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A NOTE ON CHARACTERISTIC CLASSES

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Abstract. This paper studies the relationship between the sections and the Chern or Pontrjagin classes of a vector bundle by the theory of connection. Our results are natural generalizations of the Gauss-Bonnet Theorem.

Keywords: fibre bundle, characteristic class, transgression, Poincaré dual

MSC 2000: 53C05, 55R25, 57R20

1. Introduction

Let $\pi\colon E \to M$ be an oriented Riemannian vector bundle with a Riemannian connection D, Ω its curvature matrix. Then the Euler characteristic class $e(E)$ of the bundle E can be represented by $\mathrm{Pf}\bigl(-(2\pi)^{-1}\Omega\bigr)$, where Pf is the Pfaffian polynomial. As is well-known, the Pontryagin and Chern classes can be obtained from the Euler classes. The characteristic classes are very important in the study of topology and differential geometry.

In this paper, we study the relationship between the sections and the Chern or Pontrjagin classes of a vector bundle by the theory of connection. The results are natural generalizations of the Gauss-Bonnet Theorem which concerns the relationship among the Euler class of the tangent bundle, the tangent vector fields and the Euler-Poincaré characteristic number of the manifold.

As is well known, the top Chern class of a complex vector bundle $E_{\mathbb{C}}$ and the Euler class of its realization vector bundle $E_{\mathbb{R}}$ are the same. This can be proved by the splitting principle, see [1, p. 273], [4, p. 115] or [10, p. 158]. In § 2, we give a direct proof of this fact. Then we state some known results about the characteristic classes which are needed in § 3.

In § 3, we study the relationship between the sections of the vector bundle and the Chern or Pontrjagin classes of the bundle. Using the transgression, we show that the Chern and the Pontrjagin classes can be represented by cycles in homology of the base manifolds by Poincaré dual. These cycles are determined by the generic sections of the vector bundles.

In the following, we assume that the base manifolds of vector bundles are all compact and oriented.

2. Preliminaries

The complex Euclidean space $\mathbb{C}^n$ is naturally isomorphic to a real Euclidean space $\mathbb{R}^{2n}$. The isomorphism can be given by

$$\sum z_i e_i \mapsto \sum x_i e_i + \sum y_i e_{n+i}, \quad z_i = x_i + \sqrt{-1} y_i, \quad i = 1, \dots, n,$$

where $e_1, \dots, e_n$ is a unitary basis of $\mathbb{C}^n$. The basis $e_1, e_{n+1}, \dots, e_n, e_{2n}$ of $\mathbb{R}^{2n}$ also gives an orientation on $\mathbb{R}^{2n}$. For any matrix $C = (C_{ij}) \in \underline{so}(2n)$, the Lie algebra of $SO(2n)$, let

$$T = (e_1, e_{n+1}, \dots, e_n, e_{2n}) \wedge C(e_1, e_{n+1}, \dots, e_n, e_{2n})^t.$$

The Pfaffian $\mathrm{Pf}(C)$ is defined by

$$\mathrm{Pf}(C) e_1 \wedge e_{n+1} \wedge \dots \wedge e_n \wedge e_{2n} = \frac{1}{2^n n!} T^n.$$

Let $U(n)$ be the unitary group and $\underline{u}(n)$ its Lie algebra, any element of $\underline{u}(n)$ can be represented as $A+\sqrt{-1}B$, where A, B are real matrices. The canonical representation $U(n) \to SO(2n)$ induces a representation between their Lie algebras. With the oriented bases $e_1, \dots, e_n$ and $e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}$ on $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ respectively, the map $\underline{u}(n) \to \underline{so}(2n)$ can be represented by

$$A + \sqrt{-1} B \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \quad A^t = -A, \;\; B^t = B.$$

Denote C the matrix obtained by rearranging the rows and columns of $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ according to the oriented basis $e_1, e_{n+1}, \dots, e_n, e_{2n}$ of $\mathbb{R}^{2n}$.

Lemma 2.1. $\mathrm{Pf}(C) = \det(-\sqrt{-1}(A+\sqrt{-1}B))$.

Proof. It is easy to see that T can also be represented by

$$T = (e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}) \wedge \begin{pmatrix} A & B \\ -B & A \end{pmatrix} (e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n})^t.$$

