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## IDEAL EXTENSIONS OF GRAPH ALGEBRAS

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*Abstract.* Let  $\mathcal{A}$  and  $\mathcal{B}$  be graph algebras. In this paper we present the notion of an ideal in a graph algebra and prove that an ideal extension of  $\mathcal{A}$  by  $\mathcal{B}$  always exists. We describe (up to isomorphism) all such extensions.

*Keywords:* oriented graph, graph (Shallon) algebra, congruence relation, ideal, quotient graph algebra, ideal extension

*MSC 2000:* 08A30

### 0. INTRODUCTION

In this paper we study congruence relations on graph (Shallon) algebras and introduce the notion of an ideal in the graph algebra determined by a congruence. Then the aim is, for two given graph algebras  $\mathcal{X}$  and  $\mathcal{Y}$ , to construct a graph algebra  $\mathcal{A}$  for which we can find a congruence  $\Theta$  on  $\mathcal{A}$  such that the ideal of  $\mathcal{A}$  determined by the congruence  $\Theta$  is the algebra  $\mathcal{X}$  and the quotient graph algebra  $\mathcal{A}/\Theta$  is isomorphic to the graph algebra  $\mathcal{Y}$ . (The algebra  $\mathcal{A}$  will be referred to as an ideal extension of  $\mathcal{X}$  by  $\mathcal{Y}$ .) Our objective is to answer the following questions:

- (Q1) is the ideal extension always possible?
- (Q2) is it possible to determine all ideal extensions?

We construct a class of ideal extensions of  $\mathcal{X}$  by  $\mathcal{Y}$  and denote it by  $\Gamma_{\mathcal{X},\mathcal{Y}}(Z, \varphi)$ .

The construction itself enables us to answer the question (Q1). Afterwards we show that the class  $\Gamma_{\mathcal{X},\mathcal{Y}}(Z, \varphi)$  contains all ideal extensions.

Similar ideal related extensions were carried out for other algebraic structures, for instance for lattice ordered groups (cf. [7]), semigroups (cf. [1]), ordered semigroups (cf. [2] and [5]), lattices (cf. [4]) and for partial monounary algebras (cf. [3]).

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## 1. PRELIMINARIES

Let  $A$  be a nonempty set,  $E$  a binary relation on  $A$ . Then the corresponding relational structure  $(A, E)$  is called a (directed) graph. We admit the existence of loops in a graph.

To a directed graph  $(A, E)$  there corresponds an algebra  $(A \cup \{\infty\}, \circ, \infty)$  where  $\infty \notin A$  is a nullary operation and for any  $x, y \in A \cup \{\infty\}$ , the binary operation  $\circ$  is defined by the formula

$$x \circ y = \begin{cases} x & \text{if } (x, y) \in E, \\ \infty & \text{otherwise.} \end{cases}$$

Then  $(A \cup \{\infty\}, \circ, \infty)$  is called the graph algebra corresponding to the graph  $(A, E)$ .

**1.1 Definition.** Let  $\mathcal{A} = (A \cup \{\infty\}, \circ, \infty)$  be an algebra such that  $\infty \notin A$  is a nullary operation and  $\circ$  is a binary operation satisfying  $x \circ y \in \{x, \infty\}$  for any  $x, y \in A$  and  $x \circ y = \infty$  if at least one of  $x, y$  is  $\infty$ . Then the algebra  $\mathcal{A}$  will be called a *graph* (or *Shallon*) *algebra*.

**Remark.** By writing  $\mathcal{A} = (A \cup \{\infty\}, \circ, \infty)$  we shall always mean that  $\infty \notin A$ . Thus this assumption will not be repeated.

Graph algebras were introduced by R. C. Shallon [12] and were studied, e.g. in [6], [8]–[11].

**1.2 Definition.** Let  $\mathcal{A} = (A \cup \{\infty\}, \circ, \infty)$  be a graph algebra and let  $\Theta$  be an equivalence relation on  $\mathcal{A}$ . The relation  $\Theta$  is called a *congruence* if

$$(a, b) \in \Theta \text{ and } (c, d) \in \Theta \text{ imply } (a \circ c, b \circ d) \in \Theta \text{ whenever } a, b, c, d \in A \cup \{\infty\}.$$

The class of all equivalence relations on  $\mathcal{A}$  will be denoted by  $\text{Eq}(\mathcal{A})$  and the class of all congruences on  $\mathcal{A}$  will be denoted by  $\text{Cong}(\mathcal{A})$ .

**1.3 Definition.** Let  $\mathcal{A} = (A \cup \{\infty\}, \circ, \infty)$  be a graph algebra and let  $\Theta \in \text{Eq}(\mathcal{A})$ . Denote by  $\text{Id}(\mathcal{A}, \Theta)$  the class of the equivalence  $\Theta$  containing the element  $\infty$ , i.e.

$$\text{Id}(\mathcal{A}, \Theta) = \{a \in A \cup \{\infty\} : (a, \infty) \in \Theta\}.$$

Then the subalgebra  $\mathcal{I}(\mathcal{A}) = (\text{Id}(\mathcal{A}, \Theta), \circ, \infty)$  of  $\mathcal{A}$  will be referred to as an *ideal* in the graph algebra  $\mathcal{A}$  determined by the equivalence  $\Theta$  or shortly an *ideal* in the graph algebra  $\mathcal{A}$ .

**Remark.** The ideal  $\mathcal{I}(\mathcal{A})$  in the graph algebra  $\mathcal{A}$  is a graph algebra as well.

**1.4 Notation.** Let  $\mathcal{A} = (A \cup \{\infty\}, \circ, \infty)$  be a graph algebra. We introduce “successors”  $\mathcal{S}_{\mathcal{A}}(x)$  and “predecessors”  $\mathcal{P}_{\mathcal{A}}(x)$  of an element  $x \in A$  by

$$\begin{aligned}\mathcal{S}_{\mathcal{A}}(x) &= \{y \in A : x \circ y = x\}, \\ \mathcal{P}_{\mathcal{A}}(x) &= \{y \in A : y \circ x = y\}.\end{aligned}$$

For completeness we put

$$\mathcal{S}_{\mathcal{A}}(\infty) = \mathcal{P}_{\mathcal{A}}(\infty) = \emptyset.$$

If there is no danger of confusion, we shall omit indices “ $\mathcal{A}$ ” in  $\mathcal{S}_{\mathcal{A}}(x)$  and  $\mathcal{P}_{\mathcal{A}}(x)$ .

Next, let  $x \in A$ . Then  $\mathcal{A}(x)$  will denote the set of those elements  $y$  in  $A$  for which there exists exist  $n \in \mathbb{N}$  and a sequence  $\{x_i\}_{i=0}^n$  with  $x_0, \dots, x_n \in A$  such that

$$\begin{aligned}x_0 &= x, \\ x_n &= y, \\ x_{i+1} \circ x_i &= x_{i+1} \quad \text{for } i = 0, \dots, n-1.\end{aligned}$$

For an element  $\infty$  we put  $\mathcal{A}(\infty) = \emptyset$ .

**Remark.** It is worth realizing that for  $x \in A$  we have  $\mathcal{P}_{\mathcal{A}}(x) \subseteq \mathcal{A}(x)$ .

## 2. CONGRUENCES ON GRAPH ALGEBRAS

**2.1 Theorem.** Let  $\mathcal{A} = (A \cup \{\infty\}, \circ, \infty)$  be a graph algebra,  $\Theta \in \text{Eq}(\mathcal{A})$  and let the subalgebra  $(X \cup \{\infty\}, \circ, \infty)$  of  $\mathcal{A}$  be the ideal in  $\mathcal{A}$  determined by the equivalence  $\Theta$ . Then  $\Theta \in \text{Cong}(\mathcal{A})$  if and only if the following three conditions hold:

- (i)  $\mathcal{A}(x) \subseteq X$  whenever  $x \in X \cup \{\infty\}$ ,
- (ii) if  $(x, y) \in \Theta$  then  $\mathcal{S}(x) = \mathcal{S}(y)$  whenever  $x, y \in A \setminus X$ ,
- (iii) if  $(x, y) \in \Theta$  then  $\mathcal{P}(x) \setminus X = \mathcal{P}(y) \setminus X$  whenever  $x, y \in A \cup \{\infty\}$ .

