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CONFORMALLY FLAT PSEUDO-SYMMETRIC SPACES OF CONSTANT TYPE

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Abstract. We give the complete classification of conformally flat pseudo-symmetric spaces of constant type.

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MSC 2000: 53C15, 53C25, 53C35

1. Introduction

As is well known, a Riemannian manifold (M, g) is said to be *(locally) conformally flat* if for any point $p \in M$ there exist a neighborhood U of p and a positive smooth function $f\colon U \to \mathbb{R}$ such that fg is a flat metric. The study of conformally flat Riemannian manifolds is a classical field of research in Riemannian geometry. In particular, many authors have been involved in the study of homogeneity and symmetry conditions on a conformally flat manifold. The following well-known result of P. Ryan [8] provided the complete classification of conformally flat locally symmetric spaces:

Theorem 1.1 [8]. *Let M be an n-dimensional conformally flat space with a parallel Ricci tensor. Then M has as its simply connected Riemannian covering one of the spaces*

$$\mathbb{R}^n, S^n(k), \mathbb{H}^n(-k), \mathbb{R} \times S^{n-1}(k), \mathbb{R} \times \mathbb{H}^{n-1}(-k), S^p(k) \times \mathbb{H}^{n-p}(-k),$$

where by $S^n(k)$ we denote a Euclidean n-sphere with constant curvature $k > 0$, and by $\mathbb{H}^n(-k)$ we denote an n-dimensional simply connected, connected space with constant curvature $-k < 0$.

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As concerns homogeneity, H. Takagi [11] proved that *a locally homogeneous conformally flat Riemannian manifold* (M, g) *is locally symmetric* and so, it is one of the spaces given in Ryan's classification. Indeed, in the proof of his result, Takagi only used local homogeneity to provide that (M, g) has constant Ricci eigenvalues, which is equivalent, for conformally flat manifolds, to curvature homogeneity. Therefore, as already remarked in [3], *a conformally flat curvature homogeneous space is locally symmetric.*

Coming back to symmetry conditions, semi-symmetric spaces represent a well-known and natural generalization of locally symmetric spaces. A *semi-symmetric space* is a Riemannian manifold (M, g) such that its curvature tensor R satisfies the condition

$$R(X, Y) \cdot R = 0 \tag{1.1}$$

for all vector fields X, Y on M, where $R(X, Y)$ acts as a derivation on R [9]. The curvature tensor R_p of (M, g) at a point $p \in M$ is the same as the curvature tensor of a symmetric space (which may change with the point p). Locally symmetric spaces are semi-symmetric, but in any dimension greater than two there exist examples of semi-symmetric spaces which are not locally symmetric. (The first example was given by H. Takagi in [10]. We can refer to [1] for a survey.)

In [2], the classification of conformally flat semi-symmetric spaces was obtained by proving

Theorem 1.2 [2]**.** *A conformally flat semi-symmetric space (of dimension* $n > 2$*) is either locally symmetric or it is locally irreducible and isometric to a semi-symmetric real cone.*

We shall come back to the description of semi-symmetric real cones in Section 2. Such Riemannian manifolds are the only conformally flat semi-symmetric not locally symmetric spaces. Indeed, they turn out to be also the only conformally flat not locally symmetric examples in the broader class of pseudo-symmetric spaces of constant type.

A *pseudo-symmetric space of constant type* is a Riemannian manifold (M, g) whose curvature tensor R satisfies

$$R(X, Y) \cdot R = \tilde{c}(X \wedge Y) \cdot R \tag{1.2}$$

for all vector fields X and Y on M, where $X \wedge Y$ is defined by

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$$

and $\tilde{c}$ is a real constant [4], [6]. It is evident from this definition that semi-symmetric spaces correspond to pseudo-symmetric spaces of constant type with $\tilde{c} = 0$. So, pseudo-symmetric spaces of constant type generalize the semi-symmetric ones. In dimension three, pseudo-symmetric spaces of constant type are characterized by the property that two of the Ricci eigenvalues coincide and the last one is constant ([1, Proposition 11.2]).