Let $g_i = e_i - \sqrt{-1}e_{n+i}$, $g_{n+i} = e_i + \sqrt{-1}e_{n+i}$ and $X = \frac{\sqrt{2}}{2}\begin{pmatrix} I & \sqrt{-1}I \\ \sqrt{-1}I & I \end{pmatrix} \in U(2n)$. By

$$(e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n})X = \frac{\sqrt{2}}{2}(g_{n+1}, \ldots, g_{2n}, \sqrt{-1}g_1, \ldots, \sqrt{-1}g_n),$$

$$X^{-1}(e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n})^t = \frac{\sqrt{2}}{2}(g_1, \ldots, g_n, -\sqrt{-1}g_{n+1}, \ldots, -\sqrt{-1}g_{2n})^t,$$

$$X^{-1}\begin{pmatrix} A & B \\ -B & A \end{pmatrix} X = \begin{pmatrix} A+\sqrt{-1}B & \\ & A-\sqrt{-1}B \end{pmatrix},$$

$$\begin{aligned}&(g_{n+1}, \ldots, g_{2n}) \wedge (A+\sqrt{-1}B)(g_1, \ldots, g_n)^t \\ &= (g_1, \ldots, g_n) \wedge (A-\sqrt{-1}B)(g_{n+1}, \ldots, g_{2n})^t,\end{aligned}$$

we have $T = (g_{n+1}, \ldots, g_{2n}) \wedge (A+\sqrt{-1}B)(g_1, \ldots, g_n)^t$. Then

$$\begin{aligned} T^n &= n! \det(A+\sqrt{-1}B) g_{n+1} \wedge g_1 \wedge \ldots \wedge g_{2n} \wedge g_n \\ &= n!(-2\sqrt{-1})^n \det(A+\sqrt{-1}B) e_1 \wedge e_{n+1} \wedge \ldots \wedge e_n \wedge e_{2n}. \end{aligned}$$

Hence $\mathrm{Pf}(C) = (-\sqrt{-1})^n \det(A+\sqrt{-1}B)$. □

Let $\pi\colon E_{\mathbb{C}} \to M$ be a Hermitian vector bundle with fibre $\mathbb{C}^n$, $D_{\mathbb{C}}$ a Hermitian connection on $E_{\mathbb{C}}$. The bundle $E_{\mathbb{C}}$ naturally determines a real Riemannian vector bundle $\tau\colon E_{\mathbb{R}} \to M$ with fibre $\mathbb{R}^{2n}$ and a Riemannian connection $D_{\mathbb{R}}$. If $s_1, \ldots, s_n$ is a unitary basis for the sections of $E_{\mathbb{C}}$ over an open set $U \subset M$, then $s_1, s_{n+1} = \sqrt{-1}s_1, \ldots, s_n, s_{2n} = \sqrt{-1}s_n$ form an orthonormal basis for the sections of $E_{\mathbb{R}}$ over U, see [10, p. 155]. If $D^2_{\mathbb{C}} s_i = \sum\limits_{j=1}^{n} \widetilde{\Omega}_{ij} s_j$, $\widetilde{\Omega}_{ij} = \Omega_{ij} + \sqrt{-1}\Omega_{i,n+j}$ are the curvature forms of connection $D_{\mathbb{C}}$, we have

$$\begin{aligned} D^2_{\mathbb{R}} s_i &= \sum \Omega_{ij} s_j + \sum \Omega_{i,n+j} s_{n+j}, \\ D^2_{\mathbb{R}} s_{n+i} &= -\sum \Omega_{i,n+j} s_j + \sum \Omega_{ij} s_{n+j}. \end{aligned}$$

Denote $\Omega_{E_{\mathbb{C}}}$ and $\Omega_{E_{\mathbb{R}}}$ the curvature matrices of $D_{\mathbb{C}}$, $D_{\mathbb{R}}$, respectively. By Lemma 2.1, we have

Corollary 2.2. *The top Chern class of the bundle $E_{\mathbb{C}}$ and the Euler class of $E_{\mathbb{R}}$ represented by $\Omega_{E_{\mathbb{C}}}$ and $\Omega_{E_{\mathbb{R}}}$ respectively are the same, that is,*

$$\det\Big(\frac{\sqrt{-1}}{2\pi}\Omega_{E_{\mathbb{C}}}\Big) = \mathrm{Pf}\Big(\frac{-1}{2\pi}\Omega_{E_{\mathbb{R}}}\Big).$$

From the vector bundle $\pi\colon E_{\mathbb{C}} \to M$, we can construct fibre bundles $\pi_i\colon V(E_{\mathbb{C}}, i) \to M$, $i = 1, \dots, n$. For any $p \in M$, the fibre $\pi_i^{-1}(p)$ is a complex Stiefel manifold formed by all unitary i-frames on $\pi^{-1}(p)$. For each i, we have an induced bundle $\pi_i^* E_{\mathbb{C}} \to V(E_{\mathbb{C}}, i)$ which can be decomposed by $\pi_i^* E_{\mathbb{C}} = \mathcal{E}_i \oplus F_{n-i}$. The fibre of F_{n-i} over $(s_1, \dots, s_i) \in V(E_{\mathbb{C}}, i)$ is the orthogonal complement of $s_1, \dots, s_i$ in the vector space $\pi^{-1}(p)$, the bundle $\mathcal{E}_i$ is trivial. Then we have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & F_1 & \longrightarrow & \dots & \longrightarrow & F_{n-1} & \longrightarrow & F_n = E_{\mathbb{C}} \\
 & & \downarrow & & & & \downarrow & & \downarrow \\
V(E_{\mathbb{C}}, n) & \xrightarrow{\alpha_n} & V(E_{\mathbb{C}}, n-1) & \xrightarrow{\alpha_{n-1}} & \dots & \longrightarrow & V(E_{\mathbb{C}}, 1) & \xrightarrow{\alpha_1} & M.
\end{array}$$

