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TRAVEL GROUPOIDS

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Abstract. In this paper, by a *travel groupoid* is meant an ordered pair $(V, *)$ such that V is a nonempty set and $*$ is a binary operation on V satisfying the following two conditions for all $u, v \in V$:

$$(u * v) * u = u;$$
$$\text{if } (u * v) * v = u, \text{ then } u = v.$$

Let $(V, *)$ be a travel groupoid. It is easy to show that if $x, y \in V$, then $x * y = y$ if and only if $y * x = x$. We say that $(V, *)$ is on a (finite or infinite) graph G if $V(G) = V$ and

$$E(G) = \{\{u, v\} : u, v \in V \text{ and } u \neq u * v = v\}.$$

Clearly, every travel groupoid is on exactly one graph. In this paper, some properties of travel groupoids on graphs are studied.

Keywords: travel groupoid, graph, path, geodetic graph

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By a graph we mean here a (finite or infinite) undirected graph with no multiple edges or loops. We will use the terminology of the book [1] but we extend it also to infinite graphs here. By a geodetic graph we mean a connected graph G such that there exists exactly one shortest $u - v$ path in G for all $u, v \in V(G)$.

The letters $h - n$ will serve for denoting non-negative integers.

1. TRAVEL GROUPOIDS

By a *travel groupoid* we will mean an ordered pair $(V, *)$ such that V is a nonempty set and $*$ is a binary operation on V satisfying the following axioms (t1) and (t2):

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- (t1) $(u * v) * u = u$ (for all $u, v \in V$);
 (t2) if $(u * v) * v = u$, then $u = v$ (for all $u, v \in V$).

We say that a travel groupoid $(V, *)$ is *finite* if V is finite. If $(V, *)$ is a travel groupoid, then we say that $*$ is a travel operation on V . Special kinds of travel groupoids (or travel operations) were introduced in [2] and [3].

The idea of the proof of the next proposition can be found in the proof of Lemma 2 in [3].

Proposition 1. *If $(V, *)$ is a travel groupoid, then $x * x = x$ (for each $x \in V$).*

Proof. Let $x \in V$. As follows from (t1), $(x * x) * x = x$. This implies that $((x * x) * x) * x = x * x$. By (t2), $x * x = x$, which completes the proof. \square

Proposition 2. *Let $(V, *)$ be a travel groupoid. Then*

- (1) $x * y = y$ if and only if $y * x = x$ (for all $x, y \in V$),
 (2) $x * y = x$ if and only if $x = y$ (for all $x, y \in V$)

and

- (3) $x * (x * y) = x * y$ (for all $x, y \in V$).

Proof. Clearly, (1) follows from (t1) and (2) follows from Proposition 1 and (t2).

Consider arbitrary $x, y \in V$. By (t1), $(x * y) * x = x$. As follows from (1), $x * (x * y) = x * y$. Thus (3) holds. \square

Let $(V, *)$ be a travel groupoid, and let G be a graph. We say that $(V, *)$ is *on* G or that G *has* $(V, *)$ if $V(G) = V$ and

$$E(G) = \{\{u, v\} : u, v \in V \text{ and } u \neq u * v = v\}.$$

As follows from Proposition 2, every travel groupoid is on exactly one graph.

Proposition 3. *Let $(V, *)$ be a travel groupoid on a graph G , let $u, v \in V$ and $u \neq v$. Then u and $u * v$ are adjacent vertices of G .*

Proof. The proposition follows from (1) and (3). \square

Let $(V, *)$ be a travel groupoid. If $u, v \in V$, then we define

$$(4) \quad u *^0 v = u$$

and

$$u *^{i+1} v = (u *^i v) * v \text{ for every } i \geq 0.$$

It is clear that if $j, k \geq 0$, then $(u *^j v) *^k v = u *^{j+k} v$.

Proposition 4. *Let $(V, *)$ be a travel groupoid, let $u, v \in V$, and let $k \geq 1$. If $u *^k v \neq v$, then $u *^{k-1} v \neq v$ and the elements*

$$u *^{k-1} v, \quad u *^k v, \quad \text{and} \quad u *^{k+1} v$$

are pairwise distinct.

Proof. Let $u *^k v \neq v$. If $u *^{k-1} v = v$, then Proposition 1 implies that $u *^k v = v$; a contradiction. Thus $u *^{k-1} v \neq v$. Recall that $u *^k v = (u *^{k-1} v) * v$. If $u *^k v = u *^{k-1} v$, then, by virtue of (2), $u *^{k-1} v = v$; a contradiction. Thus $u *^k v \neq u *^{k-1} v$. If $u *^k v = u *^{k+1} v$, then it follows from (2) that $u *^k v = v$; a contradiction. Thus $u *^k v \neq u *^{k+1} v$. If $u *^{k+1} v = u *^{k-1} v$, then, as follows from (t2), $u *^{k-1} v = v$; a contradiction. Thus $u *^{k-1} v \neq u *^{k+1} v$, which completes the proof. \square

Remark 1. Let $(V, *)$ be a travel groupoid, and let $u, v \in V$. If there exists $i \geq 0$ such that $u *^i v = v$, then, by virtue of (2), $u *^{i+1} v = v$. This implies that there exists at most one $k \geq 1$ such that $u *^{k-1} v \neq v$ and $u *^k v = v$.