*Proof.* First let  $\Theta$  be a congruence on  $\mathcal{A}$ .

(i) If  $x = \infty$  then  $\mathcal{A}(x) = \emptyset \subseteq X$ . Let  $x \in X$  and  $y \in \mathcal{A}(x)$ . Then  $(x, \infty) \in \Theta$  and there exist  $n \in \mathbb{N}$  and a sequence  $\{x_i\}_{i=0}^n$  with  $x_0, \dots, x_n \in A$  such that  $x_0 = x$ ,  $x_n = y$ , and  $x_{i+1} \circ x_i = x_{i+1}$  for  $i = 0, \dots, n-1$ . We shall proceed by induction with respect to  $n$ , the length of the sequence.

If  $n = 1$  then  $y \circ x = y$ . The relation  $\Theta$  is reflexive, thus  $(y, y) \in \Theta$ . Since  $\Theta$  is a congruence,  $(y \circ x, y \circ \infty) = (y, \infty) \in \Theta$ , which implies that  $y \in X$ .

Let  $n \in \mathbb{N}$ ,  $n \geq 1$ . Suppose that for all elements  $y$  in  $\mathcal{A}(x)$  with the corresponding sequence of length at the most  $n$  we have  $y \in X$ . If  $\tilde{y} \in \mathcal{A}(x)$  and  $x_0 = x$ ,  $x_{n+1} = \tilde{y}$ ,

$x_{i+1} \circ x_i = x_{i+1}$  for  $i = 0, \dots, n$  then the element  $y = x_n \in \mathcal{A}(x)$  belongs to  $X$  and by the assumption for sequences of length 1, namely for  $\tilde{x}_0 = y, \tilde{x}_1 = \tilde{y}$  and

$$\tilde{x}_1 \circ \tilde{x}_0 = \tilde{y} \circ y = x_{n+1} \circ x_n = x_{n+1} = \tilde{y} = \tilde{x}_1,$$

we obtain  $\tilde{y} \in X$ . Therefore  $\mathcal{A}(x) \subseteq X$ .

(ii) Now take  $x, y \in A \setminus X$  such that  $(x, y) \in \Theta$ . If  $x = y$  then obviously  $\mathcal{S}(x) = \mathcal{S}(y)$ . Let  $x \neq y$ . If both the sets  $\mathcal{S}(x)$  and  $\mathcal{S}(y)$  are empty, the equality  $\mathcal{S}(x) = \mathcal{S}(y)$  holds. Now suppose that at least one of  $\mathcal{S}(x)$  and  $\mathcal{S}(y)$  is nonempty. Without loss of generality let  $\mathcal{S}(x) \neq \emptyset$ .

Take  $s \in \mathcal{S}(x)$ , i.e.  $x \circ s = x$ . Then  $(x, y) \in \Theta$  and  $(s, s) \in \Theta$  imply  $(x \circ s, y \circ s) = (x, y \circ s) \in \Theta$ . By Definition 1.1,  $y \circ s \in \{y, \infty\}$ . In the case  $y \circ s = \infty$  we get  $(x, \infty) \in \Theta$ , which contradicts the assumption  $x \in A \setminus X$ . Necessarily,  $y \circ s = y$  which yields  $s \in \mathcal{S}(y)$ . We have obtained  $\mathcal{S}(x) \subseteq \mathcal{S}(y)$ . Analogously, we can infer  $\mathcal{S}(y) \subseteq \mathcal{S}(x)$ . Thus  $\mathcal{S}(y) = \mathcal{S}(x)$ .

(iii) First, consider an element  $x \in X \cup \{\infty\}$ . If  $x = \infty$  then  $\mathcal{P}(x) = \emptyset$  and thus  $\mathcal{P}(x) \setminus X = \emptyset$ . Now let  $x \neq \infty$ . Since  $\mathcal{P}(x) \subseteq \mathcal{A}(x)$ , in view of (i) we obtain  $\mathcal{P}(x) \subseteq X$ , thus for  $x \in X \cup \{\infty\}$  we get  $\mathcal{P}(x) \setminus X = \emptyset$ . The assumption  $(x, y) \in \Theta$  implies  $y \in X \cup \{\infty\}$ , thus  $\mathcal{P}(y) \setminus X = \emptyset$ . Therefore  $\mathcal{P}(x) \setminus X = \mathcal{P}(y) \setminus X$ .

Now take  $x \in A \setminus X$ . Clearly,  $(x, y) \in \Theta$  implies  $y \notin X$ . If  $\mathcal{P}(x) \setminus X = \emptyset$  then trivially  $\mathcal{P}(x) \setminus X \subseteq \mathcal{P}(y) \setminus X$ . Assume that  $\mathcal{P}(x) \setminus X \neq \emptyset$ . Then there exists an element  $z \in A \setminus X$  such that  $z \circ x = z$ . From  $(z, z) \in \Theta$  and  $(x, y) \in \Theta$  we get  $(z \circ x, z \circ y) = (z, z \circ y) \in \Theta$ . The element  $z$  does not belong to  $X$ , therefore  $z \circ y = z$ . Consequently,  $z \in \mathcal{P}(y) \setminus X$ . We have shown that  $\mathcal{P}(x) \setminus X \subseteq \mathcal{P}(y) \setminus X$ . In an analogous way we can prove the converse inclusion  $\mathcal{P}(y) \setminus X \subseteq \mathcal{P}(x) \setminus X$ . Thus  $\mathcal{P}(x) \setminus X = \mathcal{P}(y) \setminus X$ .

Conversely, suppose that  $\Theta$  is an equivalence satisfying the conditions (i)–(iii). Assume that  $a, b, c, d \in A \cup \{\infty\}$  and  $(a, b) \in \Theta, (c, d) \in \Theta$ . Taking into account that  $x \circ y \in \{x, \infty\}$ , we obtain four possible combinations to verify.

First, let  $a \circ c = a$  and  $b \circ d = b$ . In this case the result is trivial because  $(a \circ c, b \circ d) = (a, b) \in \Theta$ .

Next, let  $a \circ c = \infty$  and  $b \circ d = b \neq \infty$ . Evidently,  $d \neq \infty$ . If  $a = \infty$  then  $b \in X \cup \{\infty\}$ . Thus  $(a \circ c, b \circ d) = (\infty, b) \in \Theta$ . If  $a \neq \infty$  and  $c = \infty$  then  $d \in X \cup \{\infty\}$ . In view of (i) for  $b \in \mathcal{P}(d)$  we obtain  $b \in X$ , thus again  $(a \circ c, b \circ d) = (\infty, b) \in \Theta$ . Now let  $a \neq \infty \neq c$ . The assumptions yield  $a \notin \mathcal{P}(c)$  and  $d \in \mathcal{S}(b)$ . If  $a \notin X$ , then  $b \notin X$ . By (ii) we have  $\mathcal{S}(a) = \mathcal{S}(b)$ . Therefore  $d \in \mathcal{S}(a)$ , or inversely  $a \in \mathcal{P}(d)$ . In view of (iii) we obtain  $\mathcal{P}(c) \setminus X = \mathcal{P}(d) \setminus X$ . Since  $a \notin X$  we get  $a \in \mathcal{P}(c)$ , a contradiction. Necessarily,  $a \in X$  and consequently  $b \in X$ , which implies  $(b, \infty) \in \Theta$ . Therefore  $(a \circ c, b \circ d) = (\infty, b) \in \Theta$ .

