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CYCLES WITH A GIVEN NUMBER OF VERTICES FROM EACH
PARTITE SET IN REGULAR MULTIPARTITE TOURNAMENTS

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Abstract. If x is a vertex of a digraph D , then we denote by $d^+(x)$ and $d^-(x)$ the outdegree and the indegree of x , respectively. A digraph D is called regular, if there is a number $p \in \mathbb{N}$ such that $d^+(x) = d^-(x) = p$ for all vertices x of D .

A c -partite tournament is an orientation of a complete c -partite graph. There are many results about directed cycles of a given length or of directed cycles with vertices from a given number of partite sets. The idea is now to combine the two properties. In this article, we examine in particular, whether c -partite tournaments with r vertices in each partite set contain a cycle with exactly $r - 1$ vertices of every partite set. In 1982, Beineke and Little [2] solved this problem for the regular case if $c = 2$. If $c \geq 3$, then we will show that a regular c -partite tournament with $r \geq 2$ vertices in each partite set contains a cycle with exactly $r - 1$ vertices from each partite set, with the exception of the case that $c = 4$ and $r = 2$.

Keywords: multipartite tournaments, regular multipartite tournaments, cycles

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1. TERMINOLOGY AND INTRODUCTION

In this paper all digraphs are finite without loops and multiple arcs. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $E(D)$, respectively. If xy is an arc of a digraph D , then we write $x \rightarrow y$ and say x dominates y , and if X and Y are two disjoint vertex sets or subdigraphs of D such that every vertex of X dominates every vertex of Y , then we say that X dominates Y , denoted by $X \rightarrow Y$. Furthermore, $X \rightsquigarrow Y$ denotes the fact that there is no arc leading from Y to X . For the number of arcs from X to Y we write $d(X, Y)$.

If D is a digraph, then the *out-neighborhood* $N_D^+(x) = N^+(x)$ of a vertex x is the set of vertices dominated by x and the *in-neighborhood* $N_D^-(x) = N^-(x)$ is the

set of vertices dominating x . Therefore, if the arc $xy \in E(D)$ exists, then y is an *outer neighbor* of x and x is an *inner neighbor* of y . The numbers $d_D^+(x) = d^+(x) = |N^+(x)|$ and $d_D^-(x) = d^-(x) = |N^-(x)|$ are called the *outdegree* and the *indegree* of x , respectively. Furthermore, the numbers $\delta_D^+ = \delta^+ = \min\{d^+(x) : x \in V(D)\}$ and $\delta_D^- = \delta^- = \min\{d^-(x) : x \in V(D)\}$ are the *minimum outdegree* and the *minimum indegree*, respectively.

For a vertex set X of D , we define $D[X]$ as the subdigraph induced by X . If we replace in a digraph D every arc xy by yx , then we call the resulting digraph the *converse* of D , denoted by D^{-1} .

If we speak of a *cycle*, then we mean a directed cycle, and a cycle of length n is called an *n -cycle*. The length of a cycle C is denoted by $L(C)$. A cycle in a digraph D is *Hamiltonian* if $L(C) = |V(D)|$. A *cycle-factor* of a digraph D is a spanning subdigraph consisting of disjoint cycles. A digraph D is called *pancyclic* if it contains cycles of length n for all $n \in \{3, 4, \dots, |V(D)|\}$, and *even pancyclic* if it contains cycles of all even lengths. If $x \in V(C)$ ($x \in V(P)$, respectively) for a cycle C (a path P), then we denote the successor of x in the given cycle (path) by x^+ and the predecessor by x^- . A digraph D is *cycle complementary* if there exist two vertex-disjoint cycles C and C' such that $V(D) = V(C) \cup V(C')$.

A digraph D is *strongly connected* or *strong* if for each pair of vertices u and v , there is a path from u to v in D . A digraph D with at least $k + 1$ vertices is *k -connected* if for any set A of at most $k - 1$ vertices, the subdigraph $D - A$ obtained by deleting A is strong. The *connectivity*, denoted by $\kappa(D)$, is then defined to be the largest value of k such that D is k -connected. If $\kappa(D) = 1$ and x is a vertex of D such that $D - x$ is not strong, then we say that x is a *cut-vertex* of D .

There are several measures of how much a digraph differs from being regular. In [18], Yeo defines the *global irregularity* of a digraph D by

$$i_g(D) = \max_{x \in V(D)} \{d^+(x), d^-(x)\} - \min_{y \in V(D)} \{d^+(y), d^-(y)\}.$$

If $i_g(D) = 0$, then D is *regular* and if $i_g(D) \leq 1$, then D is called *almost regular*.

A *c -partite* or *multipartite tournament* is an orientation of a complete c -partite graph. A *tournament* is a c -partite tournament with exactly c vertices. If V_1, V_2, \dots, V_c are the partite sets of a c -partite tournament D and the vertex x of D belongs to the partite set V_i , then we define $V(x) = V_i$. If D is a c -partite tournament with the partite sets V_1, V_2, \dots, V_c such that $|V_1| \leq |V_2| \leq \dots \leq |V_c|$, then $|V_c| = \alpha(D)$ is the independence number of D .

Let $B = B(r_1, r_2, r_3, r_4)$ be the following bipartite tournament, which will be useful later. Let R_1, R_2, R_3, R_4 be pairwise disjoint independent sets of vertices with

$|R_i| = r_i$ for $1 \leq i \leq 4$. Define $V(B) = R_1 \cup R_2 \cup R_3 \cup R_4$ such that $R_i \rightarrow R_{i+1}$ for $i = 1, 2, 3$ and $R_4 \rightarrow R_1$.

There is extensive literature on cycles in multipartite tournaments, see e.g., Bang-Jensen and Gutin [1], Guo [6], Gutin [7], Volkmann [13] and Yeo [17]. Many results are about the existence of cycles of a given length as e.g. the following result of Bondy [3].

Theorem 1.1 (Bondy [3]). *Each strongly connected c -partite tournament contains a cycle of order m for each $m \in \{3, 4, \dots, c\}$.*

Other articles treat the existence of cycles containing vertices of a given number of partite sets. A good example is the following theorem of Goddard and Oellermann [5].

Theorem 1.2 (Goddard, Oellermann [5]). *If x is an arbitrary vertex of a strongly connected c -partite tournament D , then x belongs to a cycle that contains vertices from exactly q partite sets for each $q \in \{3, 4, \dots, c\}$.*

An interesting question is now to find sufficient conditions for a multipartite tournament such that we are able to combine these two categories of results, that means to solve the following problem.

Problem 1.3. Which conditions have to be fulfilled in order that a c -partite tournament with the partite sets V_1, V_2, \dots, V_c contains a cycle with exactly r_i vertices of V_i for all $1 \leq i \leq c$ and given integers $0 \leq r_i \leq |V_i|$?

In 1997, A. Yeo [16] gave a solution of this problem for regular c -partite tournaments in the case that $r_i = |V_i|$ for all $1 \leq i \leq c$.

Theorem 1.4 (Yeo [16]). *Every regular multipartite tournament D is Hamiltonian.*

Since, according to the well known result of Moon [9] that every strongly connected tournament is vertex-pancyclic, a strongly connected tournament is Hamiltonian, we note that the next theorem also treats Problem 1.3.

