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Czechoslovak Mathematical Journal, Vol. 56 (2006), No. 2, 781–788

Persistent URL: <http://dml.cz/dmlcz/128105>

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ON \mathcal{L} -STARCOMPACT SPACES

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(Received February 23, 2004)

Abstract. A space X is \mathcal{L} -starcompact if for every open cover \mathcal{U} of X , there exists a Lindelöf subset L of X such that $\text{St}(L, \mathcal{U}) = X$. We clarify the relations between \mathcal{L} -starcompact spaces and other related spaces and investigate topological properties of \mathcal{L} -starcompact spaces. A question of Hiremath [3] is answered.

Keywords: Lindelöf, star-Lindelöf and \mathcal{L} -starcompact

MSC 2000: 54D20, 54B10, 54D55

1. INTRODUCTION

By a space, we mean a topological space. Let us recall [6] that a space X is *star-Lindelöf* if for every open cover \mathcal{U} of X , there exists a countable subset B of X such that $\text{St}(B, \mathcal{U}) = X$, where $\text{St}(B, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap B \neq \emptyset\}$. It is clear that every separable space is star-Lindelöf. Also, it is not difficult to see that every T_1 -space with countable extent is star-Lindelöf. Therefore, every countably compact T_1 -space is star-Lindelöf as well as every Lindelöf space. As generalities of star-Lindelöfness, the following classes of spaces are given (see [6]):

Definition 1.1. A space X is \mathcal{L} -starcompact if for every open cover \mathcal{U} of X , there exists a Lindelöf subset L of X such that $\text{St}(L, \mathcal{U}) = X$.

Definition 1.2. A space X is $1\frac{1}{2}$ -starLindelöf if for every open cover \mathcal{U} of X , there exists a countable subset \mathcal{V} of \mathcal{U} such that $\text{St}(\bigcup \mathcal{V}, \mathcal{U}) = X$.

In [3], \mathcal{L} -starcompactness is called sLc property, and in [1], a $1\frac{1}{2}$ -star-Lindelöf space is called a star-Lindelöf space and a star-Lindelöf space is called a strongly star-Lindelöf space.

The author is supported by NSFC Project 10271056.

From the above definitions, we have the following diagram:

$$\text{star-Lindelöf} \longrightarrow \mathcal{L}\text{-starcompact} \longrightarrow 1\frac{1}{2}\text{-starLindelöf.}$$

In the following section, we give examples showing that the converses in the above diagram do not hold.

The cardinality of a set A is denoted by $|A|$. Let ω be the first infinite cardinal, ω_1 the first uncountable cardinal and \mathfrak{c} the cardinality of the set of all real numbers. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. For each ordinals α, β with $\alpha < \beta$, we write $(\alpha, \beta) = \{\gamma: \alpha < \gamma < \beta\}$, $(\alpha, \beta] = \{\gamma: \alpha < \gamma \leq \beta\}$ and $[\alpha, \beta] = \{\gamma: \alpha \leq \gamma \leq \beta\}$. Every cardinal is often viewed as a space with the usual order topology. Other terms and symbols follow [2].

2. \mathcal{L} -STARCOMPACT SPACES AND RELATED SPACES

In [3], Hiremath asked if the product of two countably compact spaces is \mathcal{L} -starcompact. However it is not difficult to see that the following well-known example gives a negative answer to the above question, we shall give the proof roughly for the sake of completeness. The symbol $\beta(X)$ means the Čech-Stone compactification of a Tychonoff space X .

Example 2.1. There exist two countably compact spaces X and Y such that $X \times Y$ is not \mathcal{L} -starcompact.

Proof. Let D be a discrete space of cardinality \mathfrak{c} . We can define $X = \bigcup_{\alpha < \omega_1} E_\alpha$, $Y = \bigcup_{\alpha < \omega_1} F_\alpha$, where E_α and F_α are the subsets of $\beta(D)$ which are defined inductively so as to satisfy the following conditions (1), (2) and (3):

- (1) $E_\alpha \cap F_\beta = D$ if $\alpha \neq \beta$;
- (2) $|E_\alpha| \leq \mathfrak{c}$ and $|F_\alpha| \leq \mathfrak{c}$;
- (3) every infinite subset of E_α (resp. F_α) has an accumulation point in $E_{\alpha+1}$ (resp. $F_{\alpha+1}$).

These sets E_α and F_α are well-defined since every infinite closed set in $\beta(D)$ has the cardinality $2^{\mathfrak{c}}$ (see [5]). Then, $X \times Y$ is not \mathcal{L} -starcompact, because the diagonal $\{(d, d): d \in D\}$ is a discrete open and closed subset of $X \times Y$ with the cardinality \mathfrak{c} and \mathcal{L} -starcompactness is preserved by open and closed subsets. \square

We end this section by giving examples which show the converses in the above diagram in §1 do not hold.