The maps in the diagram are all defined naturally. By the theory of characteristic class and Corollary 2.2,

$$\pi_{n-i}^* c_i(E_{\mathbb{C}}) = c_i(\pi_{n-i}^* E_{\mathbb{C}}) = c_i(\mathcal{E}_{n-i} \oplus F_i) = c_i(F_i) = e(F_{i\mathbb{R}}).$$

The map $\alpha_j\colon V(E_{\mathbb{C}}, j) \to V(E_{\mathbb{C}}, j-1)$ defines a fibre bundle with the fibre $S^{2n-2j+1}$. As in [10, § 14], applying the Gysin sequence to the vector bundle $F_{n-j+1} \to V(E_{\mathbb{C}}, j-1)$, we know that the pullback map

$$\alpha_j^*\colon H^k(V(E_{\mathbb{C}}, j-1), Z) \to H^k(V(E_{\mathbb{C}}, j), Z)$$

is an isomorphism for any $k < 2n-2j+1$. Since $\pi_{n-i} = \alpha_1 \dots \alpha_{n-i}\colon V(E_{\mathbb{C}}, n-i) \to M$, the maps

$$\pi_{n-i}^*\colon H^k(M, Z) \to H^k(V(E_{\mathbb{C}}, n-i), Z), \quad k < 2i+1,$$

are all isomorphisms.

Proposition 2.3. $c_i(E_{\mathbb{C}}) = \pi_{n-i}^{*-1} c_i(F_i) = \pi_{n-i}^{*-1} e(F_{i\mathbb{R}})$, $i = 1, \dots, n$.
Then $c_i(E_{\mathbb{C}}) = 0$ if the bundle $F_i \to V(E_{\mathbb{C}}, n-i)$ has a non-zero section.

For the real vector bundle $\pi\colon E \to M$, we can also construct fibre bundles $\pi_i\colon V(E, i) \to M$, $i = 1, \dots, n = \operatorname{rank} E$. For any $p \in M$, the fibre $\pi_i^{-1}(p)$ is a Stiefel manifold formed by all orthonormal frames on $\pi^{-1}(p)$. For any i, we have a pullback vector bundle $\pi_i^* E = \mathcal{E} \oplus \widetilde{F}_{n-i} \to V(E, i)$, where $\mathcal{E}$ is a trivial bundle of rank i.

Proposition 2.4. *Assume the vector bundle E is oriented, then the vector bundles $\widetilde{F}_{n-k} \to V(E,k)$ are all oriented. We have $p_i(E) = \pi^{*-1}_{n-2i}[e(\widetilde{F}_{2i}) \cdot e(\widetilde{F}_{2i})]$, where $p_i(E)$ is the ith Pontrjagin class.*

Proof. Similarly to Proposition 2.3, we have $p_i(E) = \pi^{*-1}_{n-2i} p_i(\widetilde{F}_{2i})$. By [10, Corollary 15.8], $p_i(\widetilde{F}_{2i}) = e(\widetilde{F}_{2i}) \cdot e(\widetilde{F}_{2i})$. □

3. The transgression and the Poincaré dual

In [2], [3], Chern gave an elegant proof of the Gauss-Bonnet theorem and introduced the concept of the transgression for the characteristic classes. Let $\pi\colon E \to M$ be an oriented Riemannian vector bundle with rank $2n$, $p\colon S(E) = V(E,1) \to M$ be the associated sphere bundle. The induced bundle $p^*E \to S(E)$ can be decomposed as $p^*E = \widetilde{F}_{2n-1} \oplus \mathcal{E}$. Then

$$p^* e(E) = e(\widetilde{F}_{2n-1} \oplus \mathcal{E}) = 0 \quad \text{in } H^{2n}(S(E), Z).$$

Let D be a Riemannian connection on E and p^*D the pull back connection on p^*E. Let $e_1, \ldots, e_{2n-1}, e_{2n}$ be oriented orthonormal frame fields on p^*E, $\mathcal{E}$ be generated by $e_{2n} \in S(E)$. Define another connection $\widetilde{D}$ on p^*E:

$$\widetilde{D} e_\alpha = \sum \widetilde{\omega}^\beta_\alpha e_\beta, \quad \alpha, \beta = 1, \ldots, 2n-1, \ \ \widetilde{D} e_{2n} = 0,$$

where $\widetilde{\omega}^\beta_\alpha = p^* \omega^\beta_\alpha$ are defined by $De_\alpha = \sum \omega^\beta_\alpha e_\beta + \omega^{2n}_\alpha e_{2n}$. Let $p^*\Omega$ and $\widetilde{\Omega}$ be the curvature matrices of the Riemannian connections p^*D and $\widetilde{D}$ on p^*E respectively. Then $e(E)$ can be represented by $e(\Omega) = \mathrm{Pf}\big(\frac{-1}{2\pi}\Omega\big)$ and $e(\widetilde{\Omega}) = 0$ on p^*E. By Chern-Weil methods, there is a $2n-1$ form η on $S(E)$ such that

