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SUBDIRECT DECOMPOSITIONS AND THE RADICAL OF A GENERALIZED BOOLEAN ALGEBRA EXTENSION OF A LATTICE ORDERED GROUP

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Abstract. The extension of a lattice ordered group A by a generalized Boolean algebra B will be denoted by A_B. In this paper we apply subdirect decompositions of A_B for dealing with a question proposed by Conrad and Darnel. Further, in the case when A is linearly ordered we investigate (i) the completely subdirect decompositions of A_B and those of B, and (ii) the values of elements of A_B and the radical $R(A_B)$.

Keywords: lattice ordered group, generalized Boolean algebra, extension, vector lattice, subdirect decomposition, value, radical

MSC 2000: 06F15, 06F20

1. Introduction

To each pair (A,B), where A is a lattice ordered group and B is a generalized Boolean algebra, there corresponds a lattice ordered group A_B (cf. Conrad and Darnel [3]); it is called a generalized Boolean algebra extension of A.

In [3], a series of results on A_B was proved. The relations between some properties of A_B and of B were investigated in the author's paper [10].

Let us remark that if $A = Z$ (the additive group of all integers with the natural linear order) then A_B is a Specker lattice ordered group (cf. Conrad and Darnel [4] and the author [7]). Further, if $A = R$ (the additive group of all reals with the natural linear order) then A_B is a Carathéodory vector lattice (cf. Gofman [5], and the author [6], [8], [9]).

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In [3] it was proved that if A is a vector lattice then A_B is a vector lattice as well; the following open question was proposed:

(Q) If A_B is a vector lattice, then is A a vector lattice?

In Section 3 we prove that the answer to this question is 'Yes'.

In the remaining part of the paper we assume that A is a linearly ordered group. In [10] it was shown that each direct product decomposition of A_B is finite (in the sense that it has only a finite number of nonzero direct factors) and that there is a one-to-one correspondence between internal direct product decompositions of A_B and finite internal direct product decompositions of B. We remark that internal direct product decompositions of B need not be finite.

The notion of completely subdirect decomposition of a lattice ordered group was introduced by Šik [11]. Analogously we can define this notion for generalized Boolean algebras.

In Section 4 we show that the result of [9] concerning completely subdirect decompositions of Carathéodory vector lattices remains valid for the lattice ordered group A_B; namely, we prove that there is a one-to-one correspondence between internal completely subdirect decompositions of A_B and those of B. We denote by $S(A_B)$ the system of all internal completely subdirect decompositions of A_B and we define in a natural way a binary relation $\leqslant$ on the system $S(A_B)$. We prove that under the relation $\leqslant$, $S(A_B)$ turns out to be a meet semilattice. If for each $b \in B$, the interval $[0, b]$ of B is a complete lattice, then $S(A_B)$ is a lattice.

In Section 5 we investigate the values of elements of A_B and the radical $R(A_B)$. We prove that $R(A_B)$ is determined by the set B_1 of all atoms of B.

2. Preliminaries

For lattice ordered groups we use the notation as in Birkhoff [1] and Conrad [2].

The symbol 0 can denote the zero real, the neutral element of a lattice ordered group or the least element of a generalized Boolean algebra; the meaning of this symbol will be clear from the context.

The generalized Boolean algebra is defined to be a lattice B with the least element 0 such that for each $b \in B$, the interval $[0, b]$ of B is a Boolean algebra. We always assume that B has more than one element.

We recall some notions and the notation from [3] concerning the generalized Boolean algebra extension of a latice ordered group.

We denote by Λ the set of all maximal proper filters of B. If $b \in B$, then b will be identified with the set $\Lambda(b)$ of all $\lambda \in \Lambda$ such that $b \in \lambda$.

Let A be a lattice ordered group, $A \neq \{0\}$. Consider the direct product $G_0 = \prod_{\lambda\in\Lambda} A_\lambda$, where $A_\lambda = A$ for each $\lambda \in \Lambda$. For $a \in A$ and $b \in B$ we denote by $a[b]$ the element of G_0 such that

$$a[b](\lambda) = \begin{cases} a & \text{if } \lambda \in b, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by A_B the set of all $g \in G_0$ such that either $g = 0$ or $g \neq 0$ and g can be expressed in the form

$$g = a_1[c_1] + \ldots + a_n[c_n], \tag{1}$$

where $a_1, \ldots, a_n$ are nonzero elements of A and $c_1, \ldots, c_n$ are nonzero elements of B such that $c_{i(1)} \wedge c_{i(2)} = 0$ whenever $i(1), i(2)$ are distinct elements of the set $\{1, 2, , \ldots, n\}$. Then (1) is said to be a Specker representation of g.

If, moreover, $a_{i(1)} \neq a_{i(2)}$ whenever $i(1), i(2) \in \{1, 2, \ldots, n\}$ and $i(1) \neq i(2)$, then (1) is called a standard Specker representation of g. Each nonzero element of g has a uniquely determined standard Specker representation. A_B is an ℓ-subgroup of the lattice ordered group G_0.

Let G be a lattice ordered group. In view of the definition from [1], Chapter XV, G is a vector lattice if the multiplication by scalars (= reals) in G is possible, conforming to the usual rules of vector algebra, and also the rule that, for each $r \in R$, $r \to rx$ preserves the order if $r > 0$, and inverts it if $r < 0$.

By considering a vector lattice X, the multiplication of elements of X by reals is assumed to be fixed.

Sometimes it will be convenient to distinguish between the lattice ordered group G (where the multiplication by reals is not taken into account) and the corresponding vector lattice, if it exists; in such case, this latter will be denoted by $V(G)$.

3. On the question (Q)

For the notion of a subdirect decomposition of an algebraic structure, cf., e.g., [1], Chapter VI.

Let A_B be as in Section 2.

Lemma 3.1. *A_B is a subdirect product of the indexed system $(A_\lambda)_{\lambda\in\Lambda}$.*

Proof. In view of the definition, A_B is an ℓ-subgroup of the direct product $\prod_{\lambda\in\Lambda} A_\lambda$.

Let $\lambda \in \Lambda$ and $a \in A_\lambda$. There exists $b \in B$ with $\lambda \in b$. Then $a[b]$ belongs to A_B and $(a[b])(\lambda) = a$. This completes the proof. □

Lemma 3.2. *Let G be a lattice ordered group such that the vector lattice $V(G)$ exists. Let X be an ℓ-ideal of G. Then for each $r \in R$ and each $x \in X$, the element rx belongs to X.*

P r o o f. It suffices to consider the case when $r \neq 0$ and $x \neq 0$.

a) First suppose that $x > 0$ and $r > 0$. There exists a positive integer n with $n > r$. Then we have $0 < rx < nx$. Since $nx \in X$, we obtain $rx \in X$.

b) Let $x > 0$ and $r < 0$. Then in view of a), the element $(-r)x = -(rx)$ belongs to X, whence $rx \in X$.

c) Let $x \in X$ and $r \in R$. We have $x = x^+ - x^-$, $x^+ \geqslant 0$, $x^- \geqslant 0$, thus in view of a) and b) we get $rx^+ \in X$, $rx^- \in X$; then $rx \in X$. □

Lemma 3.3. *Let G and $V(G)$ be as in* 3.2. *Let ϱ be a congruence relation on G. Then ϱ is a congruence relation on $V(G)$.*

P r o o f. There exists an ℓ-ideal X of G such that for any $x, y \in G$ we have $x\varrho y$ if and only if $x - y \in X$. For verifying that ϱ is a congruence relation on $V(G)$ it suffices to show that if $x_1, x_2 \in G$ and $x_1\varrho x_2$, then $rx_1\varrho rx_2$ for each $r \in R$.