Example 2.2. There exists an \mathcal{L} -starcompact Tychonoff space which is not star-Lindelöf.

P r o o f. Let D be a discrete space of cardinality \mathfrak{c} . Define

$$X = (\beta(D) \times (\omega + 1)) \setminus ((\beta(D) \setminus D) \times \{\omega\}).$$

Then, X is \mathcal{L} -starcompact, since $\beta(D) \times \omega$ is a Lindelöf dense subset of X .

Next, we shall show that X is not star-Lindelöf. Let us consider the open cover

$$\mathcal{U} = \{\{d\} \times (\omega + 1) : d \in D\} \cup \{\beta(D) \times \{n\} : n \in \omega\}$$

of X . Let B be a countable subset of X . Then, there exists a $d^* \in D$ such that $B \cap (\{d^*\} \times (\omega + 1)) = \emptyset$. This means that $U = \{d^*\} \times (\omega + 1)$ is the only element of \mathcal{U} containing the point $\langle d^*, \omega \rangle$, and hence $\langle d^*, \omega \rangle \notin \text{St}(B, \mathcal{V})$. \square

Example 2.3. There exists a $1\frac{1}{2}$ -starLindelöf Tychonoff space which is not \mathcal{L} -starcompact.

P r o o f. Let \mathcal{R} be a maximal almost disjoint family of infinite subsets of ω with $|\mathcal{R}| = \mathfrak{c}$. Define

$$X = \mathcal{R} \cup (\mathfrak{c} \times \omega).$$

We topologize X as follows: $\mathfrak{c} \times \omega$ has the usual product topology and is an open subspace of X . On the other hand a basic neighbourhood of $r \in \mathcal{R}$ takes the form

$$G_{\beta, K}(r) = (\{\alpha : \beta < \alpha < \mathfrak{c}\} \times (r \setminus K)) \cup \{r\}$$

for $\beta < \mathfrak{c}$ and a finite subset K of ω . To show that X is $1\frac{1}{2}$ -starLindelöf, let \mathcal{U} be an open cover of X . Let

$$M = \{n \in \omega : (\exists U \in \mathcal{U})(\exists \beta < \mathfrak{c})(\beta, \mathfrak{c}) \times \{n\} \subseteq U\}.$$

For each $n \in M$, there exist $U_n \in \mathcal{U}$ and $\beta_n < \mathfrak{c}$ such that $(\beta_n, \mathfrak{c}) \times \{n\} \subseteq U_n$. If we put $\mathcal{V}' = \{U_n : n \in M\}$, then

$$\mathcal{R} \subseteq \text{St}\left(\bigcup \mathcal{V}', \mathcal{U}\right).$$

On the other hand, for each $n < \omega$, since $\mathfrak{c} \times \{n\}$ is countably compact, we can find a finite subfamily \mathcal{V}_n of \mathcal{U} such that

$$\mathfrak{c} \times \{n\} \subseteq \text{St}\left(\bigcup \mathcal{V}_n, \mathcal{U}\right).$$

Consequently, if we put $\mathcal{V} = \mathcal{V}' \cup \bigcup \{\mathcal{V}_n : n < \omega\}$, then, \mathcal{V} is a countable subfamily of \mathcal{U} and $X = \text{St}(\bigcup \mathcal{V}, \mathcal{U})$. Hence, X is $1\frac{1}{2}$ -starLindelöf.

Next, we shall show that X is not \mathcal{L} -starcompact. Since $|\mathcal{R}| = \mathfrak{c}$, enumerate \mathcal{R} as $\{r_\alpha : \alpha < \mathfrak{c}\}$. For each $\alpha < \mathfrak{c}$, let $U_\alpha = \{r_\alpha\} \cup ((\alpha, \mathfrak{c}) \times r_\alpha)$. Consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{c}\} \cup \{\mathfrak{c} \times \omega\}$$

of X and let L be a Lindelöf subset of X . Since \mathcal{R} is discrete closed in X , $L \cap \mathcal{R}$ is countable. Hence, there exists $\beta' < \mathfrak{c}$ such that

$$(1) \quad L \cap \{r_\alpha : \alpha > \beta'\} = \emptyset.$$

On the other hand, $L \cap (\mathfrak{c} \times \{n\})$ is bounded in $\mathfrak{c} \times \{n\}$ for each $n < \omega$. Thus, there exists $\beta_n < \mathfrak{c}$ such that $\beta_n > \sup\{\alpha < \mathfrak{c} : \langle \alpha, n \rangle \in L\}$. Pick $\beta'' < \mathfrak{c}$ such that $\beta'' > \beta_n$ for each $n \in \omega$. Then,