The following theorem motivates the terms “travel groupoids” and “travel operations”.

Theorem 1. *Let $(V, *)$ be a travel groupoid on a graph G , let $u, v \in V$, and let $k \geq 1$. Assume that $u *^{k-1} v \neq v$. Then the sequence*

$$(5) \quad u *^0 v, \dots, u *^{k-1} v, u *^k v$$

is a walk in G . Moreover, if $u *^k v = v$, then the sequence (5) is an $u - v$ path in G .

Proof. Since $u *^{k-1} v \neq v$, it follows from Proposition 4 that $u *^h v \neq v$ for each h , $0 \leq h \leq k - 1$. By the definition, $u *^{h+1} v = (u *^h v) * v$ for each h , $0 \leq h \leq k - 1$. Thus, by virtue of Proposition 3, the sequence (5) is a walk in G .

Let $u *^k v = v$. Assume that there exist i and j , $1 \leq i < j \leq k$, such that $u *^i v = u *^j v$. By virtue of Proposition 4, $j < k$. Thus $v = u *^k v = (u *^j v) *^{k-j} v = (u *^i v) *^{k-j} v = u *^{k-(j-i)} v \neq v$; a contradiction. We see that the vertices $u *^0 v, \dots, u *^{k-1} v, u *^k v$ are pairwise distinct. Hence the sequence (5) is an $u - v$ path in G , which completes the proof. \square

Let G be a geodetic graph, and let d denote the distance function of G . Put $V = V(G)$. It is not difficult to see that if $u, v \in V$ and $u \neq v$, then there exists exactly one vertex $A_G(u, v)$ such that

$$d(u, A_G(u, v)) = 1 \quad \text{and} \quad d(A_G(u, v), v) = d(u, v) - 1.$$

Define a binary operation $*$ on V as follows:

$$x * y = A_G(x, y) \quad \text{if } x \neq y$$

and

$$x * y = x \quad \text{if } x = y$$

for all $x, y \in V$. We will say that $(V, *)$ is the *proper* groupoid of G .

It is clear that the proper groupoid of a geodetic graph G is a travel groupoid on G . Thus every geodetic graph has at least one travel groupoid.

Obviously, every tree is a geodetic graph. (Note that the proper groupoid of a finite tree was characterized in [3]).

Proposition 5. *Every finite tree has exactly one travel groupoid.*

Proof. Consider an arbitrary finite tree T . Put $V = V(T)$. Let $(V, *)$ be the proper groupoid of T . Suppose, to the contrary, that there exists a travel groupoid (V, \circ) of T such that (V, \circ) is different from $(V, *)$. Then there exist $u, v \in V$ such that $u \circ v \neq u * v$. By Proposition 3, both vertices $u * v$ and $u \circ v$ are adjacent to u in T . Since $u \circ v \neq u * v$, we see that the vertices $u \circ v$ and $u * v$ belong to distinct components of $T - u$. Recall that $(V, *)$ is the proper groupoid of T . This implies that the vertices $u * v$ and v belong to the same component of $T - u$. Since T contains no cycle, then, by virtue of (t1) and (t2), the vertices

$$u \circ v, u \circ^2 v, u \circ^3 v, \dots$$

are pairwise distinct, which contradicts the fact that V is finite. Thus the proposition is proved. \square

2. SIMPLE TRAVEL GROUPOIDS

We say that a travel groupoid $(V, *)$ is *simple* if it satisfies the following axiom (t3) if $v * u \neq u$, then $u * (v * u) = u * v$ (for all $u, v \in V$).

Note that the travel groupoids discussed in [2] are simple.

Remark 2. Let $(V, *)$ be a simple travel groupoid, and let $u, v \in V$ such that $v * u \neq u$. By (1), $u * v \neq v$. Thus, by (t3), $u * (v * u) = u * v$ and $v * (u * v) = v * u$.

The next remark gives an example of a travel groupoid which is not simple.

Remark 3. Let D be a directed cycle with $|V(D)| = 2n$, where $n \geq 2$. Put $V = V(D)$. Clearly, for every $u \in V$ there exists exactly one vertex, say the vertex u' , such that (u, u') is a directed edge in D . Let C denote the underlying graph of D . Obviously, C is a cycle of length $2n$. Let d denote the distance function of C . We denote by $*$ the binary operation on V defined as follows for all $v, w \in V$:

$$v * w = v \text{ if } d(v, w) = 0;$$

$$v * w = v' \text{ if } d(v, w) = n;$$

$$v * w \text{ is the only vertex } t \text{ of } C \text{ with the property that } d(v, t) = 1 \text{ and } d(t, w) = d(v, w) - 1 \text{ if } 0 < d(v, w) < n.$$

It is obvious that $(V, *)$ is a travel groupoid. Consider arbitrary $x, y \in V$ such that $d(x, y) = n$. Then $d(x * y, x * (y * x)) = 2$. Thus $(V, *)$ is not simple.