Theorem 1.5 (Volkmann [12]). *Let D be an almost regular c -partite tournament with $c \geq 4$. Then D contains a strongly connected subtournament of order p for every $p \in \{3, 4, \dots, c - 1\}$.*

In a recent article, the authors [15] settled a conjecture of Volkmann in affirmative by proving that Theorem 1.5 remains valid for $p = c$ when $c \geq 5$. Thus, we arrive at the following result.

Theorem 1.6 (Volkman, Winzen [15]). *Let D be an almost regular c -partite tournament with $c \geq 5$. Then D contains a strongly connected subtournament of order p for every $p \in \{3, 4, \dots, c\}$.*

Hence, Theorem 1.6 presents a solution of Problem 1.3 for almost regular c -partite tournaments and the case that $r_i = 1$ for all $1 \leq i \leq c$. In this article, we will treat the case that D is a regular c -tournament and $r_i = |V_i| - 1$ for all $1 \leq i \leq c$. Since the vertices of a cycle in a bipartite tournament D alternate between the two partite sets of D , Beineke and Little [2] (for a stronger form, see also Zhang [19]) gave a solution to this problem if $c = 2$.

Theorem 1.7 (Beineke, Little [2]). *A bipartite tournament is even pancyclic if it is Hamiltonian and is not isomorphic to the bipartite tournament $B(r, r, r, r)$ with $r \geq 2$.*

If we remove one vertex of each partite set in the bipartite tournament $B(r, r, r, r)$, then obviously the remaining bipartite tournament is not Hamiltonian. The case that $c = 3$ is also solved if we pay attention to the next result.

Theorem 1.8 (Volkman [14]). *Let D be a regular 3-partite tournament with $|V(D)| \geq 6$. Then D contains two complementary cycles of length 3 and $|V(D)| - 3$, unless D is isomorphic to the digraph $D_{3,2}$ of Fig. 1.*

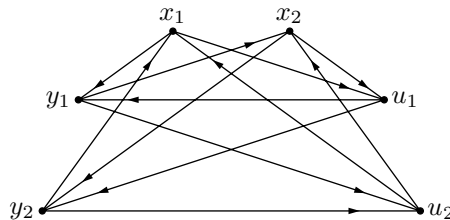


Figure 1. The 2-regular 3-partite tournament $D_{3,2}$

Since a 3-cycle contains vertices of exactly 3-partite sets and the digraph $D_{3,2}$ contains the cycle $x_2y_2u_2x_2$, we see that a regular 3-partite tournament with r vertices of each partite set always contains a cycle with exactly $r - 1$ vertices of every partite set.

In the following, we will show that all regular c -partite tournaments with r vertices in every partite set contain a cycle with exactly $r - 1$ vertices of each partite set provided $c \geq 5$ or $c = 4$ and $r \geq 3$.

2. PRELIMINARY RESULTS

The following results play an important role in our investigations.

Theorem 2.1 (Rédei [10]). *Every tournament contains a Hamiltonian path.*

Theorem 2.2 (Yeo [16]). *If D is a multipartite tournament with $\kappa(D) \geq \alpha(D)$, then D is Hamiltonian.*

Theorem 2.3 (Camion [4]). *A tournament is strongly connected if and only if it is Hamiltonian.*

Theorem 2.4 (Yeo [16]). *Let D be a $(\lfloor q/2 \rfloor + 1)$ -connected c -partite tournament such that $\alpha(D) \leq q$. If D has a cycle-factor, then D is Hamiltonian.*

Theorem 2.5 (Yeo [18]). *Let V_1, V_2, \dots, V_c be the partite sets of a c -partite tournament D such that $|V_1| \leq |V_2| \leq \dots \leq |V_c|$. If*

$$i_g(D) \leq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 2}{2},$$

then D is Hamiltonian.

Lemma 2.6 (Yeo [17], Gutin, Yeo [8]). *A digraph D has no cycle-factor if and only if its vertex set $V(D)$ can be partitioned into four subsets Y, Z, R_1 and R_2 such that*

$$R_1 \rightsquigarrow Y \quad \text{and} \quad (R_1 \cup Y) \rightsquigarrow R_2,$$

where Y is an independent set and $|Y| > |Z|$.

Lemma 2.7 (Tewes, Volkmann, Yeo [11]). *If V_1, V_2, \dots, V_c are the partite sets of a c -partite tournament D , then $||V_i| - |V_j|| \leq 2i_g(D)$ for $1 \leq i, j \leq c$.*

Since we consider only the case that $i_g(D) = 0$ in this article, we can note the following.

Remark 2.8. Let V_1, V_2, \dots, V_c be the partite sets of a regular c -partite tournament. Then Lemma 2.7 implies that $r = |V_1| = |V_2| = \dots = |V_c|$ and

$$d^+(x), d^-(x) = \frac{(c-1)r}{2}$$

for all $x \in V(D)$. That means especially that c is odd, if r is odd.

3. MAIN RESULTS

Theorem 3.1. *Let V_1, V_2, \dots, V_c be the partite sets of a regular c -partite tournament D with $c \geq 4$ and $|V_1| = |V_2| = \dots = |V_c| = r \geq 2$. Furthermore, let X be an arbitrary subset of $V(D)$ consisting of m partite sets with exactly k vertices and $c - m$ partite sets with exactly $k - 1$ vertices for $0 < m \leq c$ and $1 \leq k \leq r - 1$. If*

$$r \geq \begin{cases} \left\lceil \frac{2k(c-1)-2}{c-3} \right\rceil + k & \text{and } m = c, \\ \left\lceil \frac{2k(c-1)-1}{c-3} \right\rceil + k & \text{and } m = c-1, \\ \left\lceil \frac{(2k-3)c+3m-2k+3}{c-3} \right\rceil + k & \text{and } m \leq c-2 \end{cases}$$

then D contains a cycle C such that $V(C) = V(D) - X$.

Proof. Let $D' = D - X$ with the partite sets V'_1, V'_2, \dots, V'_c such that $|V'_1| \leq |V'_2| \leq \dots \leq |V'_c| \leq |V'_1| + 1$. Since D is regular, it follows that

$$i_g(D') \leq \begin{cases} k(c-1) & \text{if } c-1 \leq m \leq c, \\ (k-1)(c-1) + m & \text{if } m \leq c-2. \end{cases}$$

If

$$\begin{cases} k(c-1) \\ (k-1)(c-1) + m \end{cases} \leq \frac{|V(D')| - |V'_{c-1}| - 2|V'_c| + 2}{2} \begin{cases} \text{if } c-1 \leq m \leq c, \\ \text{if } m \leq c-2, \end{cases}$$

then Theorem 2.5 implies that D' is Hamiltonian, and hence the desired result. To show this, let us note that

$$\frac{|V(D')| - |V'_{c-1}| - 2|V'_c| + 2}{2} = \begin{cases} \frac{(c-3)(r-k)+2}{2} & \text{if } m = c, \\ \frac{(c-3)(r-k)+1}{2} & \text{if } m = c-1, \\ \frac{(c-3)(r-k)+c-m-1}{2} & \text{if } m \leq c-2. \end{cases}$$

If we distinguish the cases $m = c$, $m = c - 1$ and $m \leq c - 2$, then, noticing that $r \in \mathbb{N}$, equivalent transformations yield the bounds for r as in the assumptions of this theorem. This completes the proof of the theorem. \square

In the following, we will treat only the case that $m = c$ and $k = 1$. In this case Theorem 3.1 leads to the next corollary.