$$(2) \quad ((\beta'', \mathfrak{c}) \times \omega) \cap L = \emptyset.$$

Choose $\gamma < \mathfrak{c}$ such that $\gamma > \max\{\beta', \beta''\}$. Then, U_γ is the only element of \mathcal{U} containing the point r_γ and $U_\gamma \cap L = \emptyset$ by (1) and (2). It follows that $r_\gamma \notin \text{St}(L, \mathcal{U})$, which shows that X is not \mathcal{L} -starcompact. \square

Remark 1. The author does not know if each arrow in the above diagram can be reversed in the realm of normal spaces.

3. PROPERTIES OF \mathcal{L} -STARCOMPACT SPACES

Topological behavior of \mathcal{L} -starcompact spaces was extensively studied by Hiremath [3] and Ikenaga [4]. The purpose of this section is to prove some results which supplement their investigation. In [3, Example 3.6], Hiremath proved that a closed subspace of an \mathcal{L} -starcompact space need not be \mathcal{L} -starcompact. The following example shows that a regular closed subspace of an \mathcal{L} -starcompact space need not be \mathcal{L} -starcompact.

Example 3.1. There exists a star-Lindelöf (hence, an \mathcal{L} -starcompact) Tychonoff space having a regular-closed subset which is not \mathcal{L} -starcompact.

Proof. Let $S_1 = (\mathfrak{c} \times \omega) \cup \mathcal{R}$ be the same space as the space X in Example 2.3. As we have proved above, S_1 is not \mathcal{L} -starcompact. Let $S_2 = \omega \cup \mathcal{R}$ be the Isbell-Mrówka space [7], where \mathcal{R} is a maximal almost disjoint family of infinite subsets of ω with $|\mathcal{R}| = \mathfrak{c}$. Then, S_2 is \mathcal{L} -starcompact because it is separable.

Assume $S_1 \cap S_2 = \emptyset$ and let X be the quotient image of the disjoint sum $S_1 \oplus S_2$ identifying the subspace \mathcal{R} of S_1 with the subspace \mathcal{R} of S_2 . Let $\varphi: S_1 \oplus S_2 \rightarrow X$

be the quotient map. Then, $\varphi[S_1]$ is a regular-closed subspace of X which is not \mathcal{L} -starcompact.

We shall show that X is star-Lindelöf. Let \mathcal{U} be an open cover of X . For each $n \in \omega$, since $\varphi[\mathfrak{c} \times \{n\}]$ is countably compact, there exists a finite subset $F_n \subseteq \varphi[\mathfrak{c} \times \{n\}]$ such that $\varphi[\mathfrak{c} \times \{n\}] \subseteq \text{St}(F_n, \mathcal{U})$. Thus, if we put $B' = \bigcup\{F_n : n \in \omega\}$, then

$$\varphi[\mathfrak{c} \times \omega] \subseteq \text{St}(B', \mathcal{U}).$$

On the other hand, since $\varphi[S_2]$ is separable, there exists a countable subset B'' of $\varphi[S_2]$ such that $\varphi[S_2] \subseteq \text{St}(B'', \mathcal{U})$. Consequently, we can show that $\text{St}(B' \cup B'', \mathcal{U}) = X$, which shows that X is star-Lindelöf. \square

Theorem 3.2. *An open F_δ -subset of an \mathcal{L} -starcompact space is \mathcal{L} -starcompact.*

Proof. Let X be an \mathcal{L} -starcompact space and let $Y = \bigcup\{H_n : n \in \omega\}$ be an open F_δ -subset of X , where the set H_n is closed in X for each $n \in \omega$. To show that Y is \mathcal{L} -starcompact, let \mathcal{U} be an open cover of Y . We have to find a Lindelöf subset L of Y such that $\text{St}(L, \mathcal{U}) = Y$. For each $n \in \omega$, consider the open cover

$$\mathcal{U}_n = \mathcal{U} \cup \{X \setminus H_n\}$$

of X . Since X is \mathcal{L} -starcompact, there exists a Lindelöf subset L_n of X such that $\text{St}(L_n, \mathcal{U}_n) = X$. Let $M_n = L_n \cap Y$. Since Y is a F_δ -set, M_n is Lindelöf, and clearly $H_n \subseteq \text{St}(M_n, \mathcal{U})$. Thus, if we put $L = \bigcup\{M_n : n \in \omega\}$, then L is a Lindelöf subset of Y and $\text{St}(L, \mathcal{U}) = Y$. Hence, Y is \mathcal{L} -starcompact. \square

A *cozero-set* in a space X is a set of the form $f^{-1}(R \setminus \{0\})$ for some real-valued continuous function f on X . Since a cozero-set is an open F_σ -set, we have the following corollary:

Corollary 3.3. *A cozero-set of an \mathcal{L} -starcompact space is \mathcal{L} -starcompact.*

Let τ be an infinite cardinal. Recall that a space X is *Lindelöf- τ -bounded* if every subset of X of cardinality $\leq \tau$ is contained in a Lindelöf subset of X ([6]).