Lemma 1. Let $(V, *)$ be a simple travel groupoid, let $u, v, w \in V$, and let $k \geq 1$. Assume that $u *^{k-1} w \neq w$ and

$$(6) \quad v * u = w.$$

Then

$$(7) \quad u *^i v = u *^i w$$

and

$$(8) \quad v * (u *^i w) = w$$

for each i , $0 \leq i \leq k$, and

$$u *^{k-1} v \neq v.$$

Proof. We will first prove that (7) and (8) hold for each i , $0 \leq i \leq k$. We proceed by induction on i . Let first $i = 0$. Obviously, $u *^0 v = u = u *^0 w$. By (6),

$v * (u *^0 w) = w$. Let now $1 \leq i \leq k$. By the induction hypothesis,

$$(9) \quad u *^{i-1} v = u *^{i-1} w$$

and

$$(10) \quad v * (u *^{i-1} w) = w.$$

Since $u *^{k-1} w \neq w$, Proposition 4 implies that $u *^{i-1} w \neq w$. By virtue of (10), $v * (u *^{i-1} w) = w \neq u *^{i-1} w$. As follows from (t3) and Remark 2,

$$(11) \quad (u *^{i-1} w) * v = (u *^{i-1} w) * (v * (u *^{i-1} w))$$

and

$$(12) \quad v * (u *^{i-1} w) = v * ((u *^{i-1} w) * v).$$

Obviously, $u *^i v = (u *^{i-1} v) * v$. It follows from (9), (11) and (10) that $(u *^{i-1} v) * v = (u *^{i-1} w) * v = (u *^{i-1} w) * (v * (u *^{i-1} w)) = (u *^{i-1} w) * w$. Thus $u *^i v = u *^i w$ and (7) holds.

Next, as follows from (7), (9), (12) and (10), $v * (u *^i w) = v * (u *^i v) = v * ((u *^{i-1} v) * v) = v * ((u *^{i-1} w) * v) = v * (u *^{i-1} w) = w$. Thus (8) holds.

We want to prove now that $u *^{k-1} v \neq v$. Suppose, to the contrary, that $u *^{k-1} v = v$. By (7), $u *^{k-1} v = u *^{k-1} w$, and thus $w \neq u *^{k-1} w = v$. As follows from (8) and Proposition 1, $w = v * (u *^{k-1} w) = v * v = v$, which completes the proof. \square

Proposition 6. *Let $(V, *)$ be a simple travel groupoid, and let $k \geq 1$. If $x, y \in V$, $x *^{k-1} y \neq y$, and $x *^k y = y$, then $y *^{k-1} x \neq x$ and*

$$(13) \quad y *^j x = x *^{k-j} y$$

for each j , $0 \leq j \leq k$.

P r o o f. We proceed by induction on k .

Let first $k = 1$. Consider arbitrary $x, y \in V$ such that $x *^0 y \neq y$ and $x *^1 y = y$. Then $x \neq y$. We have $y *^0 x \neq x$. Obviously, (13) holds for $j = 0$. As follows from (1), (13) holds also for $j = 1$.

Let now $k \geq 2$. Consider arbitrary $x, y \in V$ such that $x *^{k-1} y \neq y$ and $x *^k y = y$. Since $x *^{k-1} y \neq y$, it follows from (2) that $x \neq y$. Put $z = x * y$. Then $z *^{k-2} y \neq y$ and $z *^{k-1} y = y$. By the induction hypothesis, $y *^{k-2} z \neq z$ and

$$(14) \quad y *^j z = z *^{(k-1)-j} y = (x * y) *^{(k-1)-j} y = x *^{k-j} y$$

for each j , $0 \leq j \leq k - 1$. Since $y *^{k-2} z \neq z$ and $x * y = z$, Lemma 1 implies that

$$(15) \quad y *^j x = y *^j z$$

for each j , $0 \leq j \leq k - 1$. Combining (14) and (15) for each j , $0 \leq j \leq k - 1$, we get (13) for each j , $0 \leq j \leq k - 1$. This means that $y *^{k-1} x = x * y$. Since $x \neq y$, (2) implies that $y *^{k-1} x \neq x$. Moreover, by (t1), $y *^k x = (y *^{k-1} x) * x = (x * y) * x = x$. Hence (13) holds also for $j = k$, which completes the proof. \square

Corollary 1. *Let $(T, *)$ be a simple travel groupoid, let $u, v \in V$, and let $k \geq 1$. If $u *^k v = v$, then $v *^k u = u$.*

Proof. The case of $u = v$ is obvious. Let $u \neq v$. Then $u *^0 v \neq v$. Since $u *^k v = v$, we see that there exists i , $1 \leq i \leq k$, such that $u *^{i-1} v \neq v$ and $u *^i v = v$. By virtue of Proposition 6, $v *^i u = u *^0 v = u$ and therefore

$$v *^k u = (v *^i u) *^{k-i} u = u *^{k-i} u = u,$$

which completes the proof. \square

Theorem 2. *Let $(V, *)$ be a simple travel groupoid on a graph G , let $u, v \in V$ and let $k \geq 1$. Assume that $u *^{k-1} v \neq v$ and $u *^k v = v$. Then the sequence*

$$v *^0 u, \dots, v *^{k-1} u, v *^k u$$

is a $v - u$ path in G .