The following possibility,  $a \circ c = a \neq \infty$  and  $b \circ d = \infty$ , is analogous to the previous one because the relation  $\Theta$  is symmetric.

Finally, let  $a \circ c = \infty$  and  $b \circ d = \infty$ . Now  $(a \circ c, b \circ d) = (\infty, \infty) \in \Theta$  because the relation  $\Theta$  is reflexive.  $\square$

### 3. EXTENSIONS OF GRAPH ALGEBRAS

Let  $\mathcal{A} = (A \cup \{\infty\}, \circ, \infty)$  be a graph algebra and  $\Theta$  a congruence relation on  $\mathcal{A}$ . In what follows, we denote by  $[x]_\Theta$  the set  $\{y \in A \cup \{\infty\} : (x, y) \in \Theta\}$ . We define a quotient graph algebra  $\mathcal{A}/\Theta = (A \cup \{\infty\}/\Theta, \bullet, \text{Id}(\mathcal{A}, \Theta))$  in a natural way, i.e. the binary operation  $\bullet$  is defined as follows:

$$[x]_\Theta \bullet [y]_\Theta = [x \circ y]_\Theta \quad \text{whenever } [x]_\Theta, [y]_\Theta \in A \cup \{\infty\}/\Theta.$$

**3.1 Definition.** Let  $\mathcal{X} = (X \cup \{\infty_{\mathcal{X}}\}, \odot, \infty_{\mathcal{X}})$ ,  $\mathcal{Y} = (Y \cup \{\infty_{\mathcal{Y}}\}, \Delta, \infty_{\mathcal{Y}})$  be graph algebras. A graph algebra  $\mathcal{A} = (A \cup \{\infty\}, \circ, \infty)$  is called an *ideal extension of the graph algebra  $\mathcal{X}$  by the graph algebra  $\mathcal{Y}$* , if  $X \subseteq A$ ,  $\infty = \infty_{\mathcal{X}}$  and there exists a congruence  $\Theta$  on  $\mathcal{A}$  such that the subalgebra  $(X \cup \{\infty\}, \circ, \infty)$  of  $\mathcal{A}$  is the ideal in  $\mathcal{A}$  determined by the equivalence  $\Theta$  and the quotient graph algebra  $\mathcal{A}/\Theta$  is isomorphic to  $\mathcal{Y}$ .

Now our aim is to describe, for given graph algebras  $\mathcal{X}$  and  $\mathcal{Y}$ , all possible ideal extensions  $\mathcal{A}$ , as well as to determine whether an ideal extension of  $\mathcal{X}$  by  $\mathcal{Y}$  always exists. Conforming with Definition 3.1, we shall take algebras  $\mathcal{A}$  with  $A \supseteq X$  and the nullary operation identical to that of  $\mathcal{X}$ . Therefore the index “ $\mathcal{X}$ ” in  $\infty_{\mathcal{X}}$  will not be necessary.

**3.2 Definition.** Let  $\mathcal{X} = (X \cup \{\infty\}, \odot, \infty)$  and  $\mathcal{Y} = (Y \cup \{\infty_{\mathcal{Y}}\}, \Delta, \infty_{\mathcal{Y}})$  be graph algebras such that  $X \cap Y = \emptyset$ .

Take an arbitrary set  $Z$  such that  $Z \cap (X \cup \{\infty\}) = Z \cap (Y \cup \{\infty_{\mathcal{Y}}\}) = \emptyset$  and any mapping  $\varphi: Z \rightarrow Y$ .

We define a graph algebra  $\mathcal{A} = (A \cup \{\infty\}, \circ, \infty)$ . The base set of  $\mathcal{A}$  is  $X \cup Y \cup Z \cup \{\infty\}$ , i.e. we put  $A = X \cup Y \cup Z$ . The operation  $\circ$  is defined in the following way:

▷ if  $a = \infty$  or  $b = \infty$  then

$$(3.2.1) \quad a \circ b = \infty;$$

▷ if  $a, b \in X$  then

$$(3.2.2) \quad a \circ b = a \odot b;$$

▷ if  $a \in X$  and  $b \in Y \cup Z$  then

$$(3.2.3) \quad a \circ b \in \{a, \infty\};$$

▷ if  $a \in Y \cup Z$  and  $b \in X$  then

$$(3.2.4) \quad a \circ b = \infty;$$

▷ if  $a, b \in Y$  then

$$(3.2.5) \quad a \circ b = \begin{cases} a & \text{if } a \Delta b = a, \\ \infty & \text{if } a \Delta b = \infty_{\mathcal{Y}}; \end{cases}$$

▷ if  $a \in Y$  and  $b \in Z$  then

$$(3.2.6) \quad a \circ b = \begin{cases} a & \text{if } \varphi(b) \in \mathcal{S}_{\mathcal{Y}}(a), \\ \infty & \text{if } \varphi(b) \notin \mathcal{S}_{\mathcal{Y}}(a); \end{cases}$$

▷ if  $a \in Z$  and  $b \in Y$  then

$$(3.2.7) \quad a \circ b = \begin{cases} a & \text{if } \varphi(a) \in \mathcal{P}_{\mathcal{Y}}(b), \\ \infty & \text{if } \varphi(a) \notin \mathcal{P}_{\mathcal{Y}}(b); \end{cases}$$

▷ if  $a, b \in Z$  then

$$(3.2.8) \quad a \circ b = \begin{cases} a & \text{if } \varphi(a) \in \mathcal{P}_{\mathcal{Y}}(\varphi(b)), \\ \infty & \text{if } \varphi(a) \notin \mathcal{P}_{\mathcal{Y}}(\varphi(b)). \end{cases}$$

We denote the class of all graph algebras  $\mathcal{A}$  constructed in this way by  $\Gamma_{\mathcal{X}, \mathcal{Y}}(Z, \varphi)$ .

**3.3 Lemma.** *Let  $\mathcal{X} = (X \cup \{\infty\}, \odot, \infty)$ ,  $\mathcal{Y} = (Y \cup \{\infty_{\mathcal{Y}}\}, \Delta, \infty_{\mathcal{Y}})$ ,  $\mathcal{A} = (A \cup \{\infty\}, \circ, \infty)$  be graph algebras such that  $X \cap Y = \emptyset$ ,  $\mathcal{A} \in \Gamma_{\mathcal{X}, \mathcal{Y}}(Z, \varphi)$  and let  $\Theta$  be an equivalence relation on  $\mathcal{A}$  satisfying the following conditions:*

- (a) *if  $x \in X \cup \{\infty\}$  and  $(x, y) \in \Theta$  then  $y \in X \cup \{\infty\}$ ,*
- (b) *if  $x, y \in X \cup \{\infty\}$  then  $(x, y) \in \Theta$ ,*
- (c)  *$(x, y) \in \Theta$  if and only if  $x = y$ , whenever  $x, y \in Y$ ,*
- (d)  *$(x, y) \in \Theta$  if and only if  $\varphi(y) = x$ , whenever  $x \in Y$ ,  $y \in Z$ ,*
- (e)  *$(x, y) \in \Theta$  if and only if  $\varphi(y) = \varphi(x)$ , whenever  $x, y \in Z$ .*

*Then  $\Theta$  is a congruence relation on  $\mathcal{A}$  and  $\text{Id}(\mathcal{A}, \Theta) = X \cup \{\infty\}$ .*

**Proof.** Let  $\Theta$  be an equivalence relation on  $\mathcal{A}$  satisfying the conditions (a)–(e). We shall verify the conditions (i)–(iii) of Theorem 2.1.