In dimension three, the problem of classifying conformally flat pseudo-symmetric spaces of constant type has been already studied and solved by N. Hashimoto and M. Sekizawa [5]. Taking into account their result and using our classification of semi-symmetric conformally flat spaces, we can solve completely the problem of classifying conformally flat pseudo-symmetric spaces of constant type by proving

Main Theorem. *A conformally flat pseudo-symmetric space of constant type (of dimension $n > 2$) is either locally symmetric or it is locally irreducible and isometric to a semi-symmetric real cone.*

The paper is organized in the following way. In Section 2, we recall some basic facts and results about conformally flat Riemannian manifolds and describe semi-symmetric real cones. Then, in Section 3, we prove our main result, combining the curvature information coming from conformal flatness and pseudo-symmetry.

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2. Preliminaries

Let (M, g) be a Riemannian manifold of dimension $n > 2$ and R its curvature tensor, taken with the sign convention $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ for all vector fields X, Y on M, where ∇ denotes the Levi Civita connection of M. By ϱ, Q and τ we denote respectively the Ricci tensor, the Ricci operator associated to ϱ through g and the scalar curvature of M. Let p be a point of M and $\{e_1, \ldots, e_n\}$ an orthonormal basis of the tangent space T_pM. The components of R and ϱ with respect to $\{e_1, \ldots, e_n\}$ are denoted respectively by R_{ijkh} and ϱ_{ik}. As is well-known, the curvature tensor of a conformally flat space satisfies

$$
\begin{aligned}
R_{ijkh} = {} & \frac{1}{n-2}(g_{ih}\varrho_{jk} + g_{jk}\varrho_{ih} - g_{ik}\varrho_{jh} - g_{jh}\varrho_{ik}) \\
& - \frac{\tau}{(n-1)(n-2)}(g_{ih}g_{jk} - g_{ik}g_{jh}).
\end{aligned}
\tag{2.1}
$$

Moreover, (2.1) characterizes the conformally flat Riemannian manifolds of dimension $n \geqslant 4$, while it is trivially satisfied by any three-dimensional manifold. Conversely, the condition

$$\nabla_i \varrho_{jk} - \nabla_j \varrho_{ik} = \frac{1}{2(n-1)}(g_{jk}\nabla_i \tau - g_{ik}\nabla_j \tau), \tag{2.2}$$

which characterizes three-dimensional conformally flat spaces, is trivially satisfied by any conformally flat Riemannian manifold of dimension greater than three.

We conclude this Section by a short description of semi-symmetric *real cones*, which will provide the only examples of conformally flat pseudo-symmetric spaces of constant type which are not locally symmetric. We can refer to [1] for more detail.

Let $(\overline{M}, \bar{g})$ be a Riemannian manifold and $\mu(t)$ the unique solution of the differential equation $\mathrm{d}\mu/\mathrm{d}t = -\mu^2$ with an initial condition $\mu(0) = \mu_0 > 0$, that is, $\mu(t) = (t + (1/\mu_0))^{-1}$. Put $\mathbb{R}_+ = \{x \in \mathbb{R};\ x > -1/\mu_0\}$ and on the product manifold $\mathbb{R}_+ \times \overline{M}$ consider the Riemannian metric

$$g = \mathrm{d}x^0 \otimes \mathrm{d}x^0 + \mu(x^0)^{-2}\pi^* g,$$

where x^0 is the natural coordinate on $\mathbb{R}_+$ and $\pi\colon \mathbb{R}_+ \times \overline{M} \to \overline{M}$ the projection onto the second factor. The manifold $(\mathbb{R}_+ \times \overline{M}, g)$ is called a *Riemannian cone over* $(\overline{M}, \bar{g})$. Let $T = \partial/\partial x^0$ denote the unit vector field tangent to $\mathbb{R}_+$ in $\mathbb{R}_+ \times \overline{M}$. The curvature tensor of $M = \mathbb{R}_+ \times \overline{M}$ is described by (see [1])

$$R(X,Y)Z = g(B_0(Y),Z)B_0(X) - g(B_0(X),Z)B_0(Y) + (\pi^*\overline{R})(X,Y)Z \tag{2.3}$$

for all tangent vectors X, Y, Z to $\overline{M}$, where $B_0(X) := \nabla_X T = \mu(X - g(X,T)T)$.

Any semi-symmetric real cone $(M = \mathbb{R}_+ \times \overline{M}, g)$ is locally isometric to some maximal cone $M_c(\widetilde{M}, \mu_0)$, where $(\widetilde{M}, \tilde{g})$ is a real space form of constant curvature c [1]. We include the case when $\dim \overline{M} = 2$. In [1], this case was excluded, since a three-dimensional real cone is a special case of three-dimensional Riemannian manifold foliated by Euclidean leaves of codimension two.