Proof. Combining Theorem 1 and Proposition 6, we get the theorem. \square

The next two lemmas will be used in Section 3.

Lemma 2. *Let $(V, *)$ be a simple travel groupoid, let $u, v \in V$, and let $j \geq 1$. Assume that $v *^j u \neq u$. Then $u * (v *^j u) = u * v$.*

Proof. We proceed by induction on j . The case of $j = 1$ immediately follows from the definition of a simple travel groupoid. Let $j \geq 2$. Since $v *^j u \neq u$, it follows from Proposition 4 that $v *^{j-1} u \neq u$. By the induction hypothesis,

$$(16) \quad u * (v *^{j-1} u) = u * v.$$

Obviously, $v *^j u = (v *^{j-1} u) * u$. Since $(v *^{j-1} u) * u \neq u$, (t3) implies that

$$(17) \quad u * ((v *^{j-1} u) * u) = u * (v *^{j-1} u).$$

Combining (16) and (17), we get $u * (v *^j u) = u * ((v *^{j-1} u) * u) = u * v$, which completes the proof. \square

Lemma 3. Let $(V, *)$ be a simple travel groupoid, let $u, v \in V$, and let $k \geq 2$. Assume that $u *^{k-1} v \neq v$ and $u *^k v = v$. Then

$$u *^i (v * u) = u *^i v$$

for each i , $0 \leq i \leq k - 1$.

Proof. Proposition 6 implies that $v *^{k-1} u \neq u$ and $v *^k u = u$. Put $w = v * u$. Then $w *^{k-2} u \neq u$ and $w *^{k-1} u = u$. By Proposition 6 again, $u *^{k-2} w \neq w$. Lemma 1 implies that $u *^i v = u *^i w = u *^i (v * u)$ for each i , $0 \leq i \leq k - 1$, which completes the proof. \square

Remark 4. Let $(V, *)$ be a simple travel groupoid on a finite graph G . It was proved in [2] and [4] that G is a geodetic graph and $(V, *)$ is its proper groupoid if and only if G is connected and $(V, *)$ satisfies the following axiom

(tg) if $w * v = v$ and $u * v \neq u * w$, then $w * (u * v) = v$ (for all $u, v, w \in V$).

The assumption that G is connected can not be deleted. There exists a simple travel groupoid satisfying (tg) on a finite disconnected graph (see Remark 2 in [2]).

3. NON-CONFUSING TRAVEL GROUPOIDS

Let $(V, *)$ be a travel groupoid, and let $u, v \in V$ such that $u \neq v$. By (2), $u * v \neq u$ and by (t2), $u *^2 v \neq u$. If there exists $i \geq 3$ such that $u *^i v = u$, then we say that the ordered pair (u, v) is a *confusing pair* in $(V, *)$.

The next lemma will be used in the next section.

Lemma 4. Let $(V, *)$ be a travel groupoid, let $u, v \in V$, $u \neq v$, and let $i \geq 3$ such that $u *^i v = u$. Then there exists j , $3 \leq j \leq i$, such that $u *^j v = u$ and the elements

$$u *^0 v, \dots, u *^{j-2} v, \quad \text{and} \quad u *^{j-1} v$$

are pairwise distinct.

Proof. Since $u \neq v$, (t2) implies that there exists j , $3 \leq j \leq i$ such that $u *^j v = u$ and all the elements

$$u *^1 v, u *^2 v, \dots, u *^{j-1} v$$

are different from u . Assume that there exist k and m , $1 \leq k < m \leq j - 1$, such that $u *^k v = u *^m v$. Then $m \geq k + 2$. It is clear that

$$u *^n v \in \{u *^k v, u *^{k+1} v, \dots, u *^{m-1} v\} \quad \text{for all } n \geq m,$$

and therefore $u *^i v \neq u$, which is a contradiction. Thus the lemma is proved. \square

Remark 5. Let $(V, *)$ be a travel groupoid on a finite graph G . It is clear that if G is not connected, then $(V, *)$ has a confusing pair.

We say that a travel groupoid $(V, *)$ is *non-confusing* if there exists no confusing pair in $(V, *)$.

Proposition 7. *Let $(V, *)$ be a finite non-confusing travel groupoid, and let $u, v \in V$ and $u \neq v$. Then there exists exactly one $k \geq 1$ such that $u *^{k-1} v \neq v$ and $u *^k v = v$.*

Proof. Define

$$u_i = u *^i v \quad \text{for all } i \geq 0.$$

Suppose, to the contrary, that

$$u_i \neq v \quad \text{for all } i \geq 0.$$

Since V is finite, there exist j and m , $0 \leq j < m$, such that $u_m = u_j$. We have

$$u_m = u_j *^{m-j} v = u_j.$$

Thus $m - j \geq 3$ and (u_j, v) is a confusing pair in $(V, *)$, which is a contradiction. We have prove that there exists $k \geq 1$ such that $u *^{k-1} v \neq v$ and $u *^k v = v$. By Remark 1, k is defined uniquely. Thus the theorem is proved. \square

Theorem 3. *Let $(V, *)$ be a finite travel groupoid on a graph G . Then $(V, *)$ is non-confusing if and only if the following statement holds for all distinct $u, v \in V$: there exists $k \geq 1$ such that the sequence*

$$u *^0 v, \dots, u *^{k-1} v, u *^k v$$

is an $u - v$ path in G .