(i) For  $x = \infty$  the condition (i) holds. Let  $x \in X$  and  $y \in A(x)$ , i.e. there exist  $n \in \mathbb{N}$  and a sequence  $\{x_i\}_{i=0}^n$  such that  $x_0 = x$ ,  $x_n = y$ , and  $x_{i+1} \circ x_i = x_{i+1}$  for  $i = 0, \dots, n-1$ . If  $x_1 \in Y \cup Z \cup \{\infty\}$  then by (3.2.1) and (3.2.4) we have  $x_1 \circ x_0 = \infty$ , a contradiction. Therefore  $x_1 \in X$ . By induction with respect to  $n$  we obtain that  $y = x_n \in X$ .

(ii) Now take  $x, y \in Y \cup Z$  such that  $(x, y) \in \Theta$ . If  $x = y$  then obviously  $S_{\mathcal{A}}(x) = S_{\mathcal{A}}(y)$ . Let  $x \neq y$ . If both the sets  $S_{\mathcal{A}}(x)$  and  $S_{\mathcal{A}}(y)$  are empty, the equality  $S_{\mathcal{A}}(x) = S_{\mathcal{A}}(y)$  holds. Now suppose that at least one of  $S_{\mathcal{A}}(x)$  and  $S_{\mathcal{A}}(y)$  is nonempty. Without loss of generality let  $S_{\mathcal{A}}(x) \neq \emptyset$ .

First, if  $x, y \in Y$  then  $(x, y) \in \Theta$  implies by (c) the equality  $x = y$ , a contradiction with the above assumption.

Next, let  $x \in Y$ ,  $y \in Z$ . Then  $(x, y) \in \Theta$  implies  $x = \varphi(y)$  (see (d)). Let  $v \in \mathcal{S}_{\mathcal{A}}(x)$ , i.e.  $x \circ v = x$ . In view of Definition 3.2 we obtain for the element  $v \in A$  that  $v$  is either in  $Y$  or in  $Z$ . If  $v \in Y$  then (3.2.5) implies  $x \Delta v = x$ . Therefore  $\varphi(y) = x \in \mathcal{P}_{\mathcal{Y}}(v)$  and in view of (3.2.7) we get  $y \circ v = y$ , i.e.  $v \in \mathcal{S}_{\mathcal{A}}(y)$ . Now let  $v \in Z$ . Then (3.2.6) yields  $\varphi(v) \in \mathcal{S}_{\mathcal{Y}}(x)$ , thus  $x \in \mathcal{P}_{\mathcal{Y}}(\varphi(v))$ . Since  $x = \varphi(y)$ , we get  $\varphi(y) \in \mathcal{P}_{\mathcal{Y}}(\varphi(v))$  and therefore by (3.2.8),  $y \circ v = y$ . Consequently,  $v \in \mathcal{S}_{\mathcal{A}}(y)$ .

Finally, if  $x, y \in Z$  then by (e) we have  $\varphi(x) = \varphi(y)$ . Again, let  $v \in \mathcal{S}_{\mathcal{A}}(x)$ . Then  $x \circ v = x$ . In view of Definition 3.2 we infer that  $v \in Y \cup Z$ . If  $v \in Y$  then  $\varphi(x) \in \mathcal{P}_{\mathcal{Y}}(v)$  by (3.2.7) and consequently  $\varphi(y) \in \mathcal{P}_{\mathcal{Y}}(v)$ . Thus  $y \circ v = y$ , i.e.  $v \in \mathcal{S}_{\mathcal{A}}(y)$ . If  $v \in Z$  then by (3.2.8) we have  $\varphi(x) \in \mathcal{P}_{\mathcal{Y}}(\varphi(v))$  and consequently  $\varphi(y) \in \mathcal{P}_{\mathcal{Y}}(\varphi(v))$ . Thus  $y \circ v = y$ , i.e.  $v \in \mathcal{S}_{\mathcal{A}}(y)$ .

In all cases we have shown that  $S_{\mathcal{A}}(x) \subseteq \mathcal{S}_{\mathcal{A}}(y)$ . In a similar way we can prove that  $S_{\mathcal{A}}(y) \subseteq \mathcal{S}_{\mathcal{A}}(x)$ . Thus  $S_{\mathcal{A}}(y) = S_{\mathcal{A}}(x)$ .

(iii) Let  $x, y \in A \cup \{\infty\}$  and  $(x, y) \in \Theta$ . Assume that  $x \in X \cup \{\infty\}$ . In view of (a) and (b) we have  $(x, y) \in \Theta$  if and only if  $y \in X \cup \{\infty\}$ . Notice that for  $x = \infty$  we have  $\mathcal{P}_{\mathcal{A}}(x) = \emptyset$  and obviously  $\mathcal{P}_{\mathcal{A}}(x) \setminus X = \emptyset$ . In the case  $x \in X$ , we get by (3.2.2) and (3.2.4) the inclusion  $\mathcal{P}_{\mathcal{A}}(x) \subseteq X$ . Consequently  $\mathcal{P}_{\mathcal{A}}(x) \setminus X = \emptyset$ . Therefore for  $x, y \in X \cup \{\infty\}$  we obtain  $\mathcal{P}_{\mathcal{A}}(x) \setminus X = \mathcal{P}_{\mathcal{A}}(y) \setminus X$ .

Let  $x, y \in Y$ . Then  $(x, y) \in \Theta$  implies  $x = y$ , thus  $\mathcal{P}_{\mathcal{A}}(x) \setminus X = \mathcal{P}_{\mathcal{A}}(y) \setminus X$  is valid.

To verify all the remaining possibilities, take  $v \in \mathcal{P}_{\mathcal{A}}(x) \setminus X$ , i.e.  $v \notin X$  and  $v \circ x = v$ . First, suppose that  $x \in Y$  and  $y \in Z$ . The assumption  $(x, y) \in \Theta$  implies  $\varphi(y) = x$ . Let  $v \in Y$ . Then in view of (3.2.5),  $v \circ x = v$  implies  $v \Delta x = v$  and  $x \in \mathcal{S}_{\mathcal{Y}}(v)$ . Since  $\varphi(y) = x$ , we obtain  $\varphi(y) \in \mathcal{S}_{\mathcal{Y}}(v)$ , i.e.  $v \in \mathcal{P}_{\mathcal{Y}}(\varphi(y))$ . Therefore by (3.2.7) we get  $v \circ y = v$ . If  $v \in Z$  then  $v \circ x = v$  yields  $\varphi(v) \in \mathcal{P}_{\mathcal{Y}}(x) = \mathcal{P}_{\mathcal{Y}}(\varphi(y))$  (see (3.2.7)). Consequently, by (3.2.8) we get  $v \circ y = v$ .

Further, if  $x, y \in Z$ , the assumption  $(x, y) \in \Theta$  by (e) yields  $\varphi(x) = \varphi(y)$ . If  $v \in Y$  then in view of (3.2.6)  $v \circ x = v$  implies  $\varphi(x) \in \mathcal{S}_{\mathcal{Y}}(v)$ , thus  $\varphi(y) \in \mathcal{S}_{\mathcal{Y}}(v)$  and again



by (3.2.6) we get  $v \circ y = v$ . Finally, if  $v \in Z$  then  $v \circ x = v$  together with (3.2.8) yields the relation  $\varphi(v) \in \mathcal{P}_{\mathcal{Y}}(\varphi(x))$ . This implies  $\varphi(v) \in \mathcal{P}_{\mathcal{Y}}(\varphi(y))$ , thus again by (3.2.8) we obtain  $v \circ y = v$ .