$$p^* e(\Omega) = -d\eta, \quad \eta = \frac{1}{(-2\pi)^n} \int_0^1 \mathrm{Pf}(\widetilde{\omega} - p^*\omega, \Omega_t, \ldots, \Omega_t)\,\mathrm{d}t,$$

where Ω_t is the curvature matrix of the connection $p^*D + t(\widetilde{D} - p^*D)$. Restricting η to each fibre of $S(E) \to M$ is the volume form of the fibre. For the computation of $\mathrm{Pf}(\widetilde{\omega} - p^*\omega, \Omega_t, \ldots, \Omega_t)$, see [7, p. 297].

When $E = TM$ is the tangent bundle of a Riemannian manifold M, the form η is the same as Chern obtained in [3].

Let $\varrho\colon [0, +\infty) \to R$ be a smooth function, $\varrho(r) = -1$ for $r \in [0,1]$, $\varrho(r) = 0$ for $r \geqslant 2$. Then the $2n$-form $\Phi = d(\varrho(|e|)\tau^*\eta)$ is a Thom form on E, where $|e|$ is the norm of $e \in E$ and $\tau\colon E - M \to S(E)$ is the projection, $e \in E - M$, $\tau(e) = e/|e|$.

For a proof, see [1, p. 132, Proposition 12.3]. For the construction of Thom form, see also Mathai and Quillen [9].

Similarly, for the complex vector bundle $\pi\colon E_{\mathbb{C}} \to M$ defined in § 2, we have

Theorem 3.1. *For any $i = 1, \ldots, n = \operatorname{rank} E_{\mathbb{C}}$, there is a $2i-1$ form η_i on $V(E_{\mathbb{C}}, n-i+1)$ such that $\pi^*_{n-i+1} c_i(\Omega_{E_{\mathbb{C}}}) = -\mathrm{d}\eta_i$.*

The theorem follows from Proposition 2.3 and the result on the Euler classes. Using the transgression form η_i, we can construct a Thom form Φ_i for the vector bundle F_i.

Let $s_1, \ldots, s_{n-i+1}$ be sections of the Hermite bundle $E_{\mathbb{C}}$ which are linearly independent on $M - Z$. Assume that Z is a set of submanifolds of M. From these sections, we have a section $\tilde{s}\colon M - Z \to V(E_{\mathbb{C}}, n-i+1)$. Then on the subset $M - Z$, we have

$$c_i(\Omega_{E_{\mathbb{C}}}) = -\mathrm{d}(\tilde{s}^*\eta_i).$$

Let U_ε be an ε-neighborhood of Z in M. For any closed $m - 2i$ form ξ on M, $m = \dim M$, we have

$$\int_M c_i(\Omega_{E_{\mathbb{C}}}) \wedge \xi = \lim_{\varepsilon \to 0} \int_{\partial U_\varepsilon} \tilde{s}^*\eta_i \wedge \xi.$$

The left-hand side of this equation is independent of the choice of the sections of the bundle $E_{\mathbb{C}}$. This equation is useful for our understanding the relationship between the characteristic classes and the sections of the vector bundles as we know for the Euler classes.

Theorem 3.2. *Let $s_1, \ldots, s_{n-i_1+1}$ be sections on $E_{\mathbb{C}}$ which are linearly independent on $M - Z$, where Z is a subset of M. If there is a nonzero Chern number $a = \int_M c_{i_1}(\Omega_{E_{\mathbb{C}}}) \ldots c_{i_k}(\Omega_{E_{\mathbb{C}}})$, $i_2 \geqslant \ldots \geqslant i_k$, $k > 1$, then the set Z cannot be contained in any submanifold of M with dimension less than $2i_2$.*

A similar result holds on real vector bundles.

Proof. If Z is contained in a submanifold N of dimension less then $2i_2$, let $U_1 \subset U_2$ be open neighborhoods of N such that N is a deformation retract of U_2. Then on U_2, we have $E_{\mathbb{C}}|_{U_2} = E_1 \oplus E_2$ and E_1 is trivial with rank $> n - i_2$. Then we can construct a connection $D_{\mathbb{C}}$ on $E_{\mathbb{C}}$ such that $c_{i_2}(\Omega_{\mathbb{C}})|_{U_1} = 0$, where $\Omega_{\mathbb{C}}$ is the curvature matrix of $D_{\mathbb{C}}$. Hence

$$\int_M c_{i_1}(\Omega_{E_{\mathbb{C}}}) \ldots c_{i_k}(\Omega_{E_{\mathbb{C}}}) = \lim_{\varepsilon \to 0} \int_{\partial U_\varepsilon} \tilde{s}^*\eta_{i_1} \wedge c_{i_2}(\Omega_{E_{\mathbb{C}}}) \ldots c_{i_k}(\Omega_{E_{\mathbb{C}}}) = 0,$$