Corollary 3.2. *Let V_1, V_2, \dots, V_c be the partite sets of a regular c -partite tournament D such that $|V_1| = |V_2| = \dots = |V_c| = r$. Furthermore, let $x_i \in V_i$ be arbitrary for all $1 \leq i \leq c$. If $c \geq 5$ and $r \geq 4$ or $c = 4$ and $r \geq 6$, then there exists a cycle C in D such that $V(C) = \bigcup_{i=1}^c (V_i - x_i)$.*

The next example shows that the condition of Corollary 3.2 that $r \geq 4$, if $c \geq 5$, is the best possible.

Example 3.3. Let D be a regular $(2p+1)$ -partite tournament with $r = 3$ vertices in each partite set. If D consists of three regular disjoint subtournaments H_1, H_2, H_3 of order $2p+1$ such that $H_1 \rightsquigarrow H_2 \rightsquigarrow H_3 \rightsquigarrow H_1$, then $D' = D - V(H_1)$ contains no Hamiltonian cycle.

Nevertheless, if $r = 3$ and thus, according to Remark 2.8, $c = 2p+1$, then there exist vertices x_1, x_2, \dots, x_c with $x_i \in V_i$ such that D contains a cycle C with $V(C) = \bigcup_{i=1}^c (V_i - x_i)$, as the following theorem demonstrates.

Theorem 3.4. *Let $V_1, V_2, \dots, V_{2p+1}$ be the partite sets of a regular $(2p+1)$ -partite tournament with $p \geq 2$ such that $|V_1| = |V_2| = \dots = |V_{2p+1}| = 3$. Then D contains a cycle with exactly 2 vertices of each partite set.*

Proof. Suppose that D contains no cycle with exactly 2 vertices of each partite set. Let T_1 be a subtournament of D with $|V(T_1)| = 2p+1$. Then we define $D' = D - V(T_1)$. Since D is regular, Remark 2.8 with $r = 3$ implies $d^+(x), d^-(x) = 3p$ and thus $d_{D'}^+(x), d_{D'}^-(x) \geq p$.

First, let D' be 2-connected. Because of $\alpha(D') = 2$, Theorem 2.2 yields that D' is Hamiltonian, a contradiction.

Secondly, let D' be not strong. Then D' can be partitioned into strong components D_1, D_2, \dots, D_t such that $D_i \rightsquigarrow D_j$ for $i < j$. The fact that $d_{D_1}^-(x) \geq p$ for all $x \in V(D_1)$ implies $|V(D_1)| \geq 2\delta_{D_1}^- + 1 \geq 2p+1$. Analogously, we observe that $|V(D_t)| \geq 2p+1$. Since $|V(D_1)| + |V(D_2)| + \dots + |V(D_t)| = 4p+2$, we deduce that $t = 2$ and $|D_1| = |D_2| = 2p+1$. This is possible only if $D_2 \rightsquigarrow T_1 \rightsquigarrow D_1$ and D_1, D_2, T_1 are regular tournaments. Hence, D is the multipartite tournament from Example 3.3. If $a_1 a_2 \dots a_{2p+1} a_1$ is a Hamiltonian cycle of T_1 , $v_1 \in V(D_1) \cap V(a_1)$ and $b_1 b_2 \dots b_{2p+1} b_1$ is a Hamiltonian cycle of D_2 such that $b_1 \in V(a_1)$, then $a_1 a_2 \dots a_{2p+1} v_1 b_2 b_3 \dots b_{2p+1} a_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction.

Thirdly, let D' be exactly 1-connected. This yields that D' contains a cut-vertex u such that $D' - \{u\}$ consists of strong components D_1, D_2, \dots, D_t with the property that $D_i \rightsquigarrow D_j$ for $i < j$. Furthermore, there are vertices $v_1 \in V(D_1)$ and $v_t \in V(D_t)$

such that $v_t \rightarrow u \rightarrow v_1$. Since $d_{D_1}^-(x) \geq p - 1$ for all $x \in V(D_1)$, we conclude that $|V(D_1)| \geq 2\delta_{D_1}^- + 1 \geq 2p - 1$. Analogously, we see that $|V(D_t)| \geq 2p - 1$. Without loss of generality, let $|V(D_1)| \leq |V(D_t)|$, since otherwise we use the converse D^{-1} of D . Now we distinguish the two possible cases $|V(D_1)| = 2p - 1$ and $|V(D_1)| = 2p$.

Case 1. Suppose that $|V(D_1)| = 2p - 1$. This is possible only if D_1 is a $(p - 1)$ -regular tournament with $u \rightarrow V(D_1)$, $V(T_1) \rightsquigarrow V(D_1)$ and $2p - 1 \leq |V(D_t)| \leq 2p + 2$. Let $C = a_1 a_2 \dots a_{2p-1} a_1$ be a Hamiltonian cycle of D_1 .

Subcase 1.1. Let $|V(D_t)| = 2p - 1$. As above, we deduce that D_t is a regular tournament with a Hamiltonian cycle $\tilde{C} = b_1 b_2 \dots b_{2p-1} b_1$ such that $V(D_t) \rightsquigarrow V(T_1)$ and $V(D_t) \rightarrow u$. The fact that $|V(D_2)| + |V(D_3)| + \dots + |V(D_{t-1})| = 3$ implies that $t = 3$ or $t = 5$.

First, let $t = 3$. In this case, D_2 is a 3-cycle $c_1 c_2 c_3 c_1$. Without loss of generality, we may suppose that $a_{2p-1} \notin V(c_1)$ and $b_1 \notin V(c_3)$. Now, $a_1 a_2 \dots a_{2p-1} c_1 c_2 c_3 b_1 b_2 \dots b_{2p-1} u a_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction.

Secondly, let $t = 5$. This yields that $|D_2| = |D_3| = |D_4| = 1$ so that $D_2 = \{v_2\}$, $D_3 = \{v_3\}$ and $D_4 = \{v_4\}$. If $v_2 \notin V(v_3)$ and $v_3 \notin V(v_4)$, then the vertices of $V(C)$ and $V(\tilde{C})$ can be chosen so that $a_{2p-1} \notin V(v_2)$ and $b_1 \notin V(v_4)$. Now, $a_1 a_2 \dots a_{2p-1} v_2 v_3 v_4 b_1 b_2 \dots b_{2p-1} u a_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction. If $v_2 \in V(v_3)$ and $v'_2 \in V(T_1) \cap V(v_3)$, then, without loss of generality, the numbering of the cycles C and \tilde{C} can be chosen so that $v_4 \notin V(b_2)$ and $a_{2p-1} \notin V(b_1)$. In this case we see that $b_1 u a_1 v_3 v_4 b_2 b_3 \dots b_{2p-1} v'_2 a_2 a_3 \dots a_{2p-1} b_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction. Analogously, we arrive at a contradiction if $v_3 \in V(v_4)$.

Subcase 1.2. Assume that $|V(D_t)| = 2p$ and thus $t = 4$ and $|V(D_2)| = |V(D_3)| = 1$. Let $D_2 = \{v_2\}$ and $D_3 = \{v_3\}$.

Subcase 1.2.1. Suppose that D_4 is Hamiltonian with a Hamiltonian cycle $C' = b_1 b_2 \dots b_{2p} b_1$.