Theorem 3.4. *Every Lindelöf- ω_1 -bounded space is star-Lindelöf.*

Proof. Let X be a Lindelöf- ω_1 -bounded space. Suppose that X is not star-Lindelöf. Then, there exists an open cover \mathcal{U} of X such that $\text{St}(B, \mathcal{U}) \neq X$ for every countable subset B of X . By induction, we can define a sequence $\{x_\alpha : \alpha < \omega_1\}$ of points of X such that

$$x_\alpha \notin \text{St}(\{x_\beta : \beta < \alpha\}, \mathcal{U}) \text{ for each } \alpha < \omega_1.$$

Since X is Lindelöf- ω_1 -bounded, the set $\{x_\alpha : \alpha < \omega_1\}$ is contained in a Lindelöf subspace $L \subseteq X$. Thus, there exists a countable subfamily $\mathcal{V} \subseteq \mathcal{U}$ which covers L . Then at least one element of \mathcal{V} contains uncountably many points x_α , which is a contradiction to the definition of the sequence $\{x_\alpha : \alpha < \omega_1\}$. Hence, X is star-Lindelöf. \square

For a space X , let $l(X)$ be the *Lindelöf number* of X , i.e., the smallest cardinal λ such that every open cover of X has an open refinement \mathcal{V} with $|\mathcal{V}| \leq \lambda$.

Theorem 3.5. *Let $\tau \geq \omega_1$. Let $X = Y \cup Z$, where Y is dense in X , Y is Lindelöf- τ -bounded and $l(Z) \leq \tau$. Then, X is \mathcal{L} -starcompact.*

Proof. Let \mathcal{U} be an open cover of X . Since Y is Lindelöf- τ -bounded, from Theorem 3.4, there exists a countable subset B of Y such that $Y \subseteq \text{St}(B, \mathcal{U})$. So it remains to find a Lindelöf subset $L' \subseteq Y$ such that $Z \subseteq \text{St}(L', \mathcal{U})$. Since $l(Z) \leq \tau$, there is a subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that $|\mathcal{V}| \leq \tau$ and $Z \subseteq \bigcup \mathcal{V}$. Pick $x_V \in V \cap Y$ for each $V \in \mathcal{V}$. Since Y is Lindelöf- τ -bounded, the subset $\{x_V : V \in \mathcal{V}\}$ of Y is included in some Lindelöf subspace $L' \subseteq Y$. Hence, $Z \subseteq \text{St}(L', \mathcal{U})$. Let $L = L' \cup B$. Then, L is a Lindelöf subspace of X and $X = \text{St}(L, \mathcal{U})$, which completes the proof. \square

In [3], Hiremath proved that a continuous image of an \mathcal{L} -starcompact space is \mathcal{L} -starcompact. By contrast, he also showed a perfect preimage of an \mathcal{L} -starcompact space need not be \mathcal{L} -starcompact. Now we give a positive result:

Theorem 3.6. *Let f be an open perfect map from a space X to an \mathcal{L} -starcompact space Y . Then, X is \mathcal{L} -starcompact.*

Proof. Since $f[X]$ is open and closed in Y , we may assume that $f[X] = Y$. Let \mathcal{U} be an open cover of X and let $y \in Y$. Since $f^{-1}(y)$ is compact, there exists a finite subcollection \mathcal{U}_y of \mathcal{U} such that $f^{-1}(y) \subseteq \bigcup \mathcal{U}_y$ and $U \cap f^{-1}(y) \neq \emptyset$ for each $U \in \mathcal{U}_y$. Pick an open neighbourhood V_y of y in Y such that $f^{-1}[V_y] \subseteq \bigcup \{U : U \in \mathcal{U}_y\}$, then we can assume that