Proof. Combining Theorem 1 and Proposition 7, we obtain the theorem. \square

The next remark gives an example of a simple travel groupoid on a finite geodetic graph. This travel groupoid has a confusing pair.

Remark 6. Let $m, n \geq 3$ be odd, and let $u_0, u_1, \dots, u_{m-1}, v, w_0, w_1, \dots, w_{n-1}$ are pairwise distinct elements. Put

$$U = \{u_0, u_1, \dots, u_{m-1}, v\} \quad \text{and} \quad W = \{w_0, w_1, \dots, w_{n-1}, v\}.$$

Obviously, $U \cap W = \{v\}$. Define $u_m = u_0$ and $w_n = w_0$. Let G_U be the graph with $V(G_U) = U$ and

$$E(G_U) = \{u_0u_1, \dots, u_{m-2}u_{m-1}, u_{m-1}u_m, u_0v\}.$$

Moreover, let G_W be the graph with $V(G_W) = W$ and

$$E(G_W) = \{w_0w_1, \dots, w_{n-2}w_{n-1}, w_{n-1}w_n, w_0v\}.$$

Since m and n are odd, we see that both G_U and G_W are geodetic graphs. At the end of Section 1, the mapping A_G was defined for a geodetic graph G . In the same way, we define the mappings A_{G_U} and A_{G_W} for the geodetic graphs G_U and G_W respectively.

Put $V = U \cup W$. We denote by $*$ the binary operation on V defined for all $x, y \in V$ as follows:

$$x * y = x \text{ if } x = y;$$

$$x * y = A_{G_U}(x, y) \text{ if } x, y \in U \text{ and } x \neq y;$$

$$x * y = A_{G_W}(x, y) \text{ if } x, y \in W \text{ and } x \neq y;$$

$$x * y = u_i \text{ if } x = u_{i-1} \text{ and } y \in W \setminus \{v\} \text{ for each } i, 0 \leq i \leq m - 1;$$

$$x * y = w_j \text{ if } x = w_{j-1} \text{ and } y \in U \setminus \{v\} \text{ for each } j, 0 \leq j \leq n - 1.$$

It is easy to see that $(V, *)$ is a simple travel groupoid. The ordered pair (u_0, w_0) is an example of a confusing pair in $(V, *)$. Let G_0 denote the graph of $(V, *)$. It is easy to see that G is a geodetic graph.

Proposition 8. *Let $(V, *)$ be a finite simple non-confusing travel groupoid, let $u, v \in V$. Then $(u *^i v) *^i u = u$ for each $i \geq 0$.*

Proof. The case of $u = v$ follows immediately from Proposition 1. Assume that $u \neq v$. According to Proposition 7, there exists $k \geq 1$ such that $u *^{k-1} v \neq v$ and $u *^k v = v$. Recall that $(V, *)$ is simple. If $i \geq k$, then $u *^i v = v$ and, by virtue of Corollary 1, $(u *^i v) *^i u = v *^i u = u$. Let $i < k$. By Proposition 6, $v *^k u = u$ and $v *^{k-i} u = u *^i v$. Thus $(u *^i v) *^i u = (v *^{k-i} u) *^i u = v *^k u = u$, which completes the proof. \square

Proposition 9. *Let $(V, *)$ be a finite simple non-confusing travel groupoid, and let $u, v, w \in V$ such that $u \neq v$. Assume that there exists $k \geq 1$ such that $u *^{k-1} w \neq v$ and $u *^k w = v$. Then $u *^{k-1} v \neq v$ and $u *^k v = v$.*

Proof. We see that $u \neq w$ (otherwise, $u *^k w = u \neq v$; a contradiction). By Proposition 7, there exists exactly one $m \geq 1$ such that $u *^{m-1} w \neq v$ and $u *^m w = v$.

If $k > m$, then $u *^{k-1} w = u *^k w$; a contradiction. Thus $k \leq m$. As follows from Proposition 6, $w *^{m-1} u \neq u$ and $w *^{m-j} u = u *^j w$ for each j , $0 \leq j \leq m$. Thus $w *^m u = u$. Since $u *^k w = v$, we get $v = w *^{m-k} u$. Hence

$$v *^{k-1} u = (w *^{m-k} u) *^{k-1} u \neq u \quad \text{and} \quad v *^k u = (w *^{m-k} u) *^k u = u.$$

If we apply Proposition 6 again, we get $u *^{k-1} v \neq v$ and $u *^k v = v$, which completes the proof. \square

Let $(V, *)$ be a finite travel groupoid on a graph G , and let $x, y \in V$. Clearly, x and y are distinct and non-adjacent vertices of G if and only if $x * y \neq y$.