At any point p of a semi-symmetric real cone M, fix an orthonormal basis of tangent vectors $\{e_0, e_1, \dots, e_r\}$ with $e_0 = T_p$ and $e_1, \dots, e_r$ tangent to the real space form $(\widetilde{M}^r, \tilde{g})$ $(r = n-1)$. Then, using (2.3) to compute the components of the curvature tensor, we get

$$\begin{cases} R_{ijkh} = 0 & \text{if } 0 \in \{i,j,k,h\}, \\ R_{ijkh} = \mu^2(c-1)(\delta_{ik}\delta_{jh} - \delta_{jk}\delta_{ih}) & \text{otherwise.} \end{cases} \tag{2.4}$$

Computing the Ricci components and the scalar curvature of M starting from (2.4), it is easy to check that (2.1) is satisfied and, if $\dim M \geqslant 4$, this implies that M is conformally flat. If $\dim M = 3$, one can check that (2.2) holds and so, M is again conformally flat. Therefore, a semi-symmetric real cone M is a conformally flat (semi-symmetric) Riemannian manifold, with scalar curvature $\tau = r(r-1)(c-1)\mu^2$. Note that τ cannot be constant, as μ depends on t and so, M is never locally symmetric.

Taking into account the definition of semi-symmetric real cones, the main result of [5] can be rewritten in the following way:

Theorem 2.1 [5]**.** *A three-dimensional conformally flat pseudo-symmetric space of constant type is either locally symmetric or it is locally isometric to a semi-symmetric real cone.*

3. Conformally flat pseudo-symmetric spaces

We first recall the definition of the nullity index at a point of a Riemannian manifold.

Definition 3.1. The *nullity vector space* of the curvature tensor at a point p of a Riemannian manifold (M, g) is given by

$$E_{0p} = \{X \in T_pM\,;\ R(X,Y)Z = 0 \text{ for all } Y, Z \in T_pM\}.$$

The *index of nullity at* p is the number $\nu(p) = \dim E_{0p}$. The *index of conullity at* p is the number $u(p) = \dim M - v(p)$.

The nullity and conullity indices are exactly the tools used by Szabó [9] in order to distinguish various locally irreducible semi-symmetric spaces, which appear in the local structure of any semi-symmetric space. When we consider a conformally flat Riemannian manifold, the nullity index can only attain some special values, as the author proved in [2]:

Theorem 3.2 [2]**.** *Let* (M, g) *be a Riemannian manifold satisfying* (2.1)*, of dimension* $n \geqslant 3$ (that is, $\dim M = 3$ or M is conformally flat)*. Then, at each point* p *of* M*, the index of nullity is either* $\nu(p) = 0, 1$ *or* n*.*

Theorem 3.2 restricts the research of conformally flat pseudo-symmetric spaces of constant type to the ones having index of nullity equal to 0, 1 or n. We are now ready to give

P r o o f o f t h e M a i n T h e o r e m. Let (M, g) be a conformally flat pseudo-symmetric space of constant type, with constant $\tilde{c}$. Taking into account Theorem 2.1 by Hashimoto and Sekizawa, we can assume that the dimension of M is $n \geqslant 4$. Our purpose is to show that necessarily $\tilde{c} = 0$. Thus, (M, g) must be conformally flat and semi-symmetric and the conclusion follows from our Theorem 1.2.

Since (M, g) is conformally flat, there exists, at any point $p \in M$, an orthonormal basis $\{e_i\}$ of the tangent space T_pM, such that the curvature components R_{ijkh} vanish whenever at least three indices are distinct (taking into account (2.1), it suffices to consider an orthonormal basis of eigenvectors of the Ricci operator).