contradicting to the fact that $a \neq 0$. □

For example, let $\mathbb{C}P^n$ be the complex projective space and $T_c\mathbb{C}P^n$ its holomorphic tangent space. It was proved in [10] that all Chern numbers of $T_c\mathbb{C}P^n$ are nonzero. Let $i_1 = n-k$, $i_2 = k$ in this case. Then for any submanifold N of $\mathbb{C}P^n$, $\dim_{\mathbb{R}} N < 2k$, there do not exist vector fields $X_1, \dots, X_{k+1} \in \Gamma(T_c\mathbb{C}P^n)$ which are linearly independent on $\mathbb{C}P^n - N$.

In the following we give a geometric proof of the fact that the Chern classes can be represented by some submanifolds of the base manifold of the bundle which is the Poincaré dual.

As in [6, Chapter 3], let $\sigma = \{s_1, \dots, s_n\}$ be a set of C^∞ sections of the bundle $E_{\mathbb{C}}$ and let the degeneracy set $D_i(\sigma)$ be defined by

$$D_i = D_i(\sigma) = \{p \in M\colon\ s_1(p) \wedge \ldots \wedge s_i(p) = 0\}, \quad i = 1, \dots, n.$$

We say that σ is generic if, for each i, s_{i+1} intersects the subspace of $E_{\mathbb{C}}$ spanned by $s_1, \dots, s_i$ transversally, so that D_{i+1} is, away from D_i, a submanifold of dimension $m-2n+2i$. Thus sections $s_1, \dots, s_{i+1}$ are linearly independent everywhere if $m+2i < 2n$. We can give each $N_{i+1} = D_{i+1} - D_i$ an orientation defined naturally. Then D_{i+1} represents a cycle in homology, called the degeneracy cycle of the sections σ. In a neighborhood of a point $p \in N_{i+1}$, complete the sections $e_1 = s_1, \dots, e_i = s_i$ to a frame for $E_{\mathbb{C}}$, and write $s_{i+1} = \sum\limits_j f_j e_j$, $f_j = f_{j,1} + \sqrt{-1} f_{j,2}$, where $f_{j,1}$, $f_{j,2}$ are real functions. N_{i+1} is then locally given by $f_{i+1} = \ldots = f_n = 0$. Let Ψ_{i+1} be the orientation on N_{i+1} near p such that the form

$$\Psi_{i+1} \wedge \mathrm{d}f_{i+1,1} \wedge \mathrm{d}f_{i+1,2} \wedge \ldots \wedge \mathrm{d}f_{n,1} \wedge \mathrm{d}f_{n,2}$$

is positive for the given orientation on M. Note that the set N_1 is discrete when $\dim M = 2n$ and $\Psi_1 = \pm 1$ in this case.

Theorem 3.3. *For $k = 1, \dots, n$, the Chern classes $c_k(E_{\mathbb{C}})$ are the Poincaré duals to the cycles D_{n-k+1}. Thus $c_k(E_{\mathbb{C}}) = 0$, if $D_{n-k+1} = D_{n-k}$ or D_{n-k+1} is a boundary.*

P r o o f. The theorem has been proved in [6] by using the Grassmann manifolds, see also [1, p. 134]. In the following we give a direct proof.

For any $k \geqslant 1$, we have $c_k(F_k) = \pi_{n-k}^* c_k(E_{\mathbb{C}})$. Thus there is a $2k-1$ form η_k on $V(E_{\mathbb{C}}, n-k+1)$ such that

$$\alpha_{n-k+1}^* c_k(\Omega_{F_k}) = -d\eta_k.$$

The restriction of η_k to each fibre of $\alpha_{n-k+1}\colon\ V(E_{\mathbb{C}}, n-k+1) \to V(E_{\mathbb{C}}, n-k)$ is the volume form. By Gram-Schmidt process, from the sections $s_1, \dots, s_{n-k+1}$, we have a section $\widetilde{\sigma}_{n-k+1}\colon M - D_{n-k+1}(\sigma) \to V(E_{\mathbb{C}}, n-k+1)$, where $\widetilde{\sigma}_{n-k+1} = \{\widetilde{s}_1, \ldots, \widetilde{s}_{n-k+1}\}$ is a Hermite frame field on $M - D_{n-k+1}(\sigma)$. Let $U_i(\varepsilon)$ be ε-neighborhoods of D_i respectively, $i = 1, \ldots, n$, $U_i(\varepsilon) \subset U_j(\varepsilon)$ if $i < j$. Then for any closed $m-2k$ form ξ, we have

$$\int_M c_k(\Omega_{E_{\mathbb{C}}}) \wedge \xi = \lim_{\varepsilon \to 0} \int_{\partial U_{n-k+1}(\varepsilon)} \widetilde{\sigma}^*_{n-k+1} \eta_k \wedge \xi.$$