The relation $x_1\varrho x_2$ yields $x_1 - x_2 \in X$; in view of 3.2 we get $r(x_1 - x_2) \in X$ and thus $rx_1\varrho rx_2$. □

Corollary 3.4. *Let G and $V(G)$ be as in* 3.2. *Then the system of all congruence relations on G coincides with the system of all congruence relations on $V(G)$.*

Lemma 3.5. *Let G and $V(G)$ be as in* 3.2. *Let G be a congruence relation on G. Put $\overline{G} = G/\varrho$. Then the vector lattice $\overline{G} = G/\varrho$ exists.*

P r o o f. Let $y \in \overline{G}$. There exists $x \in G$ such $y = \bar{x}$, where $\bar{x} = \{x_1 \in G\colon\ x_1\varrho x\}$. Let $r \in R$. We put $r\bar{x} = \overline{rx}$; then in view of 3.2 and 3.3, the mapping $\bar{x} \to \overline{rx}$ is correctly defined and in this way we obviously obtain a vector lattice $V(\overline{G})$. □

Proposition 3.6. *Let $A \neq \{0\}$ be a lattice ordered group. Further, let $B \neq \{0\}$ be a generalized Boolean algebra. Assume that $G = A_B$ is a vector lattice. Then A is a vector lattice as well.*

P r o o f. In view of 3.1, G is a subdirect product of the indexed system $(A_\lambda)_{\lambda\in\Lambda}$. Let $\lambda_0 \in \Lambda$ be fixed. In view of the well-known relation between subdirect decompositions and congruence relations (cf., e.g., [1], Chapter VI) we conclude that there exists a congruence relation ϱ_0 on G such that A_{λ_0} is isomorphic to G/ϱ_0. Then according to 3.5, A_{λ_0} is a vector lattice. Since $A_{\lambda_0} \simeq A$, we obtain that A is a vector lattice as well. □

Let Y be a nonempty subset of a vector lattice X. Assume that (i) Y is an ℓ-subgroup of the lattice ordered group X, and (ii) whenever $r \in R$ and $y \in Y$, then $ry \in Y$. We call Y a vector sublattice of X.

If G_i $(i \in I)$ are vector lattices and $G_0 = \prod_{i \in I} G_i$ then since the corresponding operations in G_0 are performed component-wise, for each $r \in R$ and each $g = (g_i)_{i \in I} \in G_0$ we have

$$rg = (rg_i)_{i \in I}; \tag{1}$$

thus G_0 is a vector lattice.

If A is a vector lattice and A_B is as above, then we consider $G = A_B$ as a vector sublattice of G_0 with $G_i = A$ for each $i \in I$. Thus according to the definition of $a[b]$ (where $a \in A$ and $b \in B$) and in view of (1), for each $r \in R$ we get

$$r(a[b]) = (ra)[b]. \tag{$*$}$$

Let G_1 be a lattice ordered group and suppose that X is a vector lattice which has the following properties:

(i) G_1 is an ℓ-subgroup of the lattice ordered group X;

(ii) whenever X_1 is a lattice ordered group such that G_1 is an ℓ-subgroup of X_1 and X_1 is an ℓ-subgroup of X with $X_1 \subset X$, then X_1 fails to be a vector sublattice of X.

Under these assumptions we say that X is a minimal vector lattice over G_1.

Again, let A and B be as above; denote $G = A_B$. Let b be a fixed element of B and

$$A_b = \{a[b]\colon\ a \in A\}.$$

Then A_b is an ℓ-subgroup of G; moreover, the mapping $a \to a[b]$ is an isomorphism of A onto A_b.

Proposition 3.7. *Let $A \neq \{0\}$ be a lattice ordered group and let $B \neq \{0\}$ be a generalized Boolean algebra. Suppose that $\bar{A}$ is a minimal vector lattice over A. Put $G = A_B$ and $\overline{G} = \bar{A}_B$. Then $\overline{G}$ is a minimal vector lattice over G.*

Proof. Since $\bar{A}$ is a vector lattice, in view of [3] we obtain that $\overline{G}$ is a vector lattice as well. Further, because A is an ℓ-subgroup of $\bar{A}$ we conclude that G is an ℓ-subgroup of $\overline{G}$.

Let X_1 be an ℓ-subgroup of $\overline{G}$ such that $G \subseteq X_1 \subset \overline{G}$. Then in view of the definition of $\overline{G}$ there exist $\bar{a} \in \bar{A}$ and $b \in B$ such that $\bar{a}[b] \notin X_1$.

In view of the above mentioned isomorphism between A and A_b, and according to the analogous isomorphism between $\bar{A}$ and $\bar{A}_b$ we obtain that $\bar{A}_b$ is a minimal vector lattice over the lattice ordered group A_b.

We denote

$$X_2 = \bar{A}_b \cap X_1.$$

Then $\bar{a}[b] \notin X_2$, whence $A_b \subseteq X_2 \subset \bar{A}_b$. This yields that X_2 fails to be a vector sublattice of the vector lattice $\bar{A}_b$. Hence there exist $r \in R$ and $p \in X_2$ with $rp \notin X_2$.

Since $p \in \bar{A}_b$ it must have the form $p = \bar{a}_1[b]$ for some $\bar{a}_1 \in \bar{A}_b$. In view of $(*)$ (applied for $\bar{A}_b$) we obtain $rp = r(\bar{a}[b]) = (r\bar{a})[b]$, whence $rp \in \bar{A}_b$. If $rp \in X_1$ then we obtain $rp \in X_2$, which is a contradiction. Thus $rp \notin X_1$. Since $p \in X_1$ we conclude that X_1 fails to be a vector sublattice of $\overline{G}$. Thus $\overline{G}$ is a minimal vector lattice over the lattice ordered group G. □

In connection with 3.7, cf. also the question proposed on p. 306 of [3], where the term 'vector hull of a lattice ordered group' has been used.

4. Completely subdirect products

Assume that a lattice ordered group G is a subdirect product of an indexed system $(X_i)_{i\in I}$ of lattice ordered groups. For $g \in G$ and $i \in I$ we denote by g_i the component of g in X_i.

Suppose that for each $i \in I$ and each $x^i \in X_i$ there exists $g \in G$ such that $g_i = x^i$ and $g_j = 0$ if $j \in I$, $j \neq i$. Then we say that the mapping $\varphi\colon g \to (g_i)_{i\in I}$ is a completely subdirect decomposition of G. (Cf. [11].)

If, moreover, for each $i \in I$, X_i is an ℓ-subgroup of G and $x_i = x^i$ whenever $x \in X_i$, then we call φ an internal completely subdirect product decomposition of G. The lattice ordered groups X_i are called internal subdirect factors of G.

The analogous terminology will be applied in the particular case when φ is a direct product decomposition of G. In this case we speak about internal direct factors of G.

The case $G = \{0\}$ being trivial we will assume that $G \neq \{0\}$ and also that all internal direct (or subdirect) factors under consideration are nonzero.