Rewriting (1.2) in a more extended and explicit way, we have that (1.2) is equivalent to

$$\begin{aligned}(3.1)\qquad & R(X,Y)R(U,V)W - R(R(X,Y)U,V)W - R(U,R(X,Y)V)W \\ &\quad - R(U,V)R(X,Y)W \\ &= \tilde{c}\{g(Y,R(U,V)W)X - g(X,R(U,V)W)Y \\ &\quad - R(g(Y,U)X - g(X,U)Y,V)W - R(U,g(Y,V)X - g(X,V)Y)W \\ &\quad - R(U,V)(g(Y,W)X - g(X,W)Y\}\end{aligned}$$

for all vector fields X, Y, U, V, W on M. We now apply (3.1) taking $X = V = e_i$, $Y = W = e_j$ and $U = e_k$ for all $i, j, k = 1, \dots, n$ such that $i \neq j \neq k \neq i$. After some standard calculations, we get

$$(3.2)\qquad (R_{ijij} + \tilde{c})(R_{jkjk} - R_{ikik}) = 0 \text{ whenever } i, j \text{ and } k \text{ are all distinct.}$$

Next, let W be a dense open subset of M such that the multiplicities of the Ricci eigenvalues remain constant on a connected neighborhood V of any point $p \in W$. According to Theorem 3.2, at each point of M the nullity index is either 0, 1 or n. If $p \in W$ and V is a connected neighborhood of p, where the Ricci eigenvalues have constant multiplicities, then the nullity index will be $\nu(q) = 0, 1$ or n for all $q \in V$. So, we have to deal with three different cases, according to the three different possible values of ν on V. If, in all these cases, we can conclude that $\tilde{c} = 0$ on V, then $\tilde{c} = 0$ on M, since it is a constant, and this will complete the proof.

a) If $\nu = n$ on V, then V is flat. In particular, V is semi-symmetric and so, $\tilde{c} = 0$.

b) If $\nu = 1$ on V, let q be a point of V and e_1 a unit vector of the nullity space E_{0q}. By the definition of the nullity space it follows at once that $\varrho(e_1, \cdot) = 0$. Therefore, we can consider an orthonormal basis $\{e_i\}$ of T_qM of Ricci eigenvectors, including e_1. Taking $i = 1$ in (3.2), since $R_{1j1j} = 0$ for all j, we then get

$$(3.3)\qquad \tilde{c}R_{jkjk} = 0 \quad \text{for all } j \neq k > 1.$$

Since $\nu(q) = 1$, e_j and e_k can never belong to the nullity space when $j, k > 1$. So, R_{jkjk} cannot identically vanish for all j and k and (3.3) implies that $\tilde{c} = 0$.

c) If $\nu = 0$ on V, we first note that if $R_{ijij} \neq -\tilde{c}$ holds for all $i \neq j$, then by (3.2) it follows that $R_{jkjk} = R_{ikik}$ whenever i, j and k are all distinct. So, V would have constant sectional curvature. In particular, V is semi-symmetric, that is, $\tilde{c} = 0$. In the sequel, we shall treat the case when there exist some indices $i \neq j$ such that $R_{ijij} = -\tilde{c}$.

Without loss of generality we can assume $R_{1212} = -\tilde{c}$. Then, applying (3.2) with $j = 1$ and $k = 2$, we get

$$(R_{1i1i} + \tilde{c})(R_{2i2i} + \tilde{c}) = 0 \quad \text{for all } i > 2.$$

In other words, either $R_{1i1i} = -\tilde{c}$ or $R_{2i2i} = -\tilde{c}$, for all $i > 2$.

Next, we start from (3.2) and take the sum over $k \neq i, j$. Recalling that the Ricci eigenvalues $\varrho_i = \varrho(e_i, e_i)$ are given by $\varrho_i = -\sum_{k \neq i} R_{ikik}$, one can easily get

$$(R_{ijij} + \tilde{c})(\varrho_i - \varrho_j) = 0 \quad \text{for all } i \neq j. \tag{3.4}$$

In particular, it follows from (3.4) that $\varrho_i = \varrho_j$ if and only if $R_{ijij} \neq -\tilde{c}$.

