ,  $N$  s.t.p.p. in  $M$ . But the converse is not true in general (see Example 2.6 (ii)). However, for the converse we can state the following: Let  $N = \bigoplus N_i$ . Also assume that  $N$  s.t.p.p. in  $M$ . If  $\sqrt{(N_i : M)} = \sqrt{(N : M)}$  for some  $i$  then  $N_i$  s.t.p.p. in  $M$ .

Let  $M = \bigoplus M_i$  be an  $R$ -module. Consider the submodule  $N = \bigoplus N_i$  of  $M$  such that  $N_i$  is a submodule of  $M_i$  for all  $i \in I$ . It can be proved that if  $N$  s.t.p.p. in  $M$  then  $N_i$  s.t.p.p. in  $M_i$  for all  $i \in I$ . For the converse, if  $N_i$  s.t.p.p. in  $M_i$  for all  $i \in I$  with  $\sqrt{N_i : M_i} = \sqrt{N : M}$ , then  $N$  s.t.p.p. in  $M$ .

Now we turn our attention to primary submodules. First, we give a characterization for primary submodules of  $M$  such that  $M = \bigoplus M_i$  is a direct sum of modules  $M_i$  ( $i \in I$ ). For each  $i$ , let  $N_i$  be a submodule of  $M_i$  and  $N = \bigoplus N_i$ .

**Theorem 2.14.** *Let  $M$  and  $N$  be as above. Assume that  $\mathfrak{P}$  is a prime ideal of  $R$ . Then  $N$  is a  $\mathfrak{P}$ -primary submodule of  $M$  if and only if  $N_i$  is a  $\mathfrak{P}$ -primary submodule of  $M_i$  whenever  $N_i \neq M_i$  for all  $i$ .*

*Proof.* Let  $N$  be a  $\mathfrak{P}$ -primary submodule of  $M$ . Since  $N \neq M$ , there is a non-empty subset  $J$  of  $I$  such that  $N_j \neq M_j$  for all  $j \in J$  and so  $N = \bigoplus N_j \oplus (\bigoplus M_i)$ . First we prove that  $\sqrt{(N_t : M_t)} = \mathfrak{P}$  for all  $t \in J$ .

Let  $r \in \mathfrak{P}$ . Choose an element  $m_t \in M_t$  but not in  $N_t$ . Let  $m = (0, \dots, m_t, \dots, 0) \in M$ . Then for a positive integer  $l$ , we have  $r^l(0, \dots, m_t, \dots, 0) \in N$  and so  $r^l m_t \in N_t$ . Hence  $r \in \sqrt{(N_t : M_t)}$  and so  $\mathfrak{P} \subseteq \sqrt{(N_t : M_t)}$ . Now take elements  $r \in \sqrt{(N_t : M_t)} \setminus \mathfrak{P}$  and  $m = (m_i) \in M$  such that

$$m = (m_i) = \begin{cases} m_i \in N_i & \text{if } i \neq t, \\ m_t \in M_t \setminus N_t & \text{if } i = t. \end{cases}$$

Then for a positive integers  $l_i$  we have  $r^{l_i}m_i \in M_i$ . Let  $k = \max\{l_i\}$  and so  $r^k m \subseteq N$ . Since  $m \notin N$ , we get that  $r \in \mathfrak{P}$ . Therefore  $\mathfrak{P} = \sqrt{(N_j : M_j)}$  for all  $j \in J$ .

Take a submodule  $N_j$  for any  $j \in J$  and  $rm_j \in N_j$  where  $r \in R$  and  $m_j \in M_j$ . Choose an element

$$m = (m_i) = \begin{cases} m_i = m_j & \text{if } i = j, \\ m_i = 0 & \text{if } i \neq j \end{cases}$$

of  $M$ . Then  $rm \in N$ . Since  $N$  is primary, it follows that either  $m \in N$  or  $r \in \mathfrak{P}$ . Hence either  $m_j \in N_j$  or  $r \in \mathfrak{P}$ . Therefore  $N_j$  is a primary submodule of  $M_j$  for all  $j \in J$ .