The definitions of a completely subdirect decomposition and of internal completely subdirect decomposition of a Boolean algebra are analogous.

Let B be a generalized Boolean algebra and let $C(B)$ be the Carathéodory vector lattice corresponding to B. In [9], the relations between internal completely subdirect decompositions of B and those of $C(B)$ have been investigated.

Now let B be as above and let A be a linearly ordered group. In the present section we will deal with the relations between internal completely subdirect decompositions of B and those of A_B.

Lemma 4.1 (Cf. [10]). *Let X be an ℓ-subgroup of a lattice ordered group G. Then the following conditions are equivalent:*

(i) *X is an internal subdirect factor of G.*
(ii) *X is an internal direct factor of G.*

Analogously, we have

Lemma 4.2 (Cf. [10]). *Let Y be an ideal of a generalized Boolean algebra. Then the following conditions are equivalent:*

(i) *X is an internal subdirect factor of B.*
(ii) *X is an internal direct factor of B.*

Now let us suppose that $A \neq \{0\}$ is a linearly ordered group and that $B \neq \{0\}$ is a generalized Boolean algebra.

Let X be a convex ℓ-subgroup of a lattice ordered group G. It is well-known that X is an internal direct factor of G if and only if, for each $0 \leqslant g \in G$, the set $\{0 \leqslant x \in X \colon x \leqslant g\}$ has a greatest element; if x_1 is the mentioned greatest element, then x_1 is the component of g in the internal direct factor X.

An analogous result holds for generalized Boolean algebras. By a simple calculation we obtain

Lemma 4.2.1. *Let X be an ideal of a generalized Boolean algebra B. Then X is an internal direct factor of B if and only if, for each $b \in B$, the set $\{x \in X \colon x \leqslant b\}$ has a greatest element; if x_1 is the mentioned greatest element, then x_1 is the component of b in the internal direct factor X.*

The proof will be omitted.

Lemma 4.2.2. *Let B be a generalized Boolean algebra and let $(X_i)_{i\in I}$ be a system of ideals of B which determines a completely subdirect product decomposition of B. For $b \in B$ let b_i be the component of b in X_i $(i \in I)$. Then $b = \bigvee_{i\in I} b_i$.*

P r o o f. Let $b \in B$. In view of 4.2.1 we have $b_i \leqslant b$ for each $i \in I$. Assume that $b_0 \in B$ such that $b_i \leqslant b_0$ for each $i \in I$. Then $b_i = (b_i)_i \leqslant (b_0)_i$ for each $i \in I$, whence $b \leqslant b_0$. Thus b is the supremum of the system $(b_i)_{i\in I}$. □

Let X be an internal direct factor of G. We denote by $\varphi(X)$ the set of all $b \in B$ such that there exists $a \in A$ with $a[b] \in X$.

Lemma 4.3 (Cf. [10]). *$\varphi(X)$ is an internal direct factor of B.*

Let Y be an internal direct factor of B. We denote by $\psi(Y)$ the set of all $g \in G$ such that either $g = 0$ or g has a Specker representation $g = a_1[c_1] + \ldots + a_n[c_n]$, where $c_1, \ldots, c_n \in B$.

Lemma 4.4 (Cf. [10]). *$\psi(Y)$ is an internal direct factor of A_B.*

Lemma 4.5 (Cf. [10]). *Let A, B be as above and let $G = A_B$.*

(i) *If X is an internal direct factor of G, then $\psi(\varphi(X)) = X$.*
(ii) *If Y is an internal direct factor of B, then $\varphi(\psi(Y)) = Y$.*

For each lattice ordered group G we denote by $F(G)$ the system of all internal direct factors of G. Similarly, for each generalized Boolean algebra B, let $F(B)$ be the system of all internal direct factors of B. Both $F(G)$ and $F(B)$ are partially ordered by the set-theoretical inclusion.

Again, let $G = A_B$. In view of the definitions of φ and ψ we have

$$X_1, X_2 \in F(G), \quad X_1 \leqslant X_2 \Rightarrow \varphi(X_1) \leqslant \varphi(X_2); \tag{1}$$

$$Y_1, Y_2 \in F(B), \quad Y_1 \leqslant Y_2 \Rightarrow \psi(Y_1) \leqslant \psi(X_2). \tag{1'}$$

According to (1), (1′), 4.2, 4.4 and 4.5 we obtain

Lemma 4.6. *Let A, B and G be as in* 4.5. *Then φ is an isomorphism of $F(G)$ onto $F(B)$; similarly, ψ is an isomorphism of $F(B)$ onto $F(G)$.*

Let $\{X_i\}_{i\in I}$ be a set of internal direct factors of a lattice ordered group G. For $g \in G$ and $i \in I$ let g_i be the component of g in X_i. If the mapping $\varphi_1\colon G \to \prod\limits_{i\in I} X_i$ (where $\varphi_1(g) = (x_i)_{i\in I}$) is an internal completely subdirect decomposition of G, then we say that the system $\alpha = \{X_i\}_{i\in I}$ determines an internal completely subdirect decomposition of G.

A similar terminology will be applied for generalized Boolean algebras.

Proposition 4.7. *Assume that $A \neq \{0\}$ is a linearly ordered group and that B is a generalized Boolean algebra. Put $G = A_B$. Let $\{X_i\}_{i\in I}$ be a set of internal direct factors of G. Then the following conditions are equivalent:*

(i) *The system $\{X_i\}_{i\in I}$ determines an internal completely subdirect decomposition of G.*
(ii) *The system $\{\varphi(X_i)\}_{i\in I}$ determines an internal completely subdirect decomposition of B.*

P r o o f. This is a consequence of 4.6 and of [10]. □

Hence there is a one-to-one correspondence between internal completely subdirect decompositions of G and those of B, where A, B and G are as in 4.7.

Under the notation as above, let $S(G)$ be the system of all internal completely subdirect product decompositions of G, and let $S(B)$ be defined analogously.

We assume that $G \neq \{0\}$ and $B \neq \{0\}$. Thus we can suppose that $S(B)$ is the set of all systems $\alpha = \{Y_i\}_{i\in I}$, where $\{Y_i\}_{i\in I}$ is a set of nonzero internal direct factors of B which determine an internal completely subdirect decomposition of B.

Let $\beta = \{Y'_j\}_{j\in J}$ be another such system. We put $\alpha \leqslant \beta$ if for each $i \in I$ there exists $j \in J$ such that $Y_i \subseteq Y'_j$.

Analogously we define the relation $\leqslant$ on the set $S(G)$.

Lemma 4.8. *The relation $\leqslant$ is a partial order on $S(B)$.*

Proof. It is obvious that the relation $\leqslant$ is reflexive and transitive. Let $\alpha, \beta \in S(B)$ such that $\alpha \leqslant \beta$ and $\beta \leqslant \alpha$. For α and β we apply the notation as above. Let $i_0 \in I$. Then there is $j(i_0) \in J$ with $Y_{i_0} \subseteq Y'_{j(i_0)}$. If $j \in J$, $j \neq j(i_0)$, then $Y'_j \cap Y'_{j(i_0)} = \{0\}$. Hence the element $j(i_0)$ is uniquely determined. Similarly, for each $j_0 \in J$ there exists a unique $i(j_0) \in I$ with $Y'_{j_0} \subseteq Y_{i(j_0)}$. Then $Y_{i_0} \subseteq Y_{i(j(i_0))}$, whence $Y_{i_0} = Y_{i(j(i_0))}$ yielding that $Y_{i_0} = Y'_{j(i_0)}$ and so the mapping $i_0 \to j(i_0)$ is a bijection. Therefore $\alpha = \beta$. □

An analogous result holds for the relation $\leqslant$ on $S(G)$.