To sum up, in all cases  $v \in \mathcal{P}_{\mathcal{A}}(y) \setminus X$  therefore  $\mathcal{P}_{\mathcal{A}}(x) \setminus X \subseteq \mathcal{P}_{\mathcal{A}}(y) \setminus X$ . The converse inclusion can be proved analogously, thus  $\mathcal{P}_{\mathcal{A}}(x) \setminus X = \mathcal{P}_{\mathcal{A}}(y) \setminus X$ .  $\square$

**3.4 Theorem.** *Let  $\mathcal{X} = (X \cup \{\infty\}, \odot, \infty)$  and  $\mathcal{Y} = (Y \cup \{\infty_{\mathcal{Y}}\}, \Delta, \infty_{\mathcal{Y}})$  be graph algebras such that  $X \cap Y = \emptyset$ . Let  $\mathcal{A} = (A \cup \{\infty\}, \circ, \infty)$  be a graph algebra from the class  $\Gamma_{\mathcal{X}, \mathcal{Y}}(Z, \varphi)$  (for an arbitrary set  $Z$  and any mapping  $\varphi: Z \rightarrow Y$ ). Then the graph algebra  $\mathcal{A}$  is the ideal extension of  $\mathcal{X}$  by  $\mathcal{Y}$ .*

*Proof.* Assume the equivalence relation  $\Theta$  on  $\mathcal{A}$  with one class  $X \cup \{\infty\}$  and all other classes containing one element  $y$  of  $Y$  each and the elements of the set  $Z$  distributed to classes according to the mapping  $\varphi$  in the following way:

$$[y]_{\Theta} = \{y\} \cup \{z \in Z: \varphi(z) = y\} \quad \text{whenever } y \in Y.$$

In this way the equivalence  $\Theta$  is uniquely determined and satisfies the conditions (a)–(e) of Lemma 3.3. Thus  $\Theta \in \text{Cong}(\mathcal{A})$ .

In view of conditions (a) and (b),  $\text{Id}(\mathcal{A}, \Theta) = X \cup \{\infty\}$ .

We shall prove that  $\mathcal{A}/\Theta \cong \mathcal{Y}$ . We recall that  $\varphi$  is an arbitrary mapping  $Z \rightarrow Y$ . Define a mapping  $\Phi: A \cup \{\infty\}/\Theta \rightarrow Y \cup \{\infty_{\mathcal{Y}}\}$  such that

$$\Phi([x]_{\Theta}) = \begin{cases} \infty_{\mathcal{Y}} & \text{if } x \in X \cup \{\infty\}, \\ x & \text{if } x \in Y, \\ \varphi(x) & \text{if } x \in Z. \end{cases}$$

First, we shall look into the correctness of  $\Phi$  by showing for  $x, y \in A \cup \{\infty\}$  the implication

$$\text{if } [x]_{\Theta} = [y]_{\Theta} \quad \text{then} \quad \Phi([x]_{\Theta}) = \Phi([y]_{\Theta}).$$

Let  $[x]_{\Theta} = [y]_{\Theta}$ , i.e.  $(x, y) \in \Theta$ . By (a) we obtain that  $x \in X \cup \{\infty\}$  and  $(x, y) \in \Theta$  yield  $y \in X \cup \{\infty\}$ , thus  $\Phi([x]_{\Theta}) = \infty_{\mathcal{Y}} = \Phi([y]_{\Theta})$ . If  $x, y \in Y$  then  $(x, y) \in \Theta$  implies  $x = y$ , thus  $\Phi([x]_{\Theta}) = x = y = \Phi([y]_{\Theta})$ . Next, if  $x \in Y$ ,  $y \in Z$  and  $(x, y) \in \Theta$  then  $x = \varphi(y)$ , therefore  $\Phi([x]_{\Theta}) = x = \varphi(y) = \Phi([y]_{\Theta})$ . Finally, if  $x, y \in Z$  then  $(x, y) \in \Theta$  yields  $\varphi(x) = \varphi(y)$ , and so  $\Phi([x]_{\Theta}) = \Phi([y]_{\Theta})$ .

Next, we shall prove injectivity of  $\Phi$ , i.e. for  $x, y \in A \cup \{\infty\}$  the implication

$$\text{if } \Phi([x]_{\Theta}) = \Phi([y]_{\Theta}) \quad \text{then} \quad [x]_{\Theta} = [y]_{\Theta}.$$

If  $x \in X \cup \{\infty\}$  then  $\Phi([y]_{\Theta}) = \Phi([x]_{\Theta}) = \infty_{\mathcal{Y}}$ , which yields  $y \in X \cup \{\infty\}$ . Therefore by (b)  $(x, y) \in \Theta$  and thus  $[x]_{\Theta} = [y]_{\Theta}$ . In view of (c)–(e) we obtain the following

results. If  $x, y \in Y$  then  $x = \Phi([x]_{\Theta}) = \Phi([y]_{\Theta}) = y$ , i.e.  $[x]_{\Theta} = [y]_{\Theta}$ . Next, if  $x \in Y, y \in Z$  then  $x = \Phi([x]_{\Theta}) = \Phi([y]_{\Theta}) = \varphi(y)$  again implies  $(x, y) \in \Theta$ . Finally, if  $x, y \in Z$  then  $\varphi(x) = \Phi([x]_{\Theta}) = \Phi([y]_{\Theta}) = \varphi(y)$ , giving the same result as above:  $[x]_{\Theta} = [y]_{\Theta}$ .

Now we shall show that  $\Phi$  is surjective. Take an arbitrary  $y \in Y \cup \{\infty_{\mathcal{Y}}\}$ . If  $y = \infty_{\mathcal{Y}}$  then  $\Phi([x]_{\Theta}) = y$  for any  $x \in X \cup \{\infty\}$ . In the case  $y \neq \infty_{\mathcal{Y}}$  we take  $y$  as a pre-image of  $y$ , i.e.  $\Phi([y]_{\Theta}) = y$ .

Finally, we have to prove that  $\Phi$  is a homomorphism, i.e. for  $x, y \in A \cup \{\infty\}$

$$\Phi([x]_{\Theta} \bullet [y]_{\Theta}) = \Phi([x]_{\Theta}) \Delta \Phi([y]_{\Theta}).$$

Since  $x \circ y \in \{x, \infty\}$ , we shall consider two cases. If  $x \circ y = \infty$  then

$$\Phi([x]_{\Theta} \bullet [y]_{\Theta}) = \Phi([x \circ y]_{\Theta}) = \Phi([\infty]_{\Theta}) = \infty_{\mathcal{Y}}.$$

To determine  $\Phi([x]_{\Theta}) \Delta \Phi([y]_{\Theta})$  we distinguish:

- ▷ if  $x \in X \cup \{\infty\}$  then  $\Phi([x]_{\Theta}) \Delta \Phi([y]_{\Theta}) = \infty_{\mathcal{Y}} \Delta \Phi([y]_{\Theta}) = \infty_{\mathcal{Y}}$  (the assumption  $y \in X \cup \{\infty\}$  leads to the same result);
- ▷ if  $x, y \in Y$  then by (3.2.5)

$$\Phi([x]_{\Theta}) \Delta \Phi([y]_{\Theta}) = \infty_{\mathcal{Y}};$$

- ▷ if  $x \in Y$  and  $y \in Z$  then  $\varphi(y) \notin \mathcal{S}_{\mathcal{Y}}(x)$  by (3.2.6), which implies

$$\Phi([x]_{\Theta}) \Delta \Phi([y]_{\Theta}) = x \Delta \varphi(y) = \infty_{\mathcal{Y}};$$

- ▷ if  $x \in Z$  and  $y \in Y$  then  $\varphi(x) \notin \mathcal{P}_{\mathcal{Y}}(y)$  by (3.2.7), thus

$$\Phi([x]_{\Theta}) \Delta \Phi([y]_{\Theta}) = \varphi(x) \Delta y = \infty_{\mathcal{Y}};$$

- ▷ if  $x, y \in Z$  then  $\varphi(x) \notin \mathcal{P}_{\mathcal{Y}}(\varphi(y))$  by (3.2.8), therefore

$$\Phi([x]_{\Theta}) \Delta \Phi([y]_{\Theta}) = \varphi(x) \Delta \varphi(y) = \infty_{\mathcal{Y}}.$$