Let $(V, *)$ be a simple non-confusing travel groupoid, and let $x, y \in V$ such that $x * y \neq y$. By virtue of Proposition 7, there exists exactly one $k \geq 2$ such that

$$(18) \quad x *^{k-1} y \neq y \quad \text{and} \quad x *^k y = y.$$

As follows from Proposition 6,

$$(19) \quad y *^{k-1} x \neq x \quad \text{and} \quad y *^k x = x.$$

Put

$$(20) \quad y = x' \quad \text{and} \quad x = y'.$$

Consider arbitrary $u, v \in V$ such that $u \in \{x, y\}$. Assume that there exists $j \geq 1$ such that $u *^{j-1} v \neq u'$ and $u *^j v = u'$. By virtue of Remark 1 and Proposition 9, $j = k$. Moreover, Proposition 9 implies that

$$(21) \quad \text{if } u *^k v = u', \text{ then } u *^{k-1} v \neq u'.$$

By the *xy-strengthening* of $*$ on V we mean the binary operation \circ on V defined for all $u, v \in V$ as follows:

$$u \circ v = u *^k v \text{ if } u \in \{x, y\} \text{ and } u *^k v = u';$$

$$u \circ v = u * v \text{ otherwise.}$$

This means that $w \circ w = w$ for every $w \in V$.

Lemma 5. *Let $(V, *)$ be a finite simple non-confusing travel groupoid on a graph G , let $x, y \in V$ such that $x * y \neq y$, and let \circ be the xy -strengthening of $*$ on V . Then (V, \circ) is a simple non-confusing travel groupoid on $G + xy$.*

Proof. There exists exactly one $k \geq 2$ such that (18) holds. Moreover, we have (19). Use the convention (20).

We first show that (V, \circ) satisfies the axioms (t1), (t2), and (t3) and that (V, \circ) is non-confusing. Consider arbitrary $r, s \in V$.

Verification of (t1). Put $t = (r \circ s) \circ r$. We will show that $t = r$. If $r = s$, then $r \circ s = r$ and therefore $t = r \circ r = r$. Assume that $r \neq s$. Then there exists $i \in \{1, k\}$ such that $r \circ s = r *^i s$. Since $r \neq s$ and $(V, *)$ is non-confusing, we get $r *^i s \neq r$. There exists $j \in \{1, k\}$ such that $t = (r *^i s) \circ r = (r *^i s) *^j r$. If $i = 1$, then (t1) and Proposition 1 imply that $t = r$. Assume that $i = k$. Then $r \in \{x, y\}$ and $r *^i s = r'$. This implies that $j = k$. We get $t = (r *^k s) *^k r = r$ again.

Verification of (t2). Obviously, there exist $i, j \in \{1, k\}$ such that $(r \circ s) \circ s = (r *^i s) *^j s = r *^{i+j} s$. Let $(r \circ s) \circ s = r$. Since $(V, *)$ is non-confusing, we get $r = s$.

We see that (V, \circ) is a travel groupoid.

Verification of (t3). Assume that $s \circ r \neq r$. As follows from (2), $s \neq r$. We will prove that $r \circ (s \circ r) = r \circ s$. If $r, s \in \{x, y\}$, then $s *^k r = r$ and therefore $s \circ r = r$, which is a contradiction. Thus

$$(22) \quad \text{at most one of } r \text{ and } s \text{ belongs to } \{x, y\}.$$

Since $(V, *)$ is non-confusing, Proposition 7 implies that there exists $m \geq 1$ such that $r *^{m-1} s \neq s$ and $r *^m s = s$. Recall that $k \geq 2$. Since $(V, *)$ is simple, it follows from Lemma 3 that

$$(23) \quad \text{if } k < m, \text{ then } r *^k (s * r) = r *^k s.$$

Recall that $s \circ r \neq r$. Since $s \circ r = s *^k r$ or $s * r$, Remark 1 implies that $s * r \neq r$. By (t3),

$$r * (s * r) = r * s.$$

Let first $r \in \{x, y\}$ and $r *^k s = r'$. Then $r \circ s = r *^k s = r'$. By (22), $s \neq r'$. Then $k < m$ and $s \circ r = s * r$. It follows from (23) that $r *^k (s * r) = r'$ and therefore

$$r \circ (s \circ r) = r \circ (s * r) = r *^k (s * r) = r *^k s = r' = r \circ s.$$

Let now $r \in \{x, y\}$ and $r *^k s \neq r'$. Then $r \circ s = r * s$. By (22), $s \neq r'$ and thus $s \circ r = s * r$. Assume that $r *^k (s * r) = r'$. Then $r \circ (s * r) = r *^k (s * r)$. As follows

from (21), $r *^{k-1} (s * r) \neq r'$. By virtue of Lemma 1, $r *^k s = r *^k (s * r) = r'$, which is a contradiction. Thus $r \circ (s * r) = r * (s * r)$ and therefore $r \circ (s \circ r) = r * s = r \circ s$.