In view of 4.7 we obtain

Lemma 4.8.1. *The partially ordered systems $S(B)$ and $S(A_B)$ are isomorphic.*

Let α and β be as above. For $b \in B$ and $i \in I$ let $b(Y_i)$ be the component of b in Y_i. The meaning of $b(Y'_j)$ is analogous. Then in view of 4.2.2 we have

$$b = \bigvee_{i\in I} b(Y_i) = \bigvee_{j\in J} b(Y'_j). \tag{1}$$

We denote by γ the system of those $Y_i \cap Y'_j$ which have more than one element. Let K be the set of all pairs (i, j) with $i \in I$, $j \in J$ such that $Y_i \cap Y'_j \in \gamma$.

Lemma 4.9. *The set K is nonempty.*

Proof. There exists $0 < b \in B$. In view of (1) we have

$$b = b \wedge \bigvee_{i\in I} b(Y_i) = \bigvee_{i\in I} (b \wedge b(Y_i)) = \bigvee_{i\in I} \bigvee_{j\in J} (b(Y'_j) \wedge b(Y_i)). \tag{2}$$

For $i \in I$ and $j \in J$, $b(Y'_j) \wedge b(Y_i) \in Y'_j \cap Y_i$. If $\gamma = \emptyset$, then $b(Y'_j) \wedge b(Y_i) = 0$ for each $i \in I$ and each $j \in J$, whence $b = 0$, which is a contradiction. □

For each $b \in B$ and each $(i,j) \in K$ we put

$$b_{ij} = b(Y_i) \wedge b(Y'_j).$$

Further, we set

$$\chi(b) = (b_{ij})_{(i,j)\in K}.$$

Lemma 4.10. *Let $b \in B$ and $b^i \in Y_i$ for each $i \in I$. Assume that $b = \bigvee\limits_{i\in I} b^i$. Then $b^i = b(Y_i)$ for each $i \in I$.*

Proof. Let $i_0 \in I$. We have

$$b^{i_0} = b^{i_0} \wedge b = b^{i_0} \wedge \Big(\bigvee_{i\in I} b(Y_i)\Big) = \bigvee_{i\in I} (b^{i_0} \wedge b(Y_i)).$$

If $i \in I$, $i \neq i_0$, then $b^{i_0} \wedge b(Y_i) = 0$, whence

$$b^{i_0} = b^{i_0} \wedge b(Y_{i_0}),$$

thus $b^{i_0} \leqslant b(Y_{i_0})$. By similar steps we prove the relation $b(Y_{i_0}) \leqslant b^{i_0}$. □

Lemma 4.11. *Let $b \in B$ and $(i,j) \in K$. Then*

$$b_{ij} = (b(Y_i))(Y'_j) = (b(Y'_j))(Y_i).$$

Proof. Put $b_i = b(Y_i)$, $b_j = b(Y'_j)$. We have

$$b_i = b_i \wedge b = b_i \wedge \Big(\bigvee_{j\in J} b_j\Big) = \bigvee_{j\in J} (b_i \wedge b_j).$$

Since $b_i \wedge b_j \in Y'_j$, in view of 4.10 (applied for the element b_i and for the subdirect decomposition β) we obtain $b_i(Y'_j) = b_i \wedge b_j$. Analogously we get $b_j(Y_i) = b_i \wedge b_j$. □

Lemma 4.12. *The mapping χ is a homomorphism of B into $\prod\limits_{(i,j)\in K} C_{ij}$, where $C_{ij} = Y_i \cap Y'_j$. Moreover, χ is a monomorphism.*

Proof. For each $i \in I$, the mapping $b \to b(Y_i)$ is a homomorphism of B into Y_i. Similarly, for each $j \in J$, the mapping $b \to b(Y'_j)$ is a homomorphism of B into Y'_j. For $(i,j) \in K$, C_{ij} is an ideal of B. According to 4.11 we conclude that the mapping $b \to b_{ij}$ is a homomorphism of B into C_{ij}. Hence χ is a homomorphism of B into $\prod\limits_{(i,j)\in K} C_{ij}$.

It remains to verify that χ is a monomorphism. Since B is a generalized Boolean algebra it suffices to show that if $b \in B$ and $\chi(b) = 0$, then $b = 0$. By way of contradiction, assume that $0 \neq b$ and $\chi(b) = 0$. Thus $b_{ij} = 0$ for each $(i,j) \in K$. According to (1) there exists $i \in I$ with $b_i > 0$. Then we have $b_i = \bigvee_{j \in J} (b_i(Y'_j))$, hence there exists $j \in J$ with $b_i(Y'_j) > 0$. Thus 4.11 yields $b_{ij} > 0$, which is a contradiction. □

Lemma 4.13. *The system $(C_{ij})_{(i,j)\in K}$ determines an internal completely subdirect decomposition of B.*

P r o o f. Let $(i,j) \in K$ and $x \in C_{ij}$. Then $x \in Y_i$, whence $x_i = x$. Further, $x \in Y'_j$, yielding $x_j = x$. Thus in view of 4.11, $x_{ij} = (x_i)_j = x_j = x$. According to 4.12, the proof is complete. □

We denote by γ the internal completely subdirect decomposition of B which is determined by the system $(C_{ij})_{(i,j)\in K}$.

Proposition 4.14. *Let α, β and γ be as above. Then in the partially ordered set $S(B)$ we have $\alpha \wedge \beta = \gamma$.*

P r o o f. Let $(i,j) \in K$. Then $C_{ij} \subseteq Y_i$ and $C_{ij} \subseteq Y'_j$, whence $\gamma \leqslant \alpha$ and $\gamma \leqslant \beta$. Let γ_1 be an element of $S(B)$ which is generated by a system $(Z_m)_{m\in M}$ of ideals of B. Assume that $\gamma_1 \leqslant \alpha$ and $\gamma_1 \leqslant \beta$. Thus for each $m \in M$ there exist $i \in I$ and $j \in J$ such that $Z_m \subseteq Y_i$ and $Z_m \subseteq Y'_j$. Then $Z_m \subseteq Y_i \cap Y'_j = C_{ij}$. We have $\{0\} \neq Z_m$, whence $C_{ij} \neq \{0\}$, thus $(i,j) \in K$. Therefore $\gamma_1 \leqslant \gamma$. This yields $\gamma = \alpha \wedge \beta$. □

Hence we obtain

Theorem 4.15. *Let B be a generalized Boolean algebra. Then the partially ordered set $S(B)$ is a meet-semilattice.*

In view of 4.15 and 4.7 we get

Theorem 4.15.1. *Let $A \neq \{0\}$ be a linearly ordered group and let $B \neq \{0\}$ be a generalized Boolean algebra. Then the partially ordered set $S(A_B)$ is a meet-semilattice.*

Let (i_1, j_1) and (i_2, j_2) be elements of K. We put $(i_1, j_1) \equiv (i_2, j_2)$ if there exist elements

$$(i^1, j^1), (i^2, j^2), \ldots, (i^n, j^n)$$

of K such that $(i^1, j^1) = (i_1, j_1)$, $(i^n, j^n) = (i_2, j_2)$ and whenever $m \in \{1, 2, \ldots, n-1\}$, then either $i^m = i^{m+1}$ or $j^m = j^{m+1}$. The relation $\equiv$ is an equivalence on the set K; let ϱ be the partition of the set K corresponding to the equivalence $\equiv$. For $(i, j) \in K$ let $\overline{(i, j)}$ be the class in ϱ containing the element (i, j).