First, let $v_2 \notin V(v_3)$. Because of

$$2p^2 \leq \sum_{x \in V(D_4)} d_{D'}^+(x) \leq \frac{2p(2p-1)}{2} + d(D_4, u) = 2p^2 - p + d(D_4, u),$$

we deduce that $d(D_4, u) \geq p \geq 2$. Hence, there exists a vertex $b_i \in V(C')$ such that $b_i \rightarrow u$ and $b_i^+ \notin V(v_3)$. Now the vertices of C can be numbered so that $a_{2p-1} \notin V(v_2)$ and $a_1 a_2 \dots a_{2p-1} v_2 v_3 b_i^+ \dots b_i^- b_i u a_1$ is a Hamiltonian cycle of D' , a contradiction.

Secondly, let $v_2 \in V(v_3)$. This implies that D_4 is a tournament. Let $v'_2 \in V(T_1) \cap V(v_3)$. If $v'_2 \rightarrow D_4$, then we observe that $d_{D_4}^+(y) \geq 3p - (|V(T_1)| - 2) - |\{u\}| = p$ for all $y \in V(D_4)$, and thus

$$2p^2 - p = |E(D_4)| \geq 2p^2,$$

a contradiction. Let $\{v'_2, u\} = \{x, y\}$ and $x \rightarrow y$. If $y' \in N^-(x) \cap V(D_4)$, then let $y'b_2b_3 \dots b_{2p}y'$ be a Hamiltonian cycle of D_4 . Summarizing our results, we see that $a_1a_2 \dots a_{2p-1}v_2b_2b_3 \dots b_{2p}y'xya_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction.

Subcase 1.2.2. Let D_4 be not Hamiltonian. Since D_4 is strongly connected, Theorem 2.3 implies that D_4 is no tournament. The fact that D_1 is a tournament and $u \rightarrow D_1$ yields that D_4 consists of vertices of exactly $2p - 1$ partite sets, and thus $v_2 \notin V(v_3)$.

Let $x \in V(T_1)$ be arbitrary. Then we observe that

$$\begin{aligned} 6p^2 &= \sum_{y \in V(D_4)} d^+(y) \leq \sum_{y \in V(D_4)} d_{D_4}^+(y) + d(D_4, u) + d(D_4, T_1) \\ &\leq 2p^2 - p - 1 + 2p - |V(u) \cap V(D_4)| - |N^+(u) \cap V(D_4)| \\ &\quad + 4p^2 - |N^+(x) \cap V(D_4)|, \end{aligned}$$

and it follows that

$$(1) \quad |V(u) \cap V(D_4)| + |N^+(x) \cap V(D_4)| + |N^+(u) \cap V(D_4)| \leq p - 1.$$

Theorem 1.2 implies that D_4 contains a cycle C' with vertices of all the $2p - 1$ partite sets of D_4 , and thus $L(C') = 2p - 1$. Let $\{v_4\} = V(D_4) - V(C')$ and $v'_4 \in V(T_1) \cap V(v_4)$. If $C' = b_1b_2 \dots b_{2p-1}b_1$ then, according to (1), there are at least $|V(C')| - (p - 1) = p \geq 2$ vertices $b_i, b_j \in V(C') - (V(u) \cup N^-(v'_4) \cup N^-(u))$ such that $\{b_i, b_j\} \rightarrow u$ and $\{b_i, b_j\} \rightsquigarrow v'_4$. Let $b_j \rightarrow v'_4$. If $v_3 \notin V(b_i^+)$, then the vertices of C can be numbered so that $a_{2p-1} \notin V(b_j^+)$ and $a_{2p-2} \notin V(v_2)$, and we see that $a_1a_2 \dots a_{2p-2}v_2v_3b_i^+ \dots b_jv'_4a_{2p-1}b_j^+ \dots b_i^-b_iua_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction. If $v_3 \in V(b_i^+)$ and thus $v_2 \notin V(b_i^+)$, $v_3 \notin V(b_j^+)$ and $V(C) \rightarrow v_3$, then the vertices of C can be numbered so that $a_{2p-2} \notin V(v_2)$. This implies that $a_1a_2 \dots a_{2p-2}v_2b_i^+ \dots b_jv'_4a_{2p-1}v_3b_j^+ \dots b_i^-b_iua_1$ is a cycle with exactly two vertices from every partite set, again a contradiction.

Subcase 1.3. Assume that $|V(D_t)| = 2p + 1$. This implies $t = 3$ and $|V(D_2)| = 1$. Let $V(D_2) = \{v_2\}$.

Subcase 1.3.1. Suppose that D_3 is Hamiltonian with a Hamiltonian cycle $C' = b_1b_2 \dots b_{2p+1}b_1$. Let $u' \in V(T_1) \cap V(u)$. If $|N^-(u) \cap V(D_3)| = 1$ and $|N^-(u') \cap V(D_3)| \leq 1$, then we conclude that $d(D_3, u) \leq 1$ and $|N^+(u') \cap V(D_3)| \geq 2p - 1$, and thus

$$\begin{aligned} (2p + 1)3p &= \sum_{y \in V(D_3)} d^+(y) = \sum_{y \in V(D_3)} d_{D_3}^+(y) + d(D_3, u) + d(D_3, T_1) \\ &\leq 2p^2 + p + 1 + (2p + 1)2p - (2p - 1) = 6p^2 + p + 2, \end{aligned}$$

a contradiction to $p \geq 2$. Hence, it follows that $|N^-(u) \cap V(D_3)| \geq 2$ or $|N^-(u') \cap V(D_3)| \geq 2$. If $|N^-(u) \cap V(D_3)| \geq 2$, then the vertices of C' can be numbered so that $b_{2p+1} \rightarrow u$ and $b_1 \notin V(v_2)$. Let $a_{2p-1} \notin V(v_2)$. Then $a_1 a_2 \dots a_{2p-1} v_2 b_1 b_2 \dots b_{2p+1} u a_1$ is a cycle with exactly 2 vertices of every partite set, a contradiction. Analogously, the case that $|N^-(u') \cap V(D_3)| \geq 2$ leads to a contradiction.

Subcase 1.3.2. Let D_3 be not Hamiltonian. Since D_3 is strongly connected, Theorem 2.3 implies that D_3 is not a tournament. Since $\{u\} \cup V(D_1)$ consists of $2p$ partite sets, it follows that D_3 consists of the vertices of exactly $2p$ partite sets and $v_2 \rightarrow D_3$. Analogously to Subcase 1.2.2, we see that

$$(2) \quad |V(u) \cap V(D_3)| + |N^+(x) \cap V(D_3)| + |N^+(u) \cap V(D_3)| \leq 2p$$

for an arbitrary vertex $x \in V(T_1)$. Theorem 1.2 implies that D_3 contains a cycle C' with the vertices of all the $2p$ partite sets of D_3 .

Hence, let $L(C') = 2p$ for $C' = b_1 b_2 \dots b_{2p} b_1$. Let us define $\{v_3\} = V(D_3) - V(C')$ and $\{v'_3\} = V(T_1) \cap V(v_3)$.

Assume that $u \rightsquigarrow C'$. Since $N^-(u) \cap V(D_3) \neq \emptyset$, it follows that $v_3 \rightarrow u$. Furthermore, (2) yields that $C' \rightsquigarrow v'_3$. If $\{\tilde{v}_3\} = V(C') \cap V(v_3)$, then let the vertices of C' be numbered so that $a_{2p-1} \notin V(v_2)$. In this case $a_1 v_3 u a_2 a_3 \dots a_{2p-1} v_2 \tilde{v}_3^+ \dots \tilde{v}_3^- v'_3 a_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction. Hence, let $N^-(u) \cap V(C') \neq \emptyset$ and $b_i \rightarrow u$.