Finally, let $r \notin \{x, y\}$. Then $r \circ s = r * s$ and $r \circ (s \circ r) = r * (s \circ r)$. Assume that $s \in \{x, y\}$ and $s *^k r = s'$. Then $s \circ r = s *^k r$. Since $s *^k r \neq r$, Lemma 2 implies that $r \circ (s \circ r) = r * (s *^k r) = r * s = r \circ s$. If

$$s \notin \{x, y\} \quad \text{or} \quad (s \in \{x, y\} \text{ and } s *^k r \neq s'),$$

then $s \circ r = s * r$ and therefore $r \circ (s \circ r) = r * (s * r) = r * s = r \circ s$.

Thus (V, \circ) is simple.

Assume that $r \neq s$ and there exists $i \geq 1$ such that $r \circ^i s = r$. Clearly, there exists $m \geq i$ such that $r \circ^i s = r *^m s$. We have that $r *^m s = r$, which contradicts the fact that $(V, *)$ is non-confusing. Thus (V, \circ) is non-confusing, too.

Recall that $x \neq y$ and $x \circ y = y$. We can see that (V, \circ) is a simple non-confusing travel groupoid on $G + xy$, which completes the proof of the lemma. \square

Theorem 4. *For every finite connected graph G there exists a simple non-confusing travel groupoid on G .*

Proof. Put $V = V(G)$ and $\beta(G) = |E(G)| - |V| + 1$. We proceed by induction on $\beta(G)$. Obviously, $\beta(G) \geq 0$. Let first $\beta(G) = 0$. Then G is a tree. It is easy to see that its proper groupoid is simple and non-confusing. Let now $\beta(G) \geq 1$. Then there exist distinct $x, y \in V$ such that x and y are adjacent in G and $G - xy$ is connected. By the induction hypothesis, there exists a simple non-confusing travel groupoid $(V, *)$ on $G - xy$. Lemma 5 implies that there exists a simple non-confusing groupoid on G , which completes the proof. \square

4. SMOOTH AND SEMI-SMOOTH TRAVEL GROUPOIDS

We say that a travel groupoid $(V, *)$ is *smooth* if it satisfies the following axiom (t4) if $u * v = u * w$, then $u * (w * v) = u * v$ (for all $u, v, w \in V$).

Moreover, we say that a travel groupoid $(V, *)$ is *semi-smooth* if it satisfies the following axiom

(t5) if $u * v = u * w$, then $u * (v * w) = u * v$ or $u * ((v * w) * w) = u * v$ (for all $u, v, w \in V$).

Obviously, every smooth travel groupoid is semi-smooth.

Proposition 10. *Every semi-smooth travel groupoid is non-confusing.*

Proof. Let $(V, *)$ be a semi-smooth travel groupoid. Obviously, there exists a graph G such that $(V, *)$ is on G . Suppose, to the contrary, that there exists a confusing pair in $(V, *)$. As follows from Lemma 4, there exist $u, w \in V$ and $k \geq 3$ such that $u \neq w$ and $u *^k w = u$, and the vertices $u *^0 w, \dots, u *^{k-2} w$ and $u *^{k-1} w$ are pairwise distinct. Define

$$u_i = u *^i w \quad \text{for } i = 0, 1, \dots, k.$$

Hence $u_0 \neq u_1 \neq u_{k-1} \neq u_k = u_0$. Obviously, $u_0 * u_1 = u_1$ and $u_0 * u_k = u_0 * u_0 \neq u_1$. Moreover, $u_0 * u_{k-1} = u_k * u_{k-1} = (u_{k-1} * w) * u_{k-1}$ and thus, by (t1), $u_0 * u_{k-1} = u_{k-1}$. This implies that there exist j , $0 \leq j \leq k-2$, such that $u_0 * u_j = u_1$ and $u_0 * u_{j+1} \neq u_1 \neq u_0 * u_{j+2}$. We have $u_0 * w = u_1 = u_0 * u_j$, $u_0 * (u_j * w) = u_0 * u_{j+1} \neq u_0 * u_j$, and $u_0 * ((u_j * w) * w) = u_0 * u_{j+2} \neq u_0 * u_j$, which contradicts (t5). Thus the proposition is proved. \square

Proposition 11. *Every complete bipartite graph has a simple smooth travel groupoid.*

Proof. Let G be complete bipartite graph. Put $V = V(G)$. There exist nonempty sets U and U' such that $U \cap U' = \emptyset$, $U \cup U' = V$ and the following statement holds for all distinct $v, w \in V$:

$$v \text{ and } w \text{ are adjacent in } G \text{ if and only if } |\{v, w\} \cap U| = 1 = |\{v, w\} \cap U'|.$$

Recall that U and U' are nonempty. Choose a vertex $u \in U$ and a vertex $u' \in U'$. We denote by $*$ the binary operation on V defined as follows:

- $x * y = x$ if $x = y$;
- $x * y = y$ if x and y are adjacent in G ;
- $x * y = u'$ if $x, y \in U$ and $x \neq y$;
- $x * y = u$ if $x, y \in U'$ and $x \neq y$.