Recall that in view of 4.13 and 4.1, for each $(i, j) \in K$ the ideal C_{ij} of B is an internal direct factor of B. Thus for each $b \in B$ there exists a uniquely determined component $b(C_{ij})$ of b in C_{ij}.

For any $(i, j) \in K$ let $D_{\overline{(i,j)}}$ be the set of all elements $b \in B$ such that $b(C_{i_1, j_1}) = 0$ whenever $(i_1, j_1) \notin \overline{(i, j)}$. Thus in view of (1) we obtain

Lemma 4.16. *Let $(i_0, j_0) \in K$ and $b \in B$. Then the following conditions are equivalent:*

(i) $b \in D_{\overline{(i_0,j_0)}}$;

(ii) $b = \bigvee\limits_{(i,j) \in \overline{(i_0,j_0)}} b(C_{ij})$.

In the remaining part of the present section we assume that the following condition is satisfied:

($*$) If $0 < b \in B$, then the interval $[0, b]$ of B is a complete lattice.

We apply the notation as above. Let $b \in B$. In view of (1) and 4.13, we have

$$b = \bigvee_{(i,j) \in K} b_{ij}.$$

Let $(i_0, j_0) \in K$. Then according to ($*$), the set $\{b_{ij}\}_{(i,j) \in \overline{(i_0,j_0)}}$ has a supremum in B; we denote it by $b_{\overline{(i_0,j_0)}}$.

Lemma 4.17. *For each $b \in B$ and each $(i_0, j_0) \in K$, $b_{\overline{(i_0,j_0)}}$ is the greatest element of the set*

$$\{x \in D_{\overline{(i_0,j_0)}} \colon x \leqslant b\}.$$

Proof. Let $b \in B$ and $(i_0, j_0) \in K$. In view of the definition of $b_{\overline{(i_0,j_0)}}$, this element belongs to the set $D_{\overline{(i_0,j_0)}}$. Let $x \in D_{\overline{(i_0,j_0)}}$, $x \leqslant b$.

From the first of the mentioned relations we obtain

$$x_{\overline{(i_0,j_0)}} = x.$$

Further, from $x \leqslant b$ we get

$$x_{\overline{(i_0,j_0)}} \leqslant b_{\overline{(i_0,j_0)}}.$$

This completes the proof. □

By applying 4.2.1 we get

Corollary 4.18. *Let $(i_0, j_0) \in K$. Then $D_{\overline{(i_0,j_0)}}$ is an internal direct factor of B. For each $b \in B$, the element $b_{\overline{(i_0,j_0)}}$ is the component of b in $D_{\overline{(i_0,j_0)}}$.*

We denote $\overline{K} = \{\overline{(i,j)} \colon (i,j) \in K\}$. For $b \in B$ we put

$$\chi_1(b) = \{b_{\bar{k}}\}_{\bar{k} \in \overline{K}}.$$

In view of 4.18, χ_1 is a homomorphism of B into $\prod\limits_{\bar{k} \in \overline{K}} D_{\bar{k}}$. Similarly as in 4.12 we can verify that χ_1 is a monomorphism. From this and from 4.17 we conclude that χ determines an internal completely subdirect decomposition of B; let us denote it by Δ.

Lemma 4.19. $\Delta = \alpha \vee \beta$.

P r o o f. Let $i_0 \in I$. There exists $j_0 \in J$ with $(i_0, j_0) \in K$. Then in view of the definition of $D_{\bar{k}}$ for $\bar{k} = \overline{(i_0, j_0)}$ we have $Y_{i_0} \subseteq D_{\bar{k}}$. Hence $\alpha \leqslant \Delta$. Similarly we have $\beta \leqslant \Delta$.

Let $\Delta_1 \in S(B)$ such that $\alpha \leqslant \Delta_1$ and $\beta \leqslant \Delta_1$. Assume that Δ_1 is determined by a system $\{E_t\}_{t \in T}$ of ideals of B. Let $i_0 \in I$. There exists $t_0 \in T$ with $Y_{i_0} \subseteq E_{t_0}$. Thus whenever $(i_0, j_0) \in K$, then $C_{i_0, j_0} \subseteq E_{t_0}$. Analogously, if $j_1 \in J$ is given and $(i_1, j_1) \in K$, then $C_{i_1, j_1} \subseteq E_{t_1}$ for some $t_1 \in T$. From this and from the definition of $D_{\bar{k}}$ for $\bar{k} \in \overline{K}$ we conclude that $D_{\bar{k}}$ is a subset of some $E_t (t \in T)$. Therefore $\Delta \leqslant \Delta_1$ and thus $\Delta = \alpha \vee \beta$. □

From 4.14, 4.19 and 4.8.1 we conclude

Theorem 4.20. *Let $A \neq \{0\}$ be a linearly ordered group and let $B \neq \{0\}$ be a generalized Boolean algebra. Suppose that the condition $(*)$ is satisfied. Then $S(A_B)$ is a lattice.*

5. The radical of A_B

In Conrad [2], there are investigated three types of radicals of a lattice ordered group G (the radical $R(G)$, the distributive radical $D(G)$ and the ideal radical $L(G)$). In the present section we deal with the radical $R(G)$ for the case when $G = A_B$, when $A \neq \{0\}$ is a linearly ordered group and B is a generalized Boolean algebra.

We recall the corresponding definitions from [2].

Let G be a lattice ordered group and $0 \neq g \in G$. A value of g is a convex ℓ-subgroup G_α of G such that G_α is maximal with respect to non-containing the element g. Put

$R_g = \bigvee G_\alpha$, where G_α runs over the system of all values of g. Further, we set

$$R(G) = \bigcap_{0 \neq g \in G} R_g.$$

Then $R(G)$ is the *radical* of G.

Again, let $0 \neq g \in G$ and let L_g be the join of all ℓ-ideals of G not containing g. Put

$$L(G) = \bigcap_{0 \neq g \in G} L_g.$$

Then $L(G)$ is the *ideal radical* of G.

A lattice ordered group is called representable if it is isomorphic to a subdirect product of linearly ordered groups.

Proposition 5.1 (Cf. [2]). *Let G be a representable lattice ordered group. Then $L(G) = R(G)$.*

Corollary 5.2. *Let $A \neq \{0\}$ be a linearly ordered group and let $B \neq \{0\}$ be a generalized Boolean algebra. Then $L(A_B) = R(A_B)$.*

Proof. In view of the definition of A_B we obtain that A_B is a subdirect product of replicas of A. Hence A_B is representable and now it suffices to apply 5.1. □

The following result is easy to verify.