Suppose now that $v'_3 \rightsquigarrow C'$. This yields that $v_2 \rightarrow v'_3$, since otherwise we observe that

$$3p = d^+(v'_3) \geq |V(D_1)| + |V(C')| - 1 + |\{v_2\}| = 4p - 1,$$

a contradiction to $p \geq 2$. If the vertices of C' are numbered so that $a_{2p-2} \notin V(v_2)$ and $a_{2p-1} \notin V(b_i^+)$, then $a_1 a_2 \dots a_{2p-2} v_2 v'_3 a_{2p-1} b_i^+ \dots b_i^- b_i u a_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction. Consequently, let $N^-(v'_3) \cap V(C') \neq \emptyset$.

Let $\{x, y\} = \{u, v'_3\}$ such that $x \rightarrow y$ and $b \in V(C')$ with $b \rightarrow x$. If $a_{2p-1} \notin V(v_2)$, then we see that $a_1 a_2 \dots a_{2p-1} v_2 b^+ \dots b^- b x y a_1$ is a cycle with exactly 2 vertices of each partite set, a contradiction.

Subcase 1.4. Assume that $|V(D_t)| = 2p + 2$. This implies that $t = 2$.

Subcase 1.4.1. Suppose that D_2 is Hamiltonian with a Hamiltonian cycle $C' = b_1 b_2 \dots b_{2p+2} b_1$. It is easy to see that the vertices of C and C' can be numbered so that $b_{2p+2} \rightarrow u$ and $a_{2p-1} \notin V(b_1)$. Now, we observe that $a_1 a_2 \dots a_{2p-1} b_1 b_2 \dots b_{2p+2} u a_1$ is a Hamiltonian cycle of D' , a contradiction.

Subcase 1.4.2. Let D_2 be not Hamiltonian. Since D_1 is a tournament and $u \notin V(x)$ for all $x \in V(D_1)$, we conclude that D_2 contains the vertices of exactly $2p + 1$ partite sets. Theorem 1.2 implies that D_2 contains a cycle C' with vertices of all the $2p + 1$

partite sets of D_2 . If $L(C') = 2p + 2$, then D_2 is Hamiltonian and Subcase 1.4.1 yields a contradiction.

Consequently, it remains to consider the case that $L(C') = 2p + 1$ with $C' = b_1 b_2 \dots b_{2p+1} b_1$. Let us define $\{v_2\} = V(D_2) - V(C')$ and $\{v'_2\} = V(T_1) \cap V(v_2)$. If $N^-(u) \cap V(C') = \emptyset$, then we observe that

$$3p = d^+(u) \geq |V(D_1)| + |V(C')| - 1 = 2p - 1 + 2p = 4p - 1,$$

a contradiction to $p \geq 2$. Hence, there exists a vertex $b_i \in V(C')$ such that $b_i \rightarrow u$. Analogously, we see that there exists a vertex $b_j \in V(C')$ such that $b_j \rightarrow v'_2$.

Noticing that either $u \rightarrow v'_2$ or $v'_2 \rightarrow u$ and that the vertices of C can be numbered so that $a_{2p-1} \notin V(b_i^+) \cup V(b_j^+)$, we observe that either $a_1 a_2 \dots a_{2p-1} b_i^+ \dots b_i^- b_i u v'_2 a_1$ or $a_1 a_2 \dots a_{2p-1} b_j^+ \dots b_j^- b_j v'_2 u a_1$ is a cycle of D with exactly 2 vertices of every partite set, a contradiction.

Case 2. Assume that $|V(D_1)| = 2p$. This implies that $|V(D_t)| = 2p$ and $t = 3$ or $|V(D_t)| = 2p + 1$ and $t = 2$. Let D_1 consist of the vertices of exactly k partite sets with $p \leq k \leq 2p$. It follows that

$$\begin{aligned} 6p^2 &= \sum_{y \in V(D_1)} d^-(y) = \sum_{y \in V(D_1)} d_{D_1}^-(y) + d(u, D_1) + d(T_1, D_1) \\ &\leq \frac{2(2p-k)(2p-2) + (2k-2p)(2p-1)}{2} + 2p - |V(u) \cap V(D_1)| \\ &\quad - |N^-(u) \cap V(D_1)| + 4p^2 - \sum_{x \in V(T_1)} |N^-(x) \cap V(D_1)| \\ &= 6p^2 + k - p - |V(u) \cap V(D_1)| - |N^-(u) \cap V(D_1)| \\ &\quad - \sum_{x \in V(T_1)} |N^-(x) \cap V(D_1)|, \end{aligned}$$

and thus

$$(3) \quad |N^-(u) \cap V(D_1)| + \sum_{x \in V(T_1)} |N^-(x) \cap V(D_1)| \leq k - p - |V(u) \cap V(D_1)|.$$

Let $y_1 \in V(D_2) \cup \dots \cup V(D_t)$ be an arbitrary vertex. We will show that there exists a Hamiltonian path in $D_1 \cup \{u, y_1\}$ with the initial vertex u and the terminal vertex y_1 . Suppose that this is not true.

First, let D_1 be Hamiltonian with a Hamiltonian cycle $C = a_1 a_2 \dots a_{2p} a_1$. If $|N^+(u) \cap V(D_1)| \geq 2$, then, without loss of generality, let $u \rightarrow a_1$ and $a_{2p} \notin V(y_1)$. But now $u a_1 a_2 \dots a_{2p} y_1$ is a Hamiltonian path in $D_1 \cup \{u, y_1\}$, a contradiction. Hence,

let $|N^+(u) \cap V(D_1)| = 1$. Together with (3), this implies

$$\begin{aligned} p - |V(u) \cap V(D_1)| &\geq k - p - |V(u) \cap V(D_1)| \\ &\geq |N^-(u) \cap V(D_1)| \geq 2p - 1 - |V(u) \cap V(D_1)|, \end{aligned}$$

a contradiction to $p \geq 2$.

Secondly, let D_1 be not Hamiltonian, and thus, according to Theorem 2.3, $k \neq 2p$. Theorem 1.2 implies that D_1 contains a cycle with vertices of all the k partite sets. Let $C = a_1 a_2 \dots a_l a_1$ be a cycle which fulfils this condition and which has the maximal cardinality of all cycles that contain vertices of all the k partite sets of D_1 . If $L(C) = 2p$, then D_1 is Hamiltonian and as above, we arrive at a contradiction. Hence, let $L(C) < 2p$ and $T'_1 = D_1 - V(C)$. It is obvious that T'_1 is a tournament and, according to Theorem 2.1, T'_1 contains a Hamiltonian path $P = b_1 b_2 \dots b_{2p-l}$.

If $|N^+(u) \cap V(C)| \leq p - 1$, then it follows that $|N^-(u) \cap V(C)| \geq k - p + 1 - |V(u) \cap V(C)|$, a contradiction to (3). Hence, we conclude that

$$(4) \quad |N^+(u) \cap V(C)| \geq p \ (\geq 2).$$

Let $u \rightarrow a_i$. If $a_i^- \rightarrow b_1$ then, noticing that $P \rightarrow y_1$, $u a_i a_i^+ \dots a_i^- b_1 b_2 \dots b_{2p-l} y_1$ is a Hamiltonian path in $D_1 \cup \{u, y_1\}$, a contradiction. Consequently, let $b_1 \rightsquigarrow a_i^-$. Suppose that $b_j \rightarrow b_1$ for some $j \geq 3$. Let $j_{\max} = \max\{j \geq 3: b_j \rightarrow b_1\}$.