$$(1) \quad V_y \subseteq \bigcap \{f[U] : U \in \mathcal{U}_y\},$$

because f is open. Taking such open set V_y for each $y \in Y$, we have an open cover $\mathcal{V} = \{V_y : y \in Y\}$ of Y . Let L be a Lindelöf subset of the \mathcal{L} -starcompact space Y such that $\text{St}(L, \mathcal{V}) = Y$. Since f is perfect, the set $f^{-1}(L)$ is a Lindelöf subset of X . To show that $\text{St}(f^{-1}(L), \mathcal{U}) = X$, let $x \in X$. Then, there exists $y \in Y$ such that $f(x) \in V_y$ and $V_y \cap L \neq \emptyset$. Since

$$x \in f^{-1}[V_y] \subseteq \bigcup \{U : U \in \mathcal{U}_y\},$$

we can choose $U \in \mathcal{U}_y$ with $x \in U$. Then $V_y \subseteq f[U]$ by (1), and hence $U \cap f^{-1}[L] \neq \emptyset$. Therefore, $x \in \text{St}(f^{-1}[L], \mathcal{U})$. Consequently, we have $\text{St}(f^{-1}(L), \mathcal{U}) = X$. \square

Corollary 3.7 (Hiremath [3]). *Let X be an \mathcal{L} -starcompact space and Y a compact space. Then, $X \times Y$ is \mathcal{L} -starcompact.*

The following theorem is a generalization of Corollary 3.7.

Theorem 3.8. *Let X be an \mathcal{L} -starcompact space and Y a locally compact, Lindelöf space. Then, $X \times Y$ is \mathcal{L} -starcompact.*

Proof. Let \mathcal{U} be an open cover of $X \times Y$. For each $y \in Y$, there exists an open neighbourhood V_y of y in Y such that $\text{cl}_Y V_y$ is compact. By Corollary 3.7, the subspace $X \times \text{cl}_Y V_y$ is \mathcal{L} -starcompact. Thus, there exists a Lindelöf subset $L_y \subseteq X \times \text{cl}_Y V_y$ such that

$$X \times \text{cl}_Y V_y \subseteq \text{St}(L_y, \mathcal{U}).$$

Since Y is Lindelöf, there exists a countable cover $\{V_{y_i} : i \in \omega\}$ of Y . Let $L = \bigcup \{L_{y_i} : i \in \omega\}$. Then, L is a Lindelöf subset of $X \times Y$ such that $\text{St}(L, \mathcal{U}) = X \times Y$. Hence, $X \times Y$ is \mathcal{L} -starcompact. \square

Hiremath [3] showed that the product of two Lindelöf spaces need not be \mathcal{L} -starcompact. In [1, Example 3.3.3], van Douwen-Reed-Roscoe-Tree also gave an example of a countably compact (and, hence, starcompact) space X and a Lindelöf space Y such that $X \times Y$ is not star-Lindelöf. Now, we shall show that the product $X \times Y$ is not \mathcal{L} -starcompact:

Example 3.3.9. *There exist a countably compact space X and a Lindelöf space Y such that $X \times Y$ is not \mathcal{L} -starcompact.*

Proof. Let $X = \omega_1$ with the usual order topology, $Y = \omega_1 + 1$ with the following topology. Each point α with $\alpha < \omega_1$ is isolated and a set U containing ω_1 is open if and only if $Y \setminus U$ is countable. Then, X is countably compact and Y is Lindelöf. Now, we show that $X \times Y$ is not \mathcal{L} -starcompact. For each $\alpha < \omega_1$, let $U_\alpha = [0, \alpha] \times [\alpha, \omega_1]$, and $V_\alpha = (\alpha, \omega_1) \times \{\alpha\}$. Consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < \omega_1\} \cup \{V_\alpha : \alpha < \omega_1\}$$

of $X \times Y$ and let L be a Lindelöf subset of $X \times Y$. Then, $\pi_X[L]$ is a Lindelöf subset of X , where $\pi_X : X \times Y \rightarrow X$ is the projection. Thus, there exists $\beta < \omega_1$ such that $L \cap ((\beta, \omega_1) \times Y) = \emptyset$. Pick α with $\alpha > \beta$. Then, $\langle \alpha, \beta \rangle \notin \text{St}(L, \mathcal{U})$ since V_β is the only element of \mathcal{U} containing $\langle \alpha, \beta \rangle$. Hence, $X \times Y$ is not \mathcal{L} -starcompact, which completes the proof. \square

Remark. In [4, Example 2], Ikenaga gave an example of a Lindelöf space X and a separable space Y such that $X \times Y$ is not star-Lindelöf. By contrast, as far as the author knows, it is open whether the product of an \mathcal{L} -starcompact space and a separable space is \mathcal{L} -starcompact.

Acknowledgements. The author would like to thank Prof. H. Ohta for his kind help and valuable suggestions.

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