We will also make use of the following classical characterization of conformally flat manifolds, proved by R. S. Kulkarni:

Theorem 3.3 [7]. *A Riemannian manifold (M, g) of dimension $n \geqslant 4$ is conformally flat if and only if for any point $p \in M$ and for any four orthonormal vectors e_1, e_2, e_3 and e_4 tangent to M at p, we have*

$$R_{1212} + R_{3434} = R_{1313} + R_{2424} = R_{1414} + R_{2323}. \tag{3.5}$$

Next, fix an index $k > 2$. As we have already remarked, either $R_{1k1k} = -\tilde{c}$ or $R_{2k2k} = -\tilde{c}$. Assume for example $R_{1k1k} = -\tilde{c}$. Then, by (3.5), taking into account that $R_{1212} = R_{1k1k} = -\tilde{c}$, we get at once

$$R_{kjkj} = R_{2j2j} \quad \text{for all } j \neq 1, 2, k. \tag{3.6}$$

We now sum over $j \neq 1, 2, k$ in (3.6). Since

$$\sum_{j \neq 1,2,k} R_{kjkj} = \varrho_i - R_{k1k1} - R_{k2k2} \quad \text{and} \quad \sum_{j \neq 1,2,k} R_{2j2j} = \varrho_2 - R_{1212} - R_{k2k2},$$

recalling that $R_{1212} = R_{1k1k} = -\tilde{c}$, we get at once $\varrho_k = \varrho_2$.

In the same way, assuming $R_{2k2k} = -\tilde{c}$, we can conclude that $\varrho_k = \varrho_1$. Therefore, for all $k > 2$, either $\varrho_k = \varrho_1$ or $\varrho_k = \varrho_2$. Note that $\varrho_1 \neq \varrho_2$, otherwise, by (2.1), V

should have constant sectional curvature and this can not occur, as we have already noted. Thus, on V we have two distinct Ricci eigenvalues $\varrho_1 \neq \varrho_2$. By reordering the Ricci eigenvectors of the orthonormal basis $\{e_i\}$ we can assume, without loss of generality, that the Ricci eigenvalues in V are

$$\varrho_1 = \ldots = \varrho_r = a \neq b = \varrho_{r+1} = \ldots = \varrho_n \tag{3.7}$$

for some integer r greater than 1 and lesser than n. If we prove that both a and b are constant on V, then V will be conformally flat and curvature homogeneous (by (2.1)). So, V will be locally symmetric [3]. In particular, $\tilde{c} = 0$ and this completes the proof.

Note that it follows from (3.7) that the scalar curvature τ can be expressed in V in the following way:

$$\tau = ra + (n - r)b. \tag{3.8}$$

We now get a different expression for the scalar curvature. Using (3.7) in (2.1), one can easily obtain

$$R_{ijij} = -\frac{a+b}{n-2} + \frac{\tau}{(n-1)(n-2)} \quad \text{if } i \leqslant r \text{ and } j > r \text{ or conversely.} \tag{3.9}$$

If $i \leqslant r$ and $j > r$, then $\varrho_i \neq \varrho_j$. So, by (3.4) it follows that $R_{ijij} = -\tilde{c}$ and (3.9) implies

$$\tau = (n-1)(a+b) - (n-1)(n-2)\tilde{c}. \tag{3.10}$$

In order to conclude that a and b are constant on V, we shall prove that $e_i(a) = e_i(b) = 0$ for all $i = 1, \ldots, n$. We differentiate both (3.8) and (3.10) with respect to e_i for any $i = 1, \ldots, r$. We get respectively

$$e_i(\tau) = re_i(a) + (n-r)e_i(b) \tag{3.11}$$

and

$$e_i(\tau) = (n-1)(e_i(a) + e_i(b)). \tag{3.12}$$

Finally, since (M, g) is conformally flat, (2.2) holds. We apply (2.2) taking $k = j$ with $i \neq j \leqslant r$. Taking into account that $\{e_i\}$ is an orthonormal basis of eigenvectors of the Ricci operator, after some standard calculations we obtain

$$\nabla_i \varrho_{jj} = e_i(\varrho_{jj}) = e_i(a) \quad \text{and} \quad \nabla_j \varrho_{ij} = 0$$

and so, by (2.2),

$$e_i(\tau) = 2(n-1)e_i(a) \quad \text{for all } i \leqslant r.$$

Comparing the expressions of $e_i(\tau)$ given by (3.11), (3.12) and (3.13), it is easy to show that $e_i(a) = e_i(b) = 0$ for all $i \leqslant r$. In the same way, taking $k = j$ with $i \neq j > r$ in (2.2), we get

$$e_i(\tau) = 2(n-1)e_i(b) \quad \text{for all } i > r$$

and, comparing (3.11), (3.12) and (3.14) we can conclude that $e_i(a) = e_i(b) = 0$ also holds for all $i > r$. Thus, a and b are constant on V and this completes the proof. □

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