First, let $a_i^- \in V(b_1)$. If $b_1 \rightarrow a_{i-2}$, then, because of the maximality of C , we deduce that $b_1 \rightsquigarrow C$, and thus

$$p - 1 \leq d_{D_1}^-(b_1) \leq |V(T'_1)| - 2 \quad \Rightarrow \quad |V(T'_1)| \geq p + 1,$$

a contradiction. Hence, let $a_{i-2} \rightarrow b_1$. Now, the maximality of C implies that $a_i^- \rightarrow \{b_2, b_3, \dots, b_{2p-l}\}$. If $j_{\max} \neq 2p-l$ then $u a_i a_i^+ \dots a_i^- b_2 b_3 \dots b_{j_{\max}} b_1 b_{j_{\max}+1} \dots b_{2p-l} y_1$ is a Hamiltonian path of $D_1 \cup \{u, y_1\}$ and if $j_{\max} = 2p-l$ then $u a_i a_i^+ \dots a_i^- b_2 b_3 \dots b_{2p-l} b_1 y_1$ is a Hamiltonian path of $D_1 \cup \{u, y_1\}$, in both cases a contradiction.

Consequently, it remains to consider the case that $b_1 \rightarrow a_i^-$. If $a_p \in V(b_1) \cap V(C)$, then the maximality of C implies that $b_1 \rightarrow \{a_{p+1}, a_{p+2}, \dots, a_i^-\}$ and thus $p \neq i$. If $b_1 \rightarrow a_{p-1}$, then analogously as above we see that $b_1 \rightsquigarrow C$, a contradiction. Again the maximality of C yields that $a_p \rightarrow \{b_2, b_3, \dots, b_{2p-l}\}$. If $j_{\max} \neq 2p-l$, then $b_2 b_3 \dots b_{j_{\max}} b_1 b_{j_{\max}+1} \dots b_{2p-l}$ is a Hamiltonian path of T'_1 and if $j_{\max} = 2p-l$, then $b_2 b_3 \dots b_{2p-l} b_1$ is a Hamiltonian path of T'_1 . Both the Hamiltonian paths have the initial vertex b_2 . Analogously as above, we see that $b_2 \rightarrow a_i^-$, $b_2 \rightarrow \{a_{q+1}, a_{q+2}, \dots, a_i^-\}$ and $a_q \rightarrow \{b_1, b_3, b_4, \dots, b_{2p-l}\}$ if $a_q \in V(b_2) \cap V(C)$ ($q \neq i$). Without loss of generality, we may suppose that $i > q > p$ (modulo l). But now, the fact that $a_q \rightarrow b_1$ and $b_1 \rightarrow \{a_{p+1}, a_{p+2}, \dots, a_{i-1}\}$ yields a contradiction.

Summarizing our results, we see that $b_1 \rightarrow \{b_2, b_3, \dots, b_{2p-l}\}$. Now, suppose that $u \rightarrow b_1$. Let $a_w \in V(y_1) \cap V(C)$ (or $a_w \in V(C) - V(b_{2p-l})$ arbitrary if $V(C) \cap V(y_1) = \emptyset$). Then it follows that $a_w \rightarrow b_{2p-l}$, since otherwise $ub_1b_2 \dots b_{2p-l}a_w a_w^+ \dots a_w^- y_1$ is a Hamiltonian path in $D_1 \cup \{u, y_1\}$, a contradiction. The maximality of C implies that $a_{w+1} \rightsquigarrow b_{2p-l}$. If $m \notin \{1, 2, \dots, l\} - \{w, w+1\}$ and $b_{2p-l} \rightarrow a_m$, then $ub_1b_2 \dots b_{2p-l}a_m a_m^+ \dots a_m^- y_1$ is a Hamiltonian path of $D_1 \cup \{u, y_1\}$, a contradiction. Altogether, we have $C \rightsquigarrow b_{2p-l}$. If $a_n \in V(b_{2p-l})$, then we conclude that $a_n \rightarrow b_{2p-l-1}$, since otherwise

$$(5) \quad ub_1b_2 \dots b_{2p-l-1}a_n a_n^+ \dots a_n^- b_{2p-l}y_1$$

is a Hamiltonian path in $D_1 \cup \{u, y_1\}$. The maximality of D_1 yields that $a_{n+1} \rightsquigarrow b_{2p-l-1}$. To get no contradiction as in (5), we deduce that $C \rightsquigarrow b_{2p-l-1}$. Successively, it follows that $C \rightsquigarrow \{b_1, b_2, \dots, b_{2p-l}\}$, a contradiction to the strong connectivity of D_1 .

Consequently, let $b_1 \rightarrow u$ and thus $d_{D_1}^-(b_1) \geq p$. Furthermore, using (4) and the results above, we conclude that

$$\begin{aligned} |N_{D_1}^+(b_1)| &\geq |N^+(u) \cap V(C)| - |V(b_1) \cap V(C)| + |V(T_1') - \{b_1\}| \\ &\geq |N^+(u) \cap V(C)| - 1 \geq p - 1. \end{aligned}$$

Altogether, we arrive at the contradiction

$$2p = |V(D_1)| = d_{D_1}^+(b_1) + d_{D_1}^-(b_1) + 2 \geq 2p + 1.$$

Hence, for an arbitrary vertex $y_1 \in V(D_1) \cup \dots \cup V(D_t)$ there exists a Hamiltonian path of $D_1 \cup \{u, y_1\}$ with the initial vertex u and the terminal vertex y_1 .

Subcase 2.1. Assume that $|V(D_t)| = 2p$ and thus $t = 3$ and $D_2 = \{v_2\}$. Observing the converse D^{-1} of D , we see that for an arbitrary vertex $y_2 \in V(D_1) \cup V(D_2)$ there exists a Hamiltonian path of $D_2 \cup \{u, y_2\}$ with the initial vertex y_2 and the terminal vertex u . Choosing $y_1 = y_2 = v_2$, we get a Hamiltonian cycle of D' , a contradiction.

Subcase 2.2. Suppose that $|V(D_t)| = 2p + 1$ and thus $t = 2$. According to (3), we have

$$\sum_{x \in V(T_1)} |N^-(x) \cap V(D_1)| \leq k - p.$$

We conclude that there are at least $k - (k - p) = p \geq 2$ vertices in $V(T_1)$ belonging to the partite sets represented in $V(D_1)$ such that they (weakly) dominate D_1 . Hence, let $w_1 \in V(T_1)$ with $w_1 \rightsquigarrow D_1$ and $x_1 \in V(D_1) \cap V(w_1)$. Let $D'' = [D' \cup \{w_1\}] - \{x_1\}$. Assume that there is a vertex $x \in V(D'')$ such that $d_{D''}^+(x) \leq p - 1$ or $d_{D''}^-(x) \leq p - 1$.