Next, let  $x \circ y = x$ , hence  $\Phi([x]_{\Theta} \bullet [y]_{\Theta}) = \Phi([x \circ y]_{\Theta}) = \Phi([x]_{\Theta})$ . Again, we distinguish the following possibilities:

- ▷ if  $x \in X \cup \{\infty\}$  then

$$\Phi([x]_{\Theta}) \Delta \Phi([y]_{\Theta}) = \infty_{\mathcal{Y}} \Delta \Phi([y]_{\Theta}) = \infty_{\mathcal{Y}} = \Phi([x]_{\Theta});$$

- ▷ the assumptions  $y \in X \cup \{\infty\}$  and  $x \notin X \cup \{\infty\}$  lead to a contradiction  $x \circ y = \infty$ ;

▷ if  $x, y \in Y$  then we get by (3.2.5)

$$\Phi([x]_{\Theta}) \Delta \Phi([y]_{\Theta}) = x \Delta y = x \circ y = x = \Phi([x]_{\Theta});$$

▷ if  $x \in Y$  and  $y \in Z$  then  $\varphi(y) \in \mathcal{S}_{\mathcal{Y}}(x)$  by (3.2.6), therefore

$$\Phi([x]_{\Theta}) \Delta \Phi([y]_{\Theta}) = x \Delta \varphi(y) = x = \Phi([x]_{\Theta});$$

▷ if  $x \in Z$  and  $y \in Y$  then  $\varphi(x) \in \mathcal{P}_{\mathcal{Y}}(y)$  by (3.2.7), thus

$$\Phi([x]_{\Theta}) \Delta \Phi([y]_{\Theta}) = \varphi(x) \Delta y = \varphi(x) = \Phi([x]_{\Theta});$$

▷ if  $x, y \in Z$  then  $\varphi(x) \in \mathcal{P}_{\mathcal{Y}}(\varphi(y))$  by (3.2.8), which yields

$$\Phi([x]_{\Theta}) \Delta \Phi([y]_{\Theta}) = \varphi(x) \Delta \varphi(y) = \varphi(x) = \Phi([x]_{\Theta}).$$

For completeness we have to note that  $\Phi(\text{Id}(\mathcal{A}, \Theta)) = \Phi([\infty]_{\Theta}) = \infty_{\mathcal{Y}}$ .

Now we can conclude that  $\Phi: A \cup \{\infty\} / \Theta \rightarrow Y \cup \{\infty_{\mathcal{Y}}\}$  is a bijective homomorphism, therefore  $\mathcal{A}$  is an ideal extension of  $\mathcal{X}$  by  $\mathcal{Y}$ .  $\square$

In this way we can construct graph algebras  $\mathcal{A}$  which are ideal extensions of  $\mathcal{X}$  by  $\mathcal{Y}$ . Therefore the answer to the question (Q1), whether the ideal extension is always possible, is affirmative. In what follows we shall show that the class  $\Gamma_{\mathcal{X}, \mathcal{Y}}(Z, \varphi)$  contains all ideal extensions. In other words: if  $\mathcal{B}$  is an ideal extension of  $\mathcal{X}$  by  $\mathcal{Y}$  then there exists a graph algebra  $\mathcal{A} \in \Gamma_{\mathcal{X}, \mathcal{Y}}(Z, \varphi)$  which is isomorphic to  $\mathcal{B}$ . Thus in the end we shall be able to claim that the reply to the question (Q2) is affirmative as well.

**3.5 Theorem.** *Let  $\mathcal{X} = (X \cup \{\infty\}, \odot, \infty)$ ,  $\mathcal{Y} = (Y \cup \{\infty_{\mathcal{Y}}\}, \Delta, \infty_{\mathcal{Y}})$ ,  $\mathcal{B} = (B \cup \{\infty\}, \diamond, \infty)$  be graph algebras,  $X \cap Y = \emptyset$  and let  $\mathcal{B}$  be the ideal extension of  $\mathcal{X}$  by  $\mathcal{Y}$ . Then there exists a graph algebra  $\mathcal{A} = (A \cup \{\infty\}, \circ, \infty)$  such that  $\mathcal{A} \in \Gamma_{\mathcal{X}, \mathcal{Y}}(Z, \varphi)$  and  $\mathcal{A} \cong \mathcal{B}$ .*

**P r o o f.** Let  $\mathcal{B}$  be the ideal extension of  $\mathcal{X}$  by  $\mathcal{Y}$ . Thus there exist  $\Sigma \in \text{Cong}(\mathcal{B})$  with  $\text{Id}(\mathcal{B}, \Sigma) = X \cup \{\infty\}$  and an isomorphism  $\Omega: B \cup \{\infty\} / \Sigma \rightarrow Y$  of the quotient graph algebra  $\mathcal{B} / \Sigma = (B \cup \{\infty\} / \Sigma, \diamond, \text{Id}(\mathcal{B}, \Sigma))$  onto  $\mathcal{Y} = (Y \cup \{\infty_{\mathcal{Y}}\}, \Delta, \infty_{\mathcal{Y}})$ .

By Axiom of Choice there is a mapping  $\nu: B \cup \{\infty\} / \Sigma \rightarrow B$  in which to each equivalence class of  $\Sigma$  there corresponds a representant of the class.

Since  $\Omega$  is bijective we can put

$$Z = B \setminus (X \cup \{\nu(\Omega^{-1}(y)): y \in Y\}) \cup \{\infty\}$$

where  $\Omega^{-1}$  is the inverse mapping to  $\Omega$  and thus  $\Omega^{-1}(y) \in B \cup \{\infty\} / \Sigma$ . Notice that the sets  $X$ ,  $\{\nu(\Omega^{-1}(y)): y \in Y\}$  and  $\{\infty\}$  are mutually disjoint because if

$$(X \cup \{\infty\}) \cap \{\nu(\Omega^{-1}(y)): y \in Y\} \neq \emptyset$$

then there is an element  $x \in X \cup \{\infty\}$  satisfying  $x = \nu(\Omega^{-1}(y))$  for some  $y \in Y$ . Therefore  $\Omega^{-1}(y) = \text{Id}(\mathcal{B}, \Sigma) = X \cup \{\infty\}$  and thus  $y = \infty_{\mathcal{Y}}$ , a contradiction to  $y \in Y$ .

Now, we shall define an algebra  $\mathcal{A}$ . Denote  $A = X \cup Y \cup Z \cup \{\infty\}$  and take a mapping  $\varphi: Z \rightarrow Y$  defined by the formula  $\varphi(x) = \Omega([x]_{\Sigma})$ .

Now consider a mapping  $\omega: A \rightarrow B$  defined in the following way:

$$\omega(x) = \begin{cases} x & \text{if } x \in X \cup Z \cup \{\infty\}, \\ \nu(\Omega^{-1}(x)) & \text{if } x \in Y. \end{cases}$$

Notice that for  $x \in Y$  the image is  $\omega(x) = \nu(\Omega^{-1}(x)) \in B \setminus (X \cup Z)$ .

We shall show that  $\omega$  is injective. Take  $x, y \in A \cup \{\infty\}$  such that  $\omega(x) = \omega(y)$ . If  $x, y \in X \cup Z \cup \{\infty\}$  then  $x = \omega(x) = \omega(y) = y$ . If both  $x, y \in Y$ , we apply injectivity of  $\nu$  and  $\Omega^{-1}$  to obtain

$$\nu(\Omega^{-1}(x)) = \nu(\Omega^{-1}(y)) \implies \Omega^{-1}(x) = \Omega^{-1}(y) \implies x = y.$$

Further, if  $x \in X \cup Z \cup \{\infty\}$  and  $y \in Y$  (or conversely) then the assumption  $\omega(x) = \omega(y)$  leads to a contradiction because  $\omega(x) \in X \cup Z \cup \{\infty\}$  and  $\omega(y) \in B \setminus (X \cup Z \cup \{\infty\})$ .

