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STAR NUMBER AND STAR ARBORICITY  
OF A COMPLETE MULTIGRAPH

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*Abstract.* Let  $G$  be a multigraph. The *star number*  $s(G)$  of  $G$  is the minimum number of stars needed to decompose the edges of  $G$ . The *star arboricity*  $sa(G)$  of  $G$  is the minimum number of star forests needed to decompose the edges of  $G$ . As usual  $\lambda K_n$  denote the  $\lambda$ -fold complete graph on  $n$  vertices (i.e., the multigraph on  $n$  vertices such that there are  $\lambda$  edges between every pair of vertices). In this paper, we prove that for  $n \geq 2$

$$(1) \quad s(\lambda K_n) = \begin{cases} \frac{1}{2}\lambda n & \text{if } \lambda \text{ is even,} \\ \frac{1}{2}(\lambda + 1)n - 1 & \text{if } \lambda \text{ is odd,} \end{cases}$$

$$(2) \quad sa(\lambda K_n) = \begin{cases} \lceil \frac{1}{2}\lambda n \rceil & \text{if } \lambda \text{ is odd, } n = 2, 3 \text{ or } \lambda \text{ is even,} \\ \lceil \frac{1}{2}\lambda n \rceil + 1 & \text{if } \lambda \text{ is odd, } n \geq 4. \end{cases}$$

*Keywords:* decomposition, star arboricity, star forest, complete multigraph

*MSC 2000:* 05C70

## 1. INTRODUCTION

A *star* is the complete bipartite graph  $K_{1,m}$  for some positive integer  $m$ . A *star forest* is a forest each component of which is a star. Let  $G$  be a multigraph. The *star number*  $s(G)$  of  $G$  is the minimum number of stars needed to decompose the edges of  $G$ . The *star arboricity*  $sa(G)$  of  $G$  is the minimum number of star forests needed to decompose the edges of  $G$ . In the literature the star number and the star arboricity were investigated for simple graphs.

For a graph  $G$ , the *independence number*  $\alpha(G)$  of  $G$  is defined to be the maximum size of a set  $A$  of vertices in  $G$  such that every pair of vertices in  $A$  are nonadjacent; the *covering number*  $\beta(G)$  of  $G$  is defined to be the minimum size of a set  $B$  of

vertices in  $G$  such that every edge of  $G$  is incident with at least one vertex in  $B$ . It is well known [5] that  $\alpha(G) + \beta(G) = |V(G)|$ . And it is easy to see that  $\beta(G) = s(G)$  if  $G$  is a simple graph. Star numbers, independence numbers and star arboricities were studied for some specific families of graphs. The star number was determined for the power of a cycle [12] (here the power of a cycle is a special case of circulant graphs). The independence numbers were determined for the following graphs: the Cartesian product of two odd cycles [8], the direct product of two paths, or two cycles, or a path and a cycle [10], and some specific family of circulant graphs [13]. The star arboricities were studied for the following graphs: complete bipartite graphs [6], [7], [15], complete regular multipartite graphs [3], cubes [15], crowns [11], and planar graphs [2], [9].

For a graph  $G$  and a positive integer  $\lambda$ , we use  $\lambda G$  to denote the graph obtained from  $G$  by replacing each edge  $e$  of  $G$  by  $\lambda$  edges with the same ends as  $e$ . Hence  $\lambda K_n$  is a multigraph on  $n$  vertices such that there are  $\lambda$  edges joining every pair of vertices. We call  $\lambda K_n$  a  $\lambda$ -fold complete graph or a complete multigraph. In this paper the star number and the star arboricity of  $\lambda K_n$  are determined. To avoid trivialities we assume that  $n \geq 2$ .

## 2. STAR NUMBER AND STAR ARBORICITY OF A COMPLETE MULTIGRAPH

The arboricity  $a(G)$  of a multigraph  $G$  is the minimum number of forests needed to decompose the edges of  $G$ . It is trivial from the definitions that  $a(G) \leq \text{sa}(G) \leq s(G)$ . The arboricity of any nontrivial multigraph is determined by the following well-known formula of Nash-Williams.

**Proposition 1** ([4], [14]). *Let  $G$  be a nontrivial multigraph. Then*

$$a(G) = \max \lceil |E(H)| / (|V(H)| - 1) \rceil$$

where the maximum is taken over all nontrivial induced subgraphs  $H$  of  $G$ .

It follows easily from Proposition 1 that  $a(\lambda K_n) = \lceil \frac{1}{2}\lambda n \rceil$ . The inequality that  $a(\lambda K_n) \geq \lceil \frac{1}{2}\lambda n \rceil$  can also be seen easily, since any forest in  $\lambda K_n$  has at most  $n - 1$  edges. To determine  $s(\lambda K_n)$  and  $\text{sa}(\lambda K_n)$ , we first consider the easy case of  $\lambda$  even. For a positive integer  $k$ , we use  $S_k$  to denote the star with  $k$  edges.

**Lemma 2.** *For an even integer  $\lambda$ ,  $\text{sa}(\lambda K_n) = s(\lambda K_n) = \frac{1}{2}\lambda n$ .*

**Proof.** By the above discussions, we have  $\frac{1}{2}\lambda n \leq a(\lambda K_n) \leq \text{sa}(\lambda K_n) \leq s(\lambda K_n)$ . It suffices to show that  $s(\lambda K_n) \leq \frac{1}{2}\lambda n$ . Trivially the edges of  $\lambda K_n$  can be decomposed

into  $\frac{1}{2}\lambda$  copies of  $2K_n$  and the edges of  $2K_n$  can be decomposed into  $n$  copies of  $S_{n-1}$ . Thus the edges of  $\lambda K_n$  can be decomposed into  $\frac{1}{2}\lambda n$  copies of  $S_{n-1}$ , which implies  $s(\lambda K_n) \leq \frac{1}{2}\lambda n$ . This completes the proof.  $\square$

Now we determine  $s(\lambda K_n)$ .

**Theorem 3.**

$$s(\lambda K_n) = \begin{cases} \frac{1}{2}\lambda n & \text{if } \lambda \text{ is even,} \\ \frac{1}{2}(\lambda + 1)n - 1 & \text{if } \lambda \text{ is odd.} \end{cases}$$

*Proof.* Due to Lemma 2, we only need to show that for an odd integer  $\lambda$ ,  $s(\lambda K_n) = \frac{1}{2}(\lambda + 1)n - 1$ .

First prove  $s(\lambda K_n) \geq \frac{1}{2}(\lambda + 1)n - 1$ . Let  $\mathcal{D}$  be an arbitrary star decomposition of  $\lambda K_n$ . We need to show  $|\mathcal{D}| \geq \frac{1}{2}(\lambda + 1)n - 1$ . Let  $v_1, v_2, \dots, v_n$  be the vertices of  $\lambda K_n$ . For  $i = 1, 2, \dots, n$ , let  $c(v_i)$  be the number of stars in  $\mathcal{D}$  which have centers at  $v_i$  (for a star with only one edge, we arbitrarily choose one end of the edge as the center of the star). For  $1 \leq i < j \leq n$ , each edge joining  $v_i$  and  $v_j$  belongs to a star in  $\