The space $\partial U_i(\varepsilon) - \partial U_{i-1}(\varepsilon)$ can be viewed as a fibre bundle over N_i with fibre $S^{2n-2i+1}$. The normal bundle $v(N_{i+1})$ of N_{i+1} in M is oriented and the orientation is given by that of M and N_{i+1}. Let $x = (x_1, \ldots, x_{m-2k}, x_{m-2k+1}, \ldots, x_m)$ be oriented coordinates in a neighborhood of $p \in N_{n-k+1}$ in M such that, restricting on N_{n-k+1}, $(x_1, \ldots, x_{m-2k})$ are oriented coordinates on N_{n-k+1}. By

$$\begin{aligned} &\Psi_{n-k+1} \wedge \mathrm{d}f_{n-k+1,1} \wedge \mathrm{d}f_{n-k+1,2} \wedge \ldots \wedge \mathrm{d}f_{n,1} \wedge \mathrm{d}f_{n,2} \\ &= \frac{\partial(f_{n-k+1,1}, f_{n-k+1,2}, \ldots, f_{n,1}, f_{n,2})}{\partial(x_{m-2k+1}, \ldots, x_m)} \Psi_{n-k+1} \wedge \mathrm{d}x_{m-2k+1} \wedge \ldots \wedge \mathrm{d}x_m, \end{aligned}$$

we have $\partial(f_{n-k+1,1}, f_{n-k+1,2}, \ldots, f_{n,1}, f_{n,2})/\partial(x_{m-2k+1}, \ldots, x_m) > 0$. As noted above, integrating along the fibres of the map $\partial U_{n-k+1}(\varepsilon) - \partial U_{n-k}(\varepsilon) \to N_{n-k+1}$ yields

$$\lim_{\varepsilon \to 0} \int_{\partial U_{n-k+1}(\varepsilon) - \partial U_{n-k}(\varepsilon)} \widetilde{\sigma}^*_{n-k+1} \eta_k \wedge \xi = \int_{N_{n-k+1}} \xi.$$

As $2n - 2i + 1 > 2k - 1$ when $i \leqslant n - k$, we have

$$\lim_{\varepsilon \to 0} \int_{\partial U_{n-k}(\varepsilon)} \widetilde{\sigma}^*_{n-k+1} \eta_k \wedge \xi = 0.$$

Hence

$$\int_M c_k(\Omega_{E_{\mathbb{C}}}) \wedge \xi = \int_{D_{n-k+1}} \xi.$$

This completes the proof of the theorem. □

Corollary 3.4. *If N_{n-k+1} is a closed submanifold of M, i.e. $\overline{N_{n-k+1}} = N_{n-k+1}$, then N_{n-k+1} is the Poincaré dual of the Chern class $c_k(E_{\mathbb{C}})$. Furthermore, if $\dim_{\mathbb{R}} M = 2n$, there is an oriented real vector bundle $F_{\mathbb{R}}$ over N_{n-k+1} such that*

$$\int_M c_k(\Omega_{E_{\mathbb{C}}}) \wedge c_{n-k}(\Omega_{E_{\mathbb{C}}}) = \int_{N_{n-k+1}} e(\Omega_{F_{\mathbb{R}}}),$$

where $\Omega_{F_{\mathbb{R}}}$ is the curvature of $F_{\mathbb{R}}$ with respect to some connection on $F_{\mathbb{R}}$.

P r o o f. By Theorem 3.3, we have

$$\int_M c_k(\Omega_{E_{\mathbb{C}}}) \wedge c_{n-k}(\Omega_{E_{\mathbb{C}}}) = \int_{N_{n-k+1}} p^* c_{n-k}(\Omega_{E_{\mathbb{C}}}),$$

where $p\colon N_{n-k+1} \to M$ is the inclusion, $\dim N_{n-k+1} = 2n - 2k$. By dimensional reason, the pull-back bundle $p^*E_{\mathbb{C}}$ can be decomposed as $p^*E_{\mathbb{C}} \cong F_{\mathbb{C}} \oplus \mathcal{E}_k$ on N_{n-k+1}, where $\mathcal{E}_k$ is a trivial bundle. Hence

$$p^* c_{n-k}(E_{\mathbb{C}}) = c_{n-k}(F_{\mathbb{C}}) = e(F_{\mathbb{R}}).$$

□

Remark. On the other hand, by the assumption of Corollary 3.4, there is a natural decomposition $p^*E_{\mathbb{C}} = \widetilde{F}_{\mathbb{C}} \oplus \mathcal{E}_{n-k}$, where $\mathcal{E}_{n-k}$ is a trivial bundle generated by the sections $s_1, \ldots, s_{n-k}$ restricted on N_{n-k+1}. If $k < \frac{1}{2}n$, we have

$$p^* c_{n-k}(E_{\mathbb{C}}) = c_{n-k}(\widetilde{F}_{\mathbb{C}}) = 0.$$

Thus, if $\int_M c_k(\Omega_{E_{\mathbb{C}}}) \wedge c_{n-k}(\Omega_{E_{\mathbb{C}}}) \neq 0$, then there do not exist generic sections on the vector bundle $E_{\mathbb{C}}$ such that N_{n-k+1} is a closed submanifold of M.

