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AN ANSWER TO A QUESTION OF CAO, REILLY AND XIONG

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Abstract. We present a simple proof of a Banach-Stone type Theorem. The method used in the proof also provides an answer to a conjecture of Cao, Reilly and Xiong.

Keywords: Riesz isomorphism, Banach lattices, Banach-Stone Theorem

MSC 2000: 46B42, 54C35

In this paper we use the standard terminology and notation of the Riesz spaces theory (see [1]). In particular, the Banach lattice (under pointwise operations, order and supremum norm) of continuous functions from a compact Hausdorff space K into a Banach lattice E is denoted by $C(K, E)$. If $E = \mathbb{R}$ then we write $C(K)$ instead of $C(K, E)$. $\mathbf{1}_K$ stands for the unit function in $C(K)$.

One version of the Banach-Stone theorem states:

Theorem 1. *Let X and Y be compact Hausdorff spaces. Then $C(X)$ and $C(Y)$ are Riesz isomorphic if and only if X and Y are homeomorphic.*

More precisely, if $\pi: C(X) \rightarrow C(Y)$ is a Riesz isomorphism then there exists a homeomorphism $\sigma: Y \rightarrow X$ and $h \in C(Y)$ such that $\pi(f)(y) = h(y)f(\sigma(y))$ and $0 < h(y)$ for each $y \in Y$. An elementary proof of this theorem can be found in [3]. This theorem is generalized in [2] as follows.

Theorem 2. *Let X and Y be compact Hausdorff spaces and E a Banach lattice. If $\pi: C(X, E) \rightarrow C(Y)$ is a Riesz isomorphism such that $\pi(f)$ has no zeros whenever f has no zero, then X and Y are homeomorphic and E is Riesz isomorphic to \mathbb{R} .*

The proof of Theorem 2 is given without using Theorem 1 in [2] and it is conjectured that Theorem 2 follows from Theorem 1. We present an elementary proof of

Theorem 2 which also yields an affirmative answer to this conjecture. First we need the following lemma.

Lemma 3. *Let X, Y and M be compact Hausdorff spaces such that $X \times M$ and Y are homeomorphic. Suppose that for a given $f \in C(X \times M)$, $f(x, m) \neq 0$ for all $(x, m) \in X \times M$ if and only if for each $x \in X$ there exists $m \in M$ such that $f(x, m) \neq 0$. Then X and Y are homeomorphic and $M = \{m\}$.*

Proof. Suppose that there exist $m_1, m_2 \in M$ and $m_1 \neq m_2$. Choose $g \in C(M)$ with $g(m_1) = 0$ and $g(m_2) = 1$. Let $f \in C(X \times M)$ with $f(x, m) = g(m)$. This is impossible, so $M = \{m\}$ and X is homeomorphic to Y . \square

Now we are ready to give an elementary proof of Theorem 2. The technique of the proof provides an answer to the conjecture mentioned above.

Proof of Theorem 2. Clearly $\pi^{-1}(\mathbf{1}_Y)$ is a strong order unit of $C(X, E)$. Then $0 < \pi^{-1}(\mathbf{1}_Y)(x)$ is a strong order unit of E for each $x \in X$. By the Kakutani Representation Theorem (see [3] for a direct and simple proof) there exists a compact Hausdorff space M such that E and $C(M)$ are Riesz isomorphic spaces. Let $a \in X$ be fixed and let $\pi_0: C(M) \rightarrow E$ be a Riesz isomorphism such that $\pi_0(\mathbf{1}_M) = \pi^{-1}(\mathbf{1}_Y)(a)$. Then $C(X, E)$, $C(X, C(M))$ and $C(X \times M)$ are Riesz isomorphic spaces under Riesz isomorphisms

$$\pi_1: C(X \times M) \rightarrow C(X, C(M)) \quad \text{and} \quad \pi_2: C(X, C(M)) \rightarrow C(X, E)$$

defined as $\pi_1(f)(x)(m) = f(x, m)$ and $\pi_2(f)(x) = \pi_0(f(x))$. By Theorem 1, there exist a homeomorphism $\sigma: Y \rightarrow X \times M$ and $h \in C(Y)$ such that $0 < h(y)$ for each $y \in Y$ and $\pi\pi_2\pi_1(f)(y) = h(y)f(\sigma(y))$. Since $\pi(f)$ has no zeros whenever f has no zeros, we have that for a given $f \in C(X \times M)$, $f(x, m) \neq 0$ for all $(x, m) \in X \times M$ whenever for each $x \in X$ there exists $m \in M$ such that $0 \neq f(x, m)$. To see this claim, let $f \in C(X \times M)$ be such that for each $x \in X$ there exists $m_x \in M$ such that $f(x, m_x) \neq 0$. Let $x_0 \in X$. Define $f_{x_0}: M \rightarrow \mathbb{R}$ by $f_{x_0}(m) = f(x_0, m)$. Then $\mathbf{1}_X \otimes \pi_0(f_{x_0}) \in C(X, E)$ is a non-zero constant function, where $\mathbf{1}_X \otimes \pi_0(f_{x_0})(x) = \pi_0(f_{x_0})$ for each $x \in X$. Choose $p \in C(X \times M)$ such that $\pi_2\pi_1(p) = \mathbf{1}_X \otimes \pi_0(f_{x_0})$. From the hypothesis we obtain

$$0 \neq \pi(\mathbf{1}_X \otimes \pi_0(f_{x_0}))(y) = \pi\pi_2\pi_1(p)(y) = h(y)p(\sigma(y)).$$

This shows that there exists $\varepsilon > 0$ such that $\varepsilon\mathbf{1}_Y \leq \pi(\mathbf{1}_X \otimes |\pi_0(f_{x_0})|)$, that is, $\pi^{-1}(\varepsilon\mathbf{1}_Y) \leq \mathbf{1}_X \otimes |\pi_0(f_{x_0})|$, hence

$$\varepsilon\pi_0(\mathbf{1}_M) = \pi^{-1}(\varepsilon\mathbf{1}_Y)(a) \leq \mathbf{1}_X \otimes |\pi_0(f_{x_0})|(a) = |\pi_0(f_{x_0})| = \pi_0(|f_{x_0}|).$$

This implies that $\varepsilon \mathbf{1}_M \leq |f_{x_0}|$. Hence $0 \neq f(x_0, m)$ for each m . From the previous lemma, we have $M = \{m\}$, hence X and Y are homeomorphic. Since $C(M)$ is a Riesz space isometrically isomorphic to \mathbb{R} , the proof is completed. \square

References

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