Lemma 5.3. *Let G be a lattice ordered group and $g \in G$. Let X be a convex ℓ-subgroup of G. Then $g \in X$ if and only if $|g| \in X$.*

In view of 5.3 we have

$$R(G) = \bigcap_{0 < g \in G} R_g. \tag{1}$$

Lemma 5.4. *Let A and B be as in 5.2. Let $0 < g \in A_B$ and suppose that g has a Specker representation*

$$g = a_1[c_1] + \ldots + a_n[c_n].$$

Let X be a convex ℓ-subgroup of $G = A_B$. Then g belongs to X if and only if all $a_i[c_i]$ $(i = 1, 2, \ldots, n)$ belong to X.

Proof. If all $a_i[c_i]$ belong to X then in view of the Specker representation we get $g \in X$. Conversely, let $g \in X$ and $i \in \{1, 2, \ldots, n\}$. Since $0 < a_i[c_i] \leqslant g$, we obtain $a_i[c_i] \in X$. □

Lemma 5.5. *Under the assumption as in* 5.4 *we have*

$$R_g = R_{a_1[c_1]} \vee \ldots \vee R_{a_n[c_n]}.$$

P r o o f. a) Let X be a value of g. Hence $g \notin X$. Thus in view of 5.4 there is $i \in \{1, 2, \ldots, n\}$ such that $a_i[c_i] \notin X$. Then there is a value Y of $a_i[c_i]$ with $X \subseteq Y$. According to the definition of R_g and of $R_{a_i[c_i]}$ we obtain $X \subseteq R_{a_i[c_i]}$ and

$$R_g \leqslant R_{a_1[c_1]} \vee \ldots \vee R_{a_n[c_n]}.$$

b) Let $i \in \{1, 2, \ldots, n\}$ and let Y_1 be a value of $a_i[c_i]$. Hence $a_i[c_i] \notin Y_1$. In view of 5.4, $g \notin Y_1$. Then there is a value X_1 of g with $Y_1 \subseteq X_1$. This yields $R_{a_i[c_i]} \leqslant R_g$. Thus we obtain

$$R_{a_1[c_1]} \vee \ldots \vee R_{a_n[c_n]} \leqslant R_g,$$

completing the proof. $\square$

Lemma 5.6. *Let A and B be as in* 5.2*; put $G = A_B$. Then*

$$R(G) = \bigcap_{0<a\in A, 0<b\in B} R_{a[b]}.$$

P r o o f. Let $0 < a \in A$, $0 < b \in B$; then $a[b] \in G$, whence

$$R(G) \subseteq \bigcap_{0<a\in A, 0<b\in B} R_{a[b]}.$$

Assume that $x \in R_{a[b]}$ for each $0 < a \in A$ and each $0 < b \in B$. Let $0 < g \in G$. Then in view of 5.5 we have $x \in R_g$, whence $x \in R(G)$. $\square$

In view of 5.6, for characterizing $R(G)$ we have to describe the ℓ-subgroups $R_{a[b]}$ for $0 < a \in A$ and $0 < b \in B$. Since A is linearly ordered, there exists a unique value A^a of the element a in A. We denote

$$A^a_b = \{a_1[b] \colon\ a_1 \in A^a\}.$$

For each $x \in G$, let $(x)^\delta$ be the orthogonal polar of x, i.e.,

$$(x)^\delta = \{y \in G \colon\ |x| \wedge |y| = 0\}.$$

Then $(x)^\delta$ is a convex ℓ-subgroup of G. For $\emptyset \neq X \subseteq G$ we put $X^\delta = \bigcap_{x\in X} (x)^\delta$.

Each linearly ordered group is projectable. Thus according to [4] the lattice ordered group G is projectable. Therefore $(a[b])^{\delta}$ is an internal direct factor of G. Thus we have

$$G = (a[b])^{\delta} \times (a[b])^{\delta\delta}. \tag{2}$$

We put

$$G_1 = \{t \in G\colon\ t((a[b])^{\delta\delta}) \in A^a_b\}.$$

Then we obtain

$$G_1 = (a[b])^{\delta} \times A^a_b. \tag{3}$$

Lemma 5.7. *Assume that b is an atom of B. Then G_1 is a value of $a[b]$.*

P r o o f. We have $a[b] \in (a[b])^{\delta\delta}$, whence

$$a[b]((a[b])^{\delta\delta}) = a[b]$$

and $a[b] \notin A^a_b$. Thus $a[b] \notin G_1$.

Let H be a convex ℓ-subgroup of G with $G_1 \subset H$. Then according to (2) we obtain $H = H_1 \times H_2$, where

$$H_1 = H \cap (a[b])^{\delta}, \quad H_2 = H \cap (a[b])^{\delta\delta}.$$

In view of (3), $(a[b])^{\delta} \subseteq G_1$, thus $(a[b])^{\delta} \subseteq H$. This yields $H_1 = (a[b])^{\delta}$ and

$$H = (a[b])^{\delta} \times H_2.$$

Since $G_1 \subset H$, by using (3) again we obtain $A^a_b \subset H_2$. Then there exists $0 < t \in H_2$ with $t \notin A^a_b$. Let

$$t = a_1[c_1] + \ldots + a_n[c_n]$$

be a Specker representation of t. Since $t \in H_2$, all $a_i[c_i]$ $(i = 1, 2, \ldots, n)$ belong to H_2. Further, since $t \notin A^a_b$, there exists $i \in \{1, 2, \ldots, n\}$ with $a_i[c_i] \notin A^a_b$.

From $a_i[c_i] \in H_2 \subseteq (a[b])^{\delta\delta}$ we get $c_i \leqslant b$. Since $0 < c_i$ and since b is an atom of B we have $c_i = b$. Then $a_i[b] \in H_2$ and $a_i[b] \notin A^a_b$. Hence $a_i \notin A^a$.

We denote by A' the set of all $a_0 \in A$ such that $a_0[b] \in H_2$. Then A' is a convex ℓ-subgroup of A and $A^a \subseteq A'$. Since $a_i \in A'$ and $a_i \notin A^a$ we obtain $A^a \subset A'$. From the fact that A^a is a value of a we get $a \in A'$. Hence $a[b] \in H_2 \subseteq H$. Therefore G_1 is a value of $a[b]$. □

Lemma 5.8. *Assume that b is an atom of B and let $0 < a \in A$. Then the lattice ordered group $(a[b])^{\delta\delta}$ is linearly ordered.*

P r o o f. Let $x_1, x_2 \in (a[b])^{\delta\delta}$. Since b is an atom of B we conclude that there exist $a_1, a_2 \in A$ with $x_1 = a_1[b]$, $x_2 = a_2[b]$. Because A is linearly ordered, the elements a_1 and a_2 are comparable and thus x_1 and x_2 are comparable as well. □

Lemma 5.9. *Let a and b be as in* 5.8. *Further, let G_1 be as above. Then G_1 is a unique value of $a[b]$.*

P r o o f. Assume that G_1' is a value of $a[b]$. Then according to (2) we have $G_1' = K_1 \times K_2$, where

$$K_1 = G_1' \cap (a[b])^{\delta}, \quad K_2 = G_1' \cap (a[b]^{\delta\delta}).$$

Put

$$G_1'' = (a[b])^{\delta} \times K_2.$$

Thus $G_1'' \supseteq G_1'$. Suppose that $G_1'' \neq G_1'$.