Notice that when $\dim M = 2n$, $\int_M c_n(E_{\mathbb{C}})$ is the intersection number of $s_1(M)$ with M in $TE_{\mathbb{C}}$, where the orientation on the fibres of $E_{\mathbb{C}}$ are determined by the complex structure.

Theorem 3.5. *Assuming* $\dim M > 2n$, *let* $i\colon S \to M$ *be an embedding which intersects transversally with* N_1, *where* S *is a* $2n$ *dimensional oriented submanifold. Then* $\int_S c_n(i^*E_{\mathbb{C}})$ *is the intersection number of* S *with* N_1.

P r o o f. It is easy to see that the section s_1 of the vector bundle $E_{\mathbb{C}}$ pulls back to a section i^*s_1 of the bundle $i^*E_{\mathbb{C}} \to S$. The zeros of the section i^*s_1 correspond exactly to the points of intersection of S with N_1. If p is a point in $S \cap N_1$, we have

$$T_pM = T_pN_1 \oplus v_p(N_1) = T_pN_1 \oplus T_pS,$$

where $v_p(N_1)$ is the normal space of N_1 at p. Furthermore, the tangent map of $s_1\colon S \to E_{\mathbb{C}}$ at p is an isomorphism of T_pS to the fibre of $E_{\mathbb{C}}$ at p. The tangent map $s_{1*p}\colon T_pS \to E_{\mathbb{C}}|_p$ preserves the orientation if and only if the orientation of $T_pS \otimes T_pN_1$ defined by those of T_pS and T_pN_1 is the same as that of T_pM, see the proof of Theorem 3.3. This completes the proof of the theorem. □

As $p_k(E) = (-1)^k c_{2k}(E \otimes \mathbb{C})$, the Pontrjagin classes are Poincaré duals to some cycles on the base manifold by Theorem 3.3. For example, let TM be the tangent bundle of a $4k$-Riemannian manifold. Then there is a decomposition: $TM \otimes \mathbb{C} = F_{2k} \oplus \mathcal{E}_{2k}$ and

$$p_k(TM) = (-1)^k c_{2k}(TM \otimes \mathbb{C}) = (-1)^k c_{2k}(F_{2k}) = (-1)^k e(F_{2k\mathbb{R}}).$$

Hence the Poincaré dual of the Pontrjagin class $p_k(TM)$ is that of the Euler class $(-1)^k e(F_{2k\mathbb{R}})$. In what follows we give some further study.

Let $\sigma = \{s_1, \dots, s_n\}$ be a set of sections of a real vector bundle E with the degeneracy set $D_i = D_i(\sigma)$ defined as in the complex case. We call σ generic if the sections σ satisfy the similar conditions. Denote $N_{i+1} = D_{i+1} - D_i$, $\dim N_{i+1} = m - n + i$, where $m = \dim M$.

Theorem 3.6. *Let $\pi \colon E \to M$ be an oriented Riemannian vector bundle and σ be a set of generic sections defined as above. If N_{n-2k+1} is a closed submanifold of M, the Poincaré dual of the Pontrjagin class $p_k(E)$ can be represented by the Poincaré dual of the Euler class of the normal bundle $v(N_{n-2k+1})$ of N_{n-2k+1} in M. Thus $p_k(E) = 0$ if the bundle $v(N_{n-2k+1})$ has a nowhere vanishing section or the cycle D_{n-2k+1} is a boundary.*

P r o o f. For each k, we have $E|_{M-D_{n-2k}} = \hat{F}_{2k} \oplus \mathcal{E}_{n-2k}$, where $\mathcal{E}_{n-2k}$ is generated by the sections $s_1, \dots, s_{2k}$. The bundle $\hat{F}_{2k}$ is oriented and s_{n-2k+1} is a transversal section of this bundle. In Proposition 12.7 of [1], Bott and Tu proved that the vector bundle $\hat{F}_{2k} \to N_{n-2k+1}$ is isomorphic to the normal bundle of N_{n-2k+1} in M. This also shows that the submanifold N_{n-2k+1} is oriented.