It can be easily verified that $(V, *)$ satisfies (t1), (t2), (t3), and (t4). Hence $(V, *)$ is a simple smooth travel groupoid. \square

Recall that every tree is a geodetic graph and that the proper groupoid of every geodetic graph is a simple travel groupoid.

Proposition 12. *The proper groupoid of every tree is a smooth travel groupoid.*

Proof is easy. \square

Note that every complete graph is geodetic. Obviously, the proper groupoid of every complete graph is a smooth travel groupoid.

Theorem 5. *Let G be a geodetic graph of diameter two, and let $(V, *)$ be the proper groupoid on G . Then $(V, *)$ is a smooth travel groupoid.*

Proof. Clearly, $(V, *)$ is a simple travel groupoid such that

$$(24) \quad x *^2 y = y \quad \text{for all } x, y \in V.$$

We will prove that $(V, *)$ is smooth. Suppose, to the contrary, that $(V, *)$ is not smooth. Then there exist $u, v, w \in V$ such that $u * v = u * w$ and

$$(25) \quad u * (v * w) \neq u * v.$$

This implies that $v \neq v * w \neq w$. By (24), $v *^2 w = w$. Recall that $(V, *)$ is simple. Since $v * w \neq w$, Proposition 6 implies that $w * v \neq v$, $w *^2 v = v$, and $w * v = v * w$. Thus the sequence

$$v, v * w = w * v, w$$

is a shortest $v - w$ path in G .

Put $t = u * v$. Then $t = u * w$. As follows from (24), $t * v = v$ and $t * w = w$. By (1), $v * t = t$. If $t = v$, then $u * w = v$ and therefore $u *^2 w = v * w \neq w$; a contradiction. Thus $t \neq v$. Since $v * t = t$, we see that v and t are adjacent in G . Let $t = w$. Since $t * v = v$, we get $w * v = v$; a contradiction. Thus $t \neq w$. Since $t * w = w$, we see that t is adjacent to w . Thus the sequence

$$v, t, w$$

is a shortest $v - w$ path in G .

Assume that $t = v * w$. Using (3), we see that $u * (v * w) = u * (u * v) = u * v$, which contradicts (25). Thus $u * v \neq v * w$. We see that G has two distinct shortest $v - w$ paths in G . This means that G is not a geodetic graph, which is a contradiction. Thus the theorem is proved. \square

We pose two questions.

Question 1. Does there exist a geodetic graph G such that the proper groupoid of G is not smooth? (If so, does there exist a geodetic groupoid G such that the proper groupoid of G is not semi-smooth?)

Question 2. Does there exist a connected graph G such that G has no smooth travel groupoid? (If so, does there exist a connected graph G such that G has no semi-smooth travel groupoid?)

5. GRAPHS WITH TRAVEL GROUPOIDS

Recall that, by Theorem 4, every finite connected graph has a simple non-confusing travel groupoid.

Theorem 6. *Let G be a finite graph. Then G has a travel groupoid if and only if G is connected or G is disconnected and no component of G is a tree.*

Proof. Assume that G is connected or G is disconnected and no component of G is a tree. If G is connected, then, by Theorem 4, there exists a travel groupoid on G . Let G be disconnected. Then every component of G contains a cycle. It is easy to see that there exists a mapping f of $V(G)$ into itself such that the following statements hold for every $u \in V(G)$:

$$u \text{ and } f(u) \text{ are adjacent vertices in } G$$

and

$$u \neq f(f(u)).$$

By virtue of Theorem 4, every component F of G has a travel groupoid, say a travel groupoid $(V(F), *F)$. For all $x, y \in V(G)$, we define

$$x * y = x *_{H} y \text{ if there exists a component } H \text{ of } G \text{ such that } x, y \in V(H)$$

and

$$x * y = f(x) \text{ if } x \text{ and } y \text{ belong to distinct components of } G.$$

It is easy to see that $(V(G), *)$ satisfies (t1) and (t2). Hence G has a travel groupoid.

Conversely, assume that G is disconnected and at least one component T of G is a tree. Suppose, to the contrary, that G has a travel groupoid, say a travel groupoid $(V, *)$, where $V = V(G)$. Consider $u \in V(T)$ and $v \in V(G) \setminus V(T)$. Since $V(T)$ is finite and T contains no cycle, we see that there exists $k \geq 1$ such that $u *^{k+1} v = u *^{k-1} v$. We have $((u *^{k-1} v) * v) * v = u *^{k-1} v$, and thus, by (t2), $u *^{k-1} v = v$. Proposition 3 implies that u and v belong to the same component of G , which is a contradiction. Thus the theorem is proved. \square

Question 3. Does there exist an infinite graph G with no finite components such that G has no travel groupoid?

Remark 7. Let $(V, *)$ be a finite travel groupoid. Put

$$X = \{(u, v, w) : u, v, w \in V \text{ and } v = u * w\}.$$

Then (V, X) is a signpost system in the sense of [5]. We say that (V, X) is the signpost system of $(V, *)$.

The signpost systems of travel groupoids create a special subclass of the class of all signpost systems. The terms “simple”, “non-confusing” and “smooth” introduced in the present paper for travel groupoids are inspired by the same terms used for signpost systems in [5].

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