Since G_1' is a value of $a[b]$ we get $a[b] \in G_1''$. Because $(a[b])(a[b])^{\delta} = 0$ we have $a[b] \in K_2$. This yields $a[b] \in G_1'$, which is a contradiction. Therefore $G_1'' = G_1'$ and hence

$$G_1' = (a[b])^{\delta} \times K_2.$$

Both A_b^a and K_2 are convex ℓ-subgroups of $(a[b])^{\delta\delta}$. According to 5.8, $(a[b])^{\delta\delta}$ is linearly ordered. Then the system of convex ℓ-subgroups of $(a[b])^{\delta\delta}$ is linearly ordered as well. This yields that G_1 and G_1' are comparable. But two distinct values of the same element cannot be comparable. Therefore $G_1' = G_1$. □

Corollary 5.10. *Let a and b be as in* 5.8. *Then $R_{a[b]} = G_1$, where G_1 is as above.*

From the definition of the partial order in G we obtain

Lemma 5.11. *Let a and b be as in* 5.8. *Then $(a[b])^{\delta}$ is the set of all $g \in G$ such that either $g = 0$, or g has a Specker representation $g = a_1[c_1] + \ldots + a_n[c_n]$ such that $a \wedge c_i = 0$ for $i = 1, 2, \ldots, n$.*

Corollary 5.12. *Let a, b be as in* 5.8 *and let $a_1 \in A$, $a_1 \neq 0$. Then $(a[b])^{\delta} = (a_1[b])^{\delta}$.*

Lemma 5.13. *Let a, b be as in 5.8 and let $a_1 \in A$, $a \leqslant a_1$. Then $R_{a[b]} \subseteq R_{a[b_1]}$.*

Proof. If A^{a_1} is defined analogously as A^a, then we have $A^a \subseteq A^{a_1}$, whence $A^a_b \subseteq A^{a_1}_b$. Hence in view of 5.9 and 5.12 we obtain $R_{a[b]} \subseteq R_{a_1[b]}$. □

Corollary 5.14. *Let a and b be as in 5.8. Let $c_1, \ldots, c_n$ be mutually orthogonal nonzero elements of B such that $b \wedge c_i = 0$ for $i = 1, 2, \ldots, n$. Let $a_1, \ldots, a_n \in A$. Then $a_1[c_1] + \ldots + a_n[c_n] \in R_{a[b]}$.*

Now let $0 < a \in A$, $0 < b \in B$; in 5.15–5.22 we suppose that b fails to be an atom of B.

Consider the Boolean algebra $[0, b]$. There exists a proper maximal ideal B^* of $[0, b]$. Let X be the set of all elements x of G such that either $x = 0$ or x has a Specker representation of the form $x = a_1[c_1] + \ldots + a_n[c_n]$ such that $c_1, \ldots, c_n$ belong to $[0, b]$ and $a_i \in A^a$ whenever $i \in \{1, 2, \ldots, n\}$ with $c_1 \notin B^*$. Then $a[b]$ does not belong to X.

The set X^δ consists of all elements $g \in G$ such that either $g = 0$ or g has a Specker representation $g = a^0_1[c^0_1] + \ldots + a^0_m[c^0_m]$ such that $c^0_j \wedge b = 0$ for $j = 1, 2, \ldots, m$.

Put $X_1 = X + X^\delta$. An easy calculation shows that X_1 is a convex ℓ-subgroup of G and that $a[b] \notin X_1$.

Lemma 5.15. *Under the assumptions as above, X_1 is a value of $a[b]$.*

Proof. By way of contradiction, assume that X_1 fails to be a value of $a[b]$. Hence there exists a convex ℓ-subgroup Y of G such that $a[b] \notin Y$ and $X_1 \subset Y$.

There is $0 < y \in Y$ with $y \notin X_1$. Let

$$y = a'_1[b_1] + \ldots + a^1_k[b_k]$$

be a Specker representation of y.

Put $b_{11} = b_1 \wedge b$ and let b_{12} be the complement of b_{11} in the interval $[0, b_1]$ of B. Hence we have

$$b_{11} \wedge b_{12} = 0, \quad b_{11} \vee b_{12} = b_1, \quad b_{11} \in [0, b], \quad b_{12} \wedge b = 0.$$

We apply the same procedure to the elements $b_2, \ldots, b_k$.

If for each $k(1) \in \{1, 2, \ldots, k\}$ we have either (i) $b_{k(1),1} \in B^*$, or (ii) $a^1_{k(1)} \in A^a$, then in view of the definition of X_1 we obtain $y \in X_1$, which is a contradiction. Hence there is $k(1) \in \{1, 2, \ldots, k\}$ such that $b_{k(1),1} \notin B^*$ and $a^1_{k(1)} \notin A^a$. We denote by b' the complement of $b_{k(1),1}$ in the Boolean algebra $[0, b]$. Then $a^1_{k(1)}[b'] \in X_1$. Further,

$$0 < a^1_{k(1)}[b_{k(1),1}] \leqslant a^1_{k(1)}[b_{k(1)}] \leqslant y,$$

whence $a^1_{k(1)}[b_{k(1),1}] \in Y$. Thus we obtain

$$a^1_{k(1)}[b'] + a^1_{k(1)}[b_{k(1),1}] \in Y.$$

Since $b' \wedge b_{k(1),1} = 0$ and $b' \vee b_{k(1),1} = b$, we have

$$a^1_{k(1)}[b'] + a^1_{k(1)}[b_{k(1),1}] = a^1_{k(1)}[b].$$

Thus $a^1_{k(1)}[b] \in Y$.

For each $a_1 \in A$ we put $f(a_1) = a_1[b]$. Then f is an isomorphism of the lattice ordered group A onto the ℓ-subgroup A_b of G. Since A^a is the unique value of a in A, we infer that A^a_b is the unique value of $a[b]$ in A_b.

We have $a^1_{k(1)} \notin A^a$. Hence $a^1_{k(1)}[b] \in A^a_b$. Therefore the convex ℓ-subgroup Y_1 of G which is generated by $a^1_{k(1)}[b]$ contains the element $a[b]$. Clearly $Y_1 \subseteq Y$ and hence $a[b] \in Y$, which is a contradiction. □

If the value X_1 of $a[b]$ is constructed as above by using the maximal proper ideal of the Boolean algebra $[0, b]$ then we say that X_1 is determined by B^*.

Again, let $0 < a \in A$, $0 < b \in B$. Suppose that b fails to be an atom of B. Let X_2 be a value of $a[b]$.

Lemma 5.16. $[0, a[b]]^\delta \subseteq X_2$.

Proof. By way of contradiction, assume that $[0, a[b]]^\delta$ fails to be a subset of X_2. Denote $Y = X_2 \vee [0, a[b]]^\delta$. Then Y is a convex ℓ-subgroup of G and $X_2 \subset Y$. Since X_2 is a value of $a[b]$ we must have $a[b] \in Y$.