By Proposition 2.4, we have $\pi^*_{n-2k} p_k(E) = [e(\widetilde{F}_{2k})]^2$. There is a $2k-1$ form η_k on $V(E, n-2k+1)$ such that $\alpha^*_{n-2k+1} e(\Omega_{2k}) = -\mathrm{d}\eta_k$, where Ω_{2k} is the curvature matrix of the naturally defined connection on the bundle $\widetilde{F}_{2k}$. From $s_1, \dots, s_{n-2k+1}$, we have a section $\widetilde{\sigma}$ of $\pi_{n-2k+1} \colon V(E, n-2k+1) \to M$ on $M - D_{n-2k+1}$. It is easy to see that, on $M - D_{n-2k+1}$,

$$p_k(E) = \widetilde{\sigma}^* \pi^*_{n-2k+1} p_k(E) = -\mathrm{d}\eta_k \wedge \widetilde{\sigma}^* \alpha^*_{n-2k+1} e(\widetilde{F}_{2k}).$$

On the submanifold N_{n-2k+1}, we have

$$\widetilde{\sigma}^* \alpha^*_{n-2k+1} e(\widetilde{F}_{2k}) = (\alpha_{n-2k+1} \circ \widetilde{\sigma})^* e(\widetilde{F}_{2k}) = e(\hat{F}_{2k}) = e(v(N_{n-2k+1})).$$

The rest of the proof is similar to that of Theorem 3.3. Let $U_{n-2k+1}(\varepsilon)$ be a ε-neighborhood of D_{n-2k+1} in M. For any $m-4k$ form ξ on M, we have

$$\begin{aligned}\int_M p_k(\Omega_E)\wedge\xi &= -\lim_{\varepsilon\to 0}\int_{M-U_{n-2k+1}(\varepsilon)} \mathrm{d}\eta_k\wedge\widetilde{\sigma}^*\alpha^*_{n-2k+1}e(\Omega_{2k})\wedge\xi\\ &= \int_{N_{n-2k+1}}\widetilde{\sigma}^*\alpha^*_{n-2k+1}e(\Omega_{2k})\wedge\xi,\end{aligned}$$

where $\widetilde{\sigma}^*\alpha^*_{n-2k+1}e(\Omega_{2k})$ is the Euler form of the vector bundle $\hat{F}_{2k}\to N_{n-2k+1}$. If the submanifold N_{n-2k+1} is closed, then $\partial N_{n-2k+1}=\emptyset$ and

$$\int_{N_{n-2k+1}}\widetilde{\sigma}^*\alpha^*_{n-2k+1}e(\Omega_{2k})\wedge\xi=\int_{N_{n-2k+1}}e(\widetilde{\Omega}_{2k})\wedge\xi,$$

where $\widetilde{\Omega}_{2k}$ is the curvature matrix of the normal bundle $v(N_{n-2k+1})$. The theorem is proved. □

In Proposition 12.8 of [1], Bott and Tu proved this kind theorem for the Euler class.

Lemma 3.7. *If* $\dim M=\operatorname{rank}E=4k$, *we can choose a set of generic sections such that* $\overline{N_i}\cap D_{i-1}=\emptyset$ *for each* $i\leqslant 2k+1$. *Thus* N_i *are closed submanifolds of* M *for* $i\leqslant 2k+1$.

Proof. We prove the lemma by induction. Assume we have chosen generic sections $s_1,\dots,s_{2k}$ such that the lemma holds for each $i\leqslant 2k$. Thus the sets N_i are all closed submanifolds of M and $N_i\cap N_j=\emptyset$ for any $i\neq j\leqslant 2k$. On $M-\bigcup N_i$, E can be decomposed as $E|_{M-\bigcup N_i}=\hat{F}_{2k}\oplus\mathcal{E}_{2k}$, where $\mathcal{E}_{2k}$ is generated by $s_1,\dots,s_{2k}$. Let U_i be a neighborhood of N_i such that N_i is a deformation retract of U_i. Since $\dim N_i=i-1<2k$, the bundle $\hat{F}_{2k}\to U_i-N_i$ has a nowhere vanishing section. Thus we can construct a section $\tilde{s}$ on $E|_{U_i}$ such that $\tilde{s}$ is nowhere zero on U_i-N_i and $\tilde{s}|_{N_i}=0$. By the partition of unity we have a section $\tilde{s}$ on E such that $s_1,\dots,s_{2k},\tilde{s}$ are linearly independent on each U_i-N_i. With a perturbation of $\tilde{s}$ on $M-\bigcup U_i$, we can get the desired section s_{2k+1}, cf. [1, p. 123]. □

Corollary 3.8. *Let* M *be an oriented manifold of dimension* $4k$ *and* $N=N_{2k+1}$ *be a closed submanifold defined as in Lemma* 3.7 *for an oriented vector bundle* E *of rank* $4k$. *Then we have*

$$\int_M p_k(E)=\chi(v(N)),$$

where $\chi(v(N))$ *is the Euler characteristic of the normal bundle* $v(N)$.

Let TM be the tangent bundle of a 4-dimensional oriented manifold. Then $\int_M \frac{1}{3}p_1(M) = \frac{1}{3}\chi(v(N))$ is the signature of the manifold M. As we know, the signature of 4-manifold M is a multiple of 8. Furthermore, if M is spin, $\mathrm{sig}(M)$ is a multiple of 16, see for example [8, p. 280]. Thus $\chi(v(N))$ is a multiple of 24 or 48 respectively.

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