Let us prove that  $\omega$  is surjective. If we take  $x \in X \cup Z \cup \{\infty\}$  then the pre-image in  $\omega$  is  $x$  itself, i.e.  $\omega(x) = x$ . Next, if  $x \in B \setminus (X \cup Z \cup \{\infty\}) = \{\nu(\Omega^{-1}(y)): y \in Y\}$  then there exists an element  $y \in Y$  such that  $x = \nu(\Omega^{-1}(y))$ . This very element  $y$  will be the pre-image of  $x$  in  $\omega$ , i.e.  $\omega(y) = \nu(\Omega^{-1}(y)) = x$ .

So far, we have proved that  $\omega$  is bijective.

Now, we shall define an operation  $\circ$  on the set  $A \cup \{\infty\}$ . For elements  $x, y \in A \cup \{\infty\}$  we put

$$x \circ y = \omega^{-1}(\omega(x) \diamond \omega(y)).$$

In what follows, we verify that the operation  $\circ$  satisfies the rules (3.2.1)–(3.2.8).

[3.2.1] Let  $x = \infty$  or  $y = \infty$ . Then  $x \circ y = \omega^{-1}(\omega(x) \diamond \omega(y)) = \omega^{-1}(\infty) = \infty$ .

[3.2.2] If  $x, y \in X$  then  $x \diamond y = x \odot y$  ( $\mathcal{B}$  is the ideal extension of  $\mathcal{X}$ ) and  $x \odot y \in \{x, \infty\}$ . Therefore

$$x \circ y = \omega^{-1}(\omega(x) \diamond \omega(y)) = \omega^{-1}(x \diamond y) = \omega^{-1}(x \odot y) = x \odot y.$$

[3.2.3] Let  $x \in X$  and  $y \in Y \cup Z$ , then  $x \circ y \in \{x, \infty\}$  is always valid regardless of the result of  $\omega^{-1}(\omega(x) \diamond \omega(y))$ .

[3.2.4] Let  $x \in Y \cup Z$  and  $y \in X$ . Suppose that  $\omega(x) \diamond \omega(y) = \omega(x)$ , i.e.  $\omega(x) \in \mathcal{P}_{\mathcal{B}}(\omega(y)) = \mathcal{P}_{\mathcal{B}}(y)$ . In view of the assumption  $\Sigma \in \text{Cong}(\mathcal{B})$  and Theorem 2.1 (i) we obtain that  $y \in X$  implies  $\omega(x) \in \mathcal{P}_{\mathcal{B}}(y) \subseteq \mathcal{B}(y) \subseteq X$ . In the case  $x \in Y$  we get  $\nu(\Omega^{-1}(x)) = \omega(x) \in X$ , then  $\Omega^{-1}(x) = \text{Id}(\mathcal{B}, \Sigma)$  and thus  $x = \infty_{\mathcal{Y}}$ , a contradiction to  $x \in Y$ . In the case  $x \in Z$  we get  $x = \omega(x) \in X$ , again a contradiction to  $x \in Y$ . Therefore  $\omega(x) \diamond \omega(y) = \infty$  and

$$x \circ y = \omega^{-1}(\omega(x) \diamond \omega(y)) = \omega^{-1}(\infty) = \infty.$$

[3.2.5] Let  $x, y \in Y$ . Since  $\Omega$  (and thus  $\Omega^{-1}$  as well) is an isomorphism, we obtain

$$\begin{aligned} [\omega(x) \diamond \omega(y)]_{\Sigma} &= [\nu(\Omega^{-1}(x)) \diamond \nu(\Omega^{-1}(y))]_{\Sigma} \\ &= [\nu(\Omega^{-1}(x))]_{\Sigma} \blacklozenge [\nu(\Omega^{-1}(y))]_{\Sigma} \\ &= \Omega^{-1}(x) \blacklozenge \Omega^{-1}(y) = \Omega^{-1}(x \triangle y). \end{aligned}$$

If  $x \triangle y = x$  then  $[\omega(x) \diamond \omega(y)]_{\Sigma} = \Omega^{-1}(x) = [\nu(\Omega^{-1}(x))]_{\Sigma} = [\omega(x)]_{\Sigma}$ . Suppose that  $\omega(x) \diamond \omega(y) = \infty$ . Then  $[\omega(x)]_{\Sigma} = \text{Id}(\mathcal{B}, \Sigma)$  and  $\omega(x) \in \text{Id}(\mathcal{B}, \Sigma)$ , a contradiction to  $\omega(x) \in B \setminus (X \cup Z)$ . Necessarily  $\omega(x) \diamond \omega(y) = \omega(x)$  and thus

$$x \circ y = \omega^{-1}(\omega(x) \diamond \omega(y)) = \omega^{-1}(\omega(x)) = x.$$

Conversely, if  $x \triangle y = \infty$  then  $[\omega(x) \diamond \omega(y)]_{\Sigma} = \Omega^{-1}(\infty_{\mathcal{Y}}) = \text{Id}(\mathcal{B}, \Sigma)$ . Thus  $\omega(x) \diamond \omega(y) \in \text{Id}(\mathcal{B}, \Sigma)$ . Since  $\omega(x) \in B \setminus (X \cup Z)$ , we can conclude that  $\omega(x) \diamond \omega(y) = \infty$  and consequently

$$x \circ y = \omega^{-1}(\omega(x) \diamond \omega(y)) = \omega^{-1}(\infty) = \infty.$$

[3.2.6] Let  $x \in Y$  and  $y \in Z$ . Then we have

$$\begin{aligned} [\omega(x) \diamond \omega(y)]_{\Sigma} &= [\nu(\Omega^{-1}(x)) \diamond y]_{\Sigma} = [\nu(\Omega^{-1}(x))]_{\Sigma} \blacklozenge [y]_{\Sigma} \\ &= \Omega^{-1}(x) \blacklozenge \Omega^{-1}(\Omega([y]_{\Sigma})) = \Omega^{-1}(x \triangle \Omega([y]_{\Sigma})) \\ &= \Omega^{-1}(x \triangle \varphi(y)). \end{aligned}$$

If  $\varphi(y) \in \mathcal{S}_{\mathcal{Y}}(x)$  then  $[\omega(x) \diamond \omega(y)]_{\Sigma} = \Omega^{-1}(x) = [\nu(\Omega^{-1}(x))]_{\Sigma} = [\omega(x)]_{\Sigma}$ . Suppose that  $\omega(x) \diamond \omega(y) = \infty$ . Then  $[\omega(x)]_{\Sigma} = \text{Id}(\mathcal{B}, \Sigma)$  and  $\omega(x) \in \text{Id}(\mathcal{B}, \Sigma)$ , a contradiction to  $\omega(x) \in B \setminus (X \cup Z)$ . Necessarily  $\omega(x) \diamond \omega(y) = \omega(x)$  and thus

$$x \circ y = \omega^{-1}(\omega(x) \diamond \omega(y)) = \omega^{-1}(\omega(x)) = x.$$

Conversely, if  $\varphi(y) \notin \mathcal{S}_{\mathcal{Y}}(x)$  then  $[\omega(x) \diamond \omega(y)]_{\Sigma} = \Omega^{-1}(\infty_{\mathcal{Y}}) = \text{Id}(\mathcal{B}, \Sigma)$ . Thus  $\omega(x) \diamond \omega(y) \in \text{Id}(\mathcal{B}, \Sigma)$ . Since  $\omega(x) \in B \setminus (X \cup Z)$ , we can conclude that  $\omega(x) \diamond \omega(y) = \infty$  and consequently

$$x \circ y = \omega^{-1}(\omega(x) \diamond \omega(y)) = \omega^{-1}(\infty) = \infty.$$