There exist $z_1, \ldots, z_n \in X_2 \cup [0, a[b]]^\delta$ such that

$$0 < a[b] = z_1 + \ldots + z_n.$$

Then it is easy to verify that without loss of generality we can suppose that $z_i > 0$ for $i = 1, 2, \ldots, n$. If $z_i \in [0, a[b]]^\delta$ for some $i \in \{1, 2, \ldots, n\}$, then we would have $z_i \wedge a[b] = 0$ which is a contradiction, since $z_i \leqslant a[b]$. Therefore all z_i belong to X_2 yielding that $a[b] \in X_2$, which is a contradiction. □

Lemma 5.17. *There exist $b_1 \in B$ with $0 < b_1 < b$ and $a_1 \in A$ with $a_1 \notin A^a$ such that $a_1[b_1] \in X_2$.*

Proof. By way of contradiction, assume that for each a_1 and b_1 with the mentioned properties we have $a_1[b_1] \notin X_2$. Let B^* be a proper maximal ideal of the Boolean algebra $[0, b]$ and let X_1 be the value of $a[b]$ which is determined by B^*. Then $X_2 \subset X_1$ and $a[b] \notin X_1$. Thus X_2 fails to be a value of $a[b]$, which is a contradiction. □

We denote by B_0 the set of all $b_1 \in B$ such that either $b_1 = 0$, or $0 < b_1 < b$ and there exists $a_1 \in A$ such that $a_1 \notin A^a$ and $a_1[b_1] \in X_2$. In view of 5.17, $B_0 \neq \emptyset$.

Lemma 5.18. *B_0 is an ideal of $[0, b]$ and $b \notin B_0$.*

P r o o f. Let $0 < b_1 \in B_0$ and $0 < b_2 \in B$, $b_2 < b_1$. There exists $0 < a_1 \in A$ with $a_1 \notin A^a$, $a_1[b_1] \in X_2$. Then $0 < a_1[b_2] < a_1[b_1]$, whence $a_1[b_2] \in X_2$ and thus $b_2 \in B_0$.

Let $0 < b_1 \in B_0$, $0 < b_2 \in B_0$. Then there exist $a_i \in A$ such that $0 < a_i \notin A^a$, $a_i[b_i] \in X_2$ for $i = 1, 2$. Put $a_3 = a_1 \wedge a_2$. Hence without loss of generality we can suppose that $a_3 = a_2$ and then

$$a_2[b_1] \vee a_2[b_2] = a_2[b_1 \vee b_2] \in X_2.$$

Thus $b_1 \vee b_2 \in B_0$. Therefore B_0 is an ideal of $[0, b]$. Assume that $0 < a_4 \in A$, $a_4 \notin A^a$ and $a_4[b] \in X_2$. Let A^1 be the convex ℓ-subgroup of A generated by a_4. Since $a_4 \notin A^a$ we have $A^a \subset A^1$ and hence $a \in A^1$. Then there is $n \in N$ with $a \leqslant na_4$. We get $0 < a[b] \leqslant na_4[b] \in X_2$ yielding $a[b] \in X_2$, which is a contradiction. □

Lemma 5.19. *B_0 is a proper maximal ideal of $[0, b]$ and X_2 is generated by B_0.*

P r o o f. By way of contradiction, assume that B_0 fails to be a proper maximal ideal of $[0, b]$. Then in view of 5.17 and 5.18, there exists a proper maximal ideal B^* of $[0, b]$ such that $B_0 \subset B^*$. Let X_1 be as above. Then $X_2 \subset X_1$, which is a contradiction. Thus we have $B^* = B_0$.

Let $a_1 \in A$ and $b_1 \in B^*$. If $a_1[b_1] \notin X_2$, then $X_2 \subset X_1$, which is impossible. From this we conclude that $X_2 = X_1$. □

Corollary 5.20. *There is a one-to-one correspondence between values of $a[b]$ and proper maximal ideals of the Boolean algebra $[0, b]$.*

Lemma 5.21. *Let $a_1 \in A$. There exist values X_1 and X_2 of $a[b]$ such that $a_1[b] \in X_1 \vee X_2$.*

P r o o f. It suffices to consider the case $a_1 > 0$. Let X_1 be as above. There exists $b_1 \in [0, b]$ such that $b_1 < b$ and $b_1 \notin B^*$. Further, there exists a proper maximal ideal B_1^* of $[0, b]$ such that $b_1 \in B_1^*$. Also, there exists a value X_2 of $a[b]$ which is determined by B_1^*.

Let b_1' be the complement of b_1 in the Boolean algebra $[0,b]$. Since $b_1 \notin B^*$ we get $b_1' \in B^*$. In view of the definition of B^* we have $a_1[b_1'] \in X_1$. Similarly, $a_1[b_1] \in X_2$. Then

$$a_1[b_1'] \vee a_1[b_1] = a_1[b_1' \vee b_1] = a_1[b].$$

Since $a_1[b_1'] \vee a_1[b_1] \in X_1 \vee X_2$, the proof is complete. □

Lemma 5.22. *Let $0 \neq g \in G$. Then $g \in R_{a[b]}$.*

P r o o f. By applying the Specker representation of g we conclude that it suffices to verify the validity of the relation $a_1[b_1] \in R_{a[b]}$ for each $0 < a_1 \in A$ and each $0 < b_1 \in B$. Put $b_{11} = b_1 \wedge b$ and let b_{12} be the complement of b_{11} in the interval $[0,b_1]$ of B. Then $b_{12} \wedge b = 0$ and hence in view of 5.16 we get $a_1[b_{12}] \in X$ for each value X of $a[b]$.

Further, in view of 5.21, there exist values X_1 and X_2 of $a[b]$ such that $a_1[b_{11}] \in X_1 \vee X_2$. Hence

$$a_1[b_1] = a_1[b_{11}] \vee a_1[b_{12}] \in X_1 \vee X_2.$$

Therefore $a_1[b_1] \in R_{a[b]}$. □

We denote by B_1 the set of all atoms of B. From 5.6 and 5.22 we obtain

Proposition 5.23. *If $B_1 = \emptyset$, then $R(G) = G$. If $B_1 \neq \emptyset$, then $R(G) = \cap R_{a[b]}$, where $0 < a \in A$ and $b \in B_1$.*

Let $b \in B_1$ and $0 < a \in A$. In view of 5.10 we have

$$R_{a[b]} = (a[b])^\delta \times A_b^a.$$

Recall that $A_b^a = \{a_1[b]\}_{a_1 \in A^a}$. Since $A^a \subset [-a,a]$, we get

$$\bigcap_{0<a\in A} A^a = \{0\},$$

whence

$$\bigcap_{0<a\in A} A_b^a = \{0\}.$$

Further, 5.12 yields $(a[b])^\delta = (a_0[b])^\delta$ for each $0 < a_0 \in A$. Denote

$$R_b = \bigcap_{0<a\in A} R_{a[b]}.$$

Then for each $0 < a \in A$ we have

$$R_b = (a[b])^{\delta} \times \{0\} = (a[b])^{\delta},$$
$$R(G) = \bigcap_{0<b\in B_1} R_b = \bigcap_{0<b\in B_1} (a[b])^{\delta}. \qquad (+)$$

Thus in view of 5.22 we obtain

Theorem 5.24. *Let $A \neq \{0\}$ be a linearly ordered group, $B \neq \{0\}$ be a generalized Boolean algebra. Let B_1 be the set of all atoms of B.* (i) *If $B_1 = \emptyset$, then $R(G) = G$.* (ii) *If $B_1 \neq \emptyset$, then $R(G)$ is given by the relation* (+).

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