Conversely, assume that  $N = \bigoplus N_j \oplus (\bigoplus M_i)$  is such that for all  $j \in J \subseteq I$ ,  $N_j$  is a  $\mathfrak{P}$ -primary submodule in  $M_j$ . Take elements  $r \in R$  and  $m \in M$  such that  $rm \in N$ . Then  $m = (m_i) \in \bigoplus M_i$  and so  $rm = (rm_i) \in \bigoplus N_j \oplus (\bigoplus M_i)$ . Now assume that for some  $j \in J$ , we have  $m_j \notin N_j$ . Then  $rm_j \in N_j$  and so  $r \in \mathfrak{P}$ . This means that  $N$  is a  $\mathfrak{P}$ -primary submodule of  $M$ .  $\square$

**Corollary 2.15.** *Let  $N$  and  $M$  be as in Theorem 2.14. Also suppose that  $N$  is a primary submodule of  $M$ . Then  $N$  s.t.p.p. in  $M$  if and only if  $N_i$  s.t.p.p. in  $M_i$  for all  $i \in I$ .*

For the rest of this section, we assume  $R$  to be a principal ideal domain and  $M = R \oplus R$ . We close this paper by giving equivalent conditions to the prime property. Let  $N$  be a submodule of  $M$ . If  $N$  is generated by  $(a, b)$  then clearly  $M\text{-rad}_R N = \gcd\{a, b\}R$  and it does not satisfy the prime property. Now assume  $N$  is generated by  $\{(a, b), (c, d)\}$  and let  $\Delta = ad - bc$ . Then it is routine to check that there is an element  $k$  of  $R$  such that  $\Delta = k \gcd(a, b, c, d)$  and  $(N : M) = kR$  where  $\gcd(a, b, c, d)$  denotes the greatest common divisor of the elements  $a, b, c$  and  $d$ . Let  $\Delta = p^t$  where  $p$  is a prime element of  $R$  and  $t \in \mathbb{N}$ . Then  $N$  is a prime submodule whenever  $t = 1$ . Otherwise  $N$  is a primary submodule of  $M$ .

**Theorem 2.16.** *Let  $N$  be a submodule of  $M = R \oplus R$  generated by  $\{(a, b), (c, d)\}$  and let  $(N : M) = kR$  for some  $k \in R$ . Let  $k = p_1^{t_1} \dots p_n^{t_n}$  and  $s = p_1 \dots p_n$  where for each  $i$ ,  $p_i$  is a prime element in  $R$  and  $t_i \in \mathbb{N}$  for all  $i = 1, 2, \dots, n$ . Then the following statements are equivalent.*

- (1)  $s$  divides  $a, b, c$  and  $d$ .
- (2)  $N$  s.t.p.p. in  $M$ .
- (3)  $M\text{-rad}_R N = RE_M(N) = \sqrt{kRM}$ .

**Proof.** It is sufficient to prove the equivalence of (1) and (2).

(1)  $\Rightarrow$  (2): Let  $pR$  be a prime ideal containing  $(N : M)$ . Then  $p = p_i$  for some  $1 \leq i \leq n$  and so there exist  $t_1, t_2, t_3, t_4 \in R$  such that  $(a, b) = p(t_1, t_2)$ ,  $(c, d) = p(t_3, t_4)$ . Hence  $N$  s.t.p.p. in  $M$ .

(2)  $\Rightarrow$  (1): Let  $p$  be a prime element of  $R$  such that  $p$  divides  $s$ . Then  $kR \subseteq sR \subseteq pR$ . Hence  $(a, b), (c, d) \in pRM$ . It follows that  $(a, b) = p(t_1, t_2)$ ,  $(c, d) = p(t_3, t_4)$  for  $t_1, t_2, t_3, t_4 \in R$ . Therefore,  $s$  divides  $a, b, c$  and  $d$ .  $\square$

Now we close this paper by the following observation.

Let  $M, N$  and  $\Delta$  be as above. If  $\Delta = p^t$  where  $p$  is a prime element of  $R$  and  $p$  divides  $a, b, c$  and  $d$ , then  $M\text{-rad}_R N = pM$  is a prime submodule of  $M$ .

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