[3.2.7] Let  $x \in Z$  and  $y \in Y$ . In a similar way we obtain

$$\begin{aligned} [\omega(x) \diamond \omega(y)]_{\Sigma} &= [x \diamond \nu(\Omega^{-1}(y))]_{\Sigma} = [x]_{\Sigma} \blacklozenge [\nu(\Omega^{-1}(y))]_{\Sigma} \\ &= \Omega^{-1}(\Omega([x]_{\Sigma})) \blacklozenge \Omega^{-1}(y) = \Omega^{-1}(\Omega([x]_{\Sigma}) \Delta y) \\ &= \Omega^{-1}(\varphi(x) \Delta y). \end{aligned}$$

If  $\varphi(x) \in \mathcal{P}_{\mathcal{Y}}(y)$  then  $[\omega(x) \diamond \omega(y)]_{\Sigma} = \Omega^{-1}(\varphi(x)) = \Omega^{-1}(\Omega([x]_{\Sigma})) = [x]_{\Sigma}$ . Suppose that  $\omega(x) \diamond \omega(y) = \infty$ . Then  $[x]_{\Sigma} = \text{Id}(\mathcal{B}, \Sigma)$  and  $x \in \text{Id}(\mathcal{B}, \Sigma)$ , a contradiction to  $x \in Z$ . Necessarily  $\omega(x) \diamond \omega(y) = \omega(x)$  and thus

$$x \circ y = \omega^{-1}(\omega(x) \diamond \omega(y)) = \omega^{-1}(\omega(x)) = x.$$

Conversely, if  $\varphi(x) \notin \mathcal{P}_{\mathcal{Y}}(y)$  then  $[\omega(x) \diamond \omega(y)]_{\Sigma} = \Omega^{-1}(\infty_{\mathcal{Y}}) = \text{Id}(\mathcal{B}, \Sigma)$ . Thus  $\omega(x) \diamond \omega(y) \in \text{Id}(\mathcal{B}, \Sigma)$ . Since  $\omega(x) = x \in Z$ , we can conclude that  $\omega(x) \diamond \omega(y) = \infty$  and consequently

$$x \circ y = \omega^{-1}(\omega(x) \diamond \omega(y)) = \omega^{-1}(\infty) = \infty.$$

[3.2.8] Finally, take  $x, y \in Z$ . In this case

$$\begin{aligned} [\omega(x) \diamond \omega(y)]_{\Sigma} &= [x \diamond y]_{\Sigma} = \Omega^{-1}(\Omega([x \diamond y]_{\Sigma})) = \Omega^{-1}(\Omega([x]_{\Sigma} \blacklozenge [y]_{\Sigma})) \\ &= \Omega^{-1}(\Omega([x]_{\Sigma}) \Delta \Omega([y]_{\Sigma})) = \Omega^{-1}(\varphi(x) \Delta \varphi(y)). \end{aligned}$$

If  $\varphi(x) \in \mathcal{P}_{\mathcal{Y}}(\varphi(y))$  then  $[\omega(x) \diamond \omega(y)]_{\Sigma} = \Omega^{-1}(\varphi(x)) = \Omega^{-1}(\Omega([x]_{\Sigma})) = [x]_{\Sigma}$ . Since  $x \in Z$  we can conclude that  $\omega(x) \diamond \omega(y)$  cannot be  $\infty$ . Therefore

$$x \circ y = \omega^{-1}(\omega(x) \diamond \omega(y)) = \omega^{-1}(\omega(x)) = x.$$

Conversely, if  $\varphi(x) \notin \mathcal{P}_{\mathcal{Y}}(\varphi(y))$  then we get  $[\omega(x) \diamond \omega(y)]_{\Sigma} = \Omega^{-1}(\infty_{\mathcal{Y}}) = \text{Id}(\mathcal{B}, \Sigma)$  and this time  $\omega(x) \diamond \omega(y)$  cannot be  $\omega(x) = x \in Z$ . Thus

$$x \circ y = \omega^{-1}(\omega(x) \diamond \omega(y)) = \omega^{-1}(\infty) = \infty.$$

In this way we have verified that  $\mathcal{A} \in \Gamma_{\mathcal{X}, \mathcal{Y}}(Z, \varphi)$ . In view of Theorem 3.4 the algebra  $\mathcal{A}$  is an ideal extension of  $\mathcal{X}$  by  $\mathcal{Y}$ .

Finally, we shall show that the graph algebras  $\mathcal{A} = (A \cup \{\infty\}, \circ, \infty)$  and  $\mathcal{B} = (B \cup \{\infty\}, \diamond, \infty)$  are isomorphic. Recall that the mapping  $\omega$  is bijective. Furthermore, this mapping preserves both operations, i.e. for any  $x, y \in A \cup \{\infty\}$  we have

$$\omega(x \circ y) = \omega(\omega^{-1}(\omega(x) \diamond \omega(y))) = \omega(x) \diamond \omega(y)$$

and for the nullary operation we get  $\omega(\infty) = \infty$ . Thus  $\omega$  is an isomorphism of  $\mathcal{A}$  onto  $\mathcal{B}$ .  $\square$

**Remark.** For the graph algebra  $\mathcal{A}$  defined in the proof of the previous theorem we could consider the binary relation  $\Theta$  on  $\mathcal{A}$  defined by

$$(x, y) \in \Theta \iff (\omega(x), \omega(y)) \in \Sigma \quad \text{whenever } x, y \in A \cup \{\infty\}.$$

Since  $\Sigma \in \text{Eq}(\mathcal{B})$  we immediately get  $\Theta \in \text{Eq}(\mathcal{A})$ . Now take  $a, b, c, d \in A \cup \{\infty\}$  such that  $(a, b) \in \Theta$  and  $(c, d) \in \Theta$ . Then  $(\omega(a), \omega(b)) \in \Sigma$  and  $(\omega(c), \omega(d)) \in \Sigma$ . Since  $\Sigma$  is a congruence, we obtain

$$\begin{aligned} (\omega(a) \diamond \omega(c), \omega(b) \diamond \omega(d)) &= (\omega(\omega^{-1}(\omega(a) \diamond \omega(c))), \omega(\omega^{-1}(\omega(b) \diamond \omega(d)))) \\ &= (\omega(a \circ c), \omega(b \circ d)) \in \Sigma, \end{aligned}$$

which implies that  $(a \circ c, b \circ d) \in \Theta$ . Thus  $\Theta \in \text{Cong}(\mathcal{A})$ . Therefore

$$\mathcal{A} / \Theta \cong \mathcal{Y} \cong \mathcal{B} / \Sigma.$$

**3.6 Remark.** The class  $\Gamma_{\mathcal{X}, \mathcal{Y}}(Z, \varphi)$  contains all ideal extensions (up to isomorphism) of the graph algebra  $\mathcal{X}$  by the graph algebra  $\mathcal{Y}$ .

**3.7 Remark.** In Definition 3.2 we define a new algebra  $\mathcal{A}$  to be really an “extension” of the algebra  $\mathcal{X}$ . Therefore for the elements  $a, b \in X$  we define the result of  $a \circ b$  to be  $a \odot b$  (see (3.2.2)). Nevertheless, the quotient graph algebra  $\mathcal{A} / \Theta$  does not depend on the operation “ $\odot$ ” in  $\mathcal{X}$ . Thus instead of (3.2.2) and (3.2.3) we could put one common condition

$$\text{if } a \in X \text{ and } b \in A \text{ then } a \circ b \in \{a, \infty\}.$$

In this case we do not require  $\mathcal{X}$  to be a subalgebra of  $\mathcal{A}$ . Under this assumption we would construct a larger class  $\Gamma_{\mathcal{X}, \mathcal{Y}}(Z, \varphi)$ .

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