This yields the contradiction $3p = d_D^+(x), d_D^-(x) \leq p-1 + |V(T_1)| - 1 = 3p-1$. Hence, let $d_{D''}^+(x), d_{D''}^-(x) \geq p \geq 2$ for all $x \in V(D'')$, and thus

$$(6) \quad d(D_2, w_1) \geq p - 1 \geq 1.$$

If $D_1 - \{x_1\}$ is not strongly connected, then let $D'_1, D'_2, \dots, D'_{t'}$ be the strong components of $D_1 - \{x_1\}$ such that $D'_i \rightsquigarrow D'_j$ for $i < j$. If $D'_1 \rightsquigarrow u$, then it follows that $d_{D'_1}^-(y) \geq p - 1$ for all $y \in V(D'_1)$, and thus $|V(D'_1)| \geq 2p - 1$, a contradiction to $|V(D_1)| = 2p$. Consequently, we may assume that there is a vertex $y \in D'_1$ such that $u \rightarrow y$ provided $D_1 - \{x_1\}$ is not strongly connected. If $D_1 - \{x_1\}$ is strongly connected and $D_1 - \{x_1\} \rightsquigarrow u$, then we see that

$$2p - 1 \leq |V(u) \cap V(D_1)| + |N^-(u) \cap V(D_1)| \leq k - p \Rightarrow 3p - 1 \leq k \leq 2p,$$

a contradiction to $p \geq 2$. Consequently, we observe that there is a vertex $y \in V(D_1) - \{x_1\}$ such that $u \rightarrow y$ provided $D_1 - \{x_1\}$ is strong.

The above results guarantee that D'' is strong. If D'' is 2-connected, then Theorem 2.2 yields that D'' is Hamiltonian, a contradiction. Hence D'' is exactly 1-connected. Obviously, the vertices u and w_1 are no cut-vertices of D'' . Since $D_1 - \{x_1\} \rightsquigarrow D_2$, $x_1 \in V(w_1)$, $N^-(w_1) \cap V(D_2) \neq \emptyset$, $N^-(u) \cap V(D_2) \neq \emptyset$, $w_1 \rightsquigarrow D_1 - \{x_1\}$ and $d_{D''}^+(x), d_{D''}^-(x) \geq 2$ for all $x \in V(D'')$ furthermore, there is also no cut-vertex of D'' in $D_1 - \{x_1\}$. Hence, let $x' \in V(D_2)$ be a cut-vertex of D'' . Because of (6), $N^+(u) \cap V(D'_1) \neq \emptyset$, $w_1 \rightsquigarrow D_1$, $d_{D''}^-(u) \geq 2$ and $d_{D''}^-(w_1) \geq 2$, so if the vertex x' were no cut-vertex of D_2 , then necessarily $x' \rightarrow \{w_1, u\} \rightsquigarrow D_2 - \{x'\} =: \hat{D}$ and $u \rightarrow w_1$. Since $d_{D''}^-(w_1) \geq p$ this implies that $p = 2$, and thus $|\hat{D}| = 4$. Let $y \in \hat{D} - V(w_1)$ be such that $d_{\hat{D}}^+(y) = 1$. Then we observe that $d^+(y) \leq 1 + |\{x'\}| + |V(T_1)| - 2 = 5$, a contradiction. Hence, x' is a cut-vertex of D_2 . Let $D''_1, D''_2, \dots, D''_{t''}$ be the strong components of $D_2 - \{x'\}$ such that $D''_i \rightsquigarrow D''_j$ for $i < j$.

Suppose that there is a vertex $y \in V(D''_{t''})$ with $y \rightarrow u$. Since $N^+(u) \cap V(D'_1) \neq \emptyset$, $w_1 \rightsquigarrow D_1$ and (6) is valid, we conclude that $D'' - \{x'\}$ is strongly connected, a contradiction. Consequently, let $u \rightsquigarrow D''_{t''}$. This yields that $d_{D''_{t''}}^+(x) \geq 3p - (|V(T_1)| - 1) - |\{x'\}| = p - 1$ for all $x \in V(D''_{t''})$ and thus $|V(D''_{t''})| \geq 2\delta_{D''_{t''}}^+ + 1 \geq 2p - 1$. To get no contradiction, it follows that $t'' = 2$, $|V(D''_2)| = 2p - 1$ and $D''_2 \rightsquigarrow T_1 \cup \{x'\}$. Since $D_1 - \{x_1\} \rightsquigarrow D_2$, $d_{D''}^-(u) \geq 2$ and $w_1 \rightsquigarrow D_1 - \{x_1\}$, we deduce that $D'' - \{x'\}$ is strongly connected, a contradiction.

This completes the proof of the theorem. □

Combining Corollary 3.2 with Theorems 1.6 and 3.4 we can see that we have found a solution of Problem 1.3 for regular multipartite tournaments and $r_i = |V_i| - 1$ for all $1 \leq i \leq c$.

Corollary 3.5. *Let V_1, V_2, \dots, V_c be the partite sets of a regular c -partite tournament D with $c \geq 5$ such that $|V_1| = |V_2| = \dots = |V_c| = r \geq 2$. Then D contains a cycle with exactly $r - 1$ vertices of each partite set.*

Now, we shall prove the main theorem of this article.

Theorem 3.6. *Let V_1, V_2, \dots, V_c be the partite sets of a regular c -partite tournament D with $|V_1| = |V_2| = \dots = |V_c| = r \geq 2$. If $c \geq 5$ or $c = 4$ and $r \geq 4$ or $c = 3$ or $c = 2$ and D is not isomorphic to $B(\frac{1}{2}r, \frac{1}{2}r, \frac{1}{2}r, \frac{1}{2}r)$, then D contains a cycle with exactly $r - 1$ vertices from each partite set.*

Proof. If $c \geq 5$, then Corollary 3.5 yields the desired result. Since, according to Theorem 1.4, D is Hamiltonian, the result for $c = 2$ follows directly from Theorem 1.7 while for the case $c = 3$ we use Theorems 1.1 and 1.8. Hence, let $c = 4$. According to Remark 2.8, r has to be even. If $r \geq 6$, then Corollary 3.2 leads to the desired result.

Consequently, it remains to consider the case that $c = r = 4$. Suppose that D does not contain any cycle with exactly 3 vertices of every partite set. Let T_1 be a subtournament of D of order 4 and $D' = D - V(T_1)$. This implies that $\alpha(D') = 3$. With respect to Remark 2.8, we observe that $d^+(x), d^-(x) = 6$ for all $x \in V(D)$ and $d_{D'}^+(x), d_{D'}^-(x) \geq 3$ for all $x \in V(D')$. Now we distinguish four different cases.

Case 1. Let $\kappa(D') \geq 3$. In this case Theorem 2.2 yields that D' is Hamiltonian, a contradiction.

Case 2. Assume that $\kappa(D') = 0$. Let D_1, D_2, \dots, D_t be the strong components of D such that $D_i \rightsquigarrow D_j$ for $i < j$. Since $d_{D_1}^-(x) \geq 3$ for all $x \in V(D_1)$, we deduce that $|V(D_1)| \geq 7$. Analogously, we conclude that $|V(D_t)| \geq 7$. Hence, we arrive at the contradiction $12 = |V(D')| \geq |V(D_1)| + |V(D_t)| \geq 14$.

Case 3. Suppose that $\kappa(D') = 1$. Let u be a cut-vertex of D' such that $D' - u$ consists of the strong components D_1, D_2, \dots, D_t with $D_i \rightsquigarrow D_j$ for $i < j$. This implies that $d_{D_1}^-(x) \geq 2$ for all $x \in V(D_1)$ and thus, since $c = 4$, we conclude that $|V(D_1)| \geq 6$. Analogously, we observe that $|V(D_t)| \geq 6$, a contradiction to $|V(D')| = 12$.

Case 4. Assume that $\kappa(D') = 2$.

First, let D' contain a cycle-factor. In this case, because of $\alpha(D') = 3$, Theorem 2.4 yields that D' is Hamiltonian, a contradiction.

Secondly, let D' contain no cycle-factor. Now, Lemma 2.6 implies that $V(D')$ can be partitioned into four subsets Y, Z, R_1 and R_2 such that $R_1 \rightsquigarrow Y$ and $(R_1 \cup Y) \rightsquigarrow R_2$, where Y is an independent set and $|Y| > |Z|$.

If $|Z| \leq 1$, then we deduce that $\kappa(D') \leq 1$, a contradiction to $\kappa(D') = 2$. If $|Z| \geq 3$, then Y has to be an independent set with $|Y| \geq 4$, a contradiction to

$\alpha(D') = 3$. Hence, let $|Z| = 2$ and $|Y| = 3$, which means that Y is a partite set of D' . Without loss of generality, let $|R_1| \leq |R_2|$.

Assume that $|R_1| = 0$. This yields that $|R_2| = 7$ and thus $d_{D'}^+(y) \geq 7$ for all $y \in Y$, a contradiction to $d^+(x), d^-(x) = 6$ for all $x \in V(D)$.

Now, let $1 \leq |R_1| \leq 2$. In this case we see that there is a vertex $x \in R_1$ with $d_{D[R_1]}^-(x) = 0$ and thus $d_{D'}^-(x) \leq |Z| = 2$, a contradiction.

Finally, let $|R_1| = 3$. Because $d_{D'}^-(x) \geq 3$ for all $x \in R_1$, we conclude that $D[R_1]$ is a 3-cycle and $Z \rightarrow R_1$. Since $D' - Y$ and R_1 consist of vertices of 3 partite sets, this is impossible. This completes the proof of the theorem. \square

For the case that $c = 4$ and $r = 2$, Theorem 3.6 is not true in general as the following example (see also [12]) demonstrates.

Example 3.7. Let $V_i = \{v'_i, v''_i\}$ for $i = 1, 2, 3, 4$ be the partite sets of a 4-partite tournament such that $v'_1 \rightarrow v'_2 \rightarrow v'_3 \rightarrow v'_1, v''_1 \rightarrow v''_2 \rightarrow v''_3 \rightarrow v''_1$,

$$\begin{aligned} \{v'_1, v'_2, v'_3\} &\rightarrow v'_4 \rightarrow \{v''_1, v''_2, v''_3\} \rightarrow v''_4 \rightarrow \{v'_1, v'_2, v'_3\}, \\ v'_1 &\rightarrow v''_3 \rightarrow v'_2 \rightarrow v''_1 \rightarrow v'_3 \rightarrow v''_2 \rightarrow v'_1 \end{aligned}$$

(see also Fig. 2). Now it is a simple matter to check that the resulting 4-partite tournament is 3-regular without a cycle containing exactly $r - 1 = 1$ vertices of every partite set.

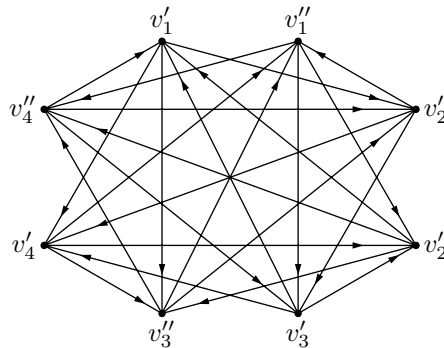


Figure 2. A regular 4-partite tournament without a strong subtournament of order 4

The results of the Theorems 1.4 and 1.6 and Corollary 3.5 lead us to the following conjecture.

Conjecture 3.8. *Let V_1, V_2, \dots, V_c be the partite sets of a regular c -partite tournament D with $c \geq 5$ such that $|V_1| = |V_2| = \dots = |V_c| = r \geq 2$. Then D contains a cycle with exactly m vertices of each partite set for every $m \in \{1, 2, \dots, r\}$.*

Note that, according to Theorem 3.1, Conjecture 3.8 is valid for a given m if c and r are sufficiently large.

References

- [1] *J. Bang-Jensen, G. Gutin: Digraphs: Theory, Algorithms and Applications.* Springer-Verlag, London, 2000. [Zbl 0985.05002](#)
- [2] *L. W. Beineke, C. Little: Cycles in bipartite tournaments.* J. Combinat. Theory Ser. B *32* (1982), 140–145. [Zbl 0465.05035](#)
- [3] *J. A. Bondy: Disconnected orientations and a conjecture of Las Vergnas.* J. London Math. Soc. *14* (1976), 277–282. [Zbl 0344.05124](#)
- [4] *P. Camion: Chemins et circuits hamiltoniens des graphes complets.* C. R. Acad. Sci. Paris *249* (1959), 2151–2152. [Zbl 0092.15801](#)
- [5] *W. D. Goddard, O. R. Oellermann: On the cycle structure of multipartite tournaments.* In: Graph Theory Combinat. Appl. 1 (Y. Alavi, G. Chartrand, O. R. Oellermann, and A. J. Schenk, eds.). Wiley-Interscience, New York, 1991, pp. 525–533. [Zbl 0840.05026](#)
- [6] *Y. Guo: Semicomplete multipartite digraphs: a generalization of tournaments.* Habilitation thesis. RWTH Aachen, 1998.
- [7] *G. Gutin: Cycles and paths in semicomplete multipartite digraphs, theorems and algorithms: a survey.* J. Graph Theory *19* (1995), 481–505. [Zbl 0839.05043](#)
- [8] *G. Gutin, A. Yeo: Note on the path covering number of a semicomplete multipartite digraph.* J. Combinat. Math. Combinat. Comput. *32* (2000), 231–237. [Zbl 0949.05066](#)
- [9] *J. W. Moon: On subtournaments of a tournament.* Canad. Math. Bull. *9* (1966), 297–301. [Zbl 0141.41204](#)
- [10] *L. Rédei: Ein kombinatorischer Satz.* Acta Litt. Sci. Szeged *7* (1934), 39–43. [Zbl 0009.14606](#)
- [11] *M. Tewes, L. Volkmann, and A. Yeo: Almost all almost regular c -partite tournaments with $c \geq 5$ are vertex pancyclic.* Discrete Math. *242* (2002), 201–228. [Zbl 0993.05083](#)
- [12] *L. Volkmann: Strong subtournaments of multipartite tournaments.* Australas. J. Combin. *20* (1999), 189–196. [Zbl 0935.05051](#)
- [13] *L. Volkmann: Cycles in multipartite tournaments: results and problems.* Discrete Math. *245* (2002), 19–53. [Zbl 0996.05063](#)
- [14] *L. Volkmann: All regular multipartite tournaments that are cycle complementary.* Discrete Math. *281* (2004), 255–266. [Zbl 1049.05043](#)
- [15] *L. Volkmann, S. Winzen: Almost regular c -partite tournaments contain a strong subtournament of order c when $c \geq 5$.* Submitted.
- [16] *A. Yeo: One-diregular subgraphs in semicomplete multipartite digraphs.* J. Graph Theory *24* (1997), 175–185. [Zbl 0865.05045](#)
- [17] *A. Yeo: Semicomplete multipartite digraphs.* Ph.D. Thesis. Odense University, 1998.
- [18] *A. Yeo: How close to regular must a semicomplete multipartite digraph be to secure Hamiltonicity?* Graphs Combin. *15* (1999), 481–493. [Zbl 0939.05059](#)

- [19] *K.-M. Zhang*: Vertex even-pancyclicity in bipartite tournaments. Nanjing Daxue Xuebao Shuxue Bannian Kan 1 (1984), 85–88. [Zbl 0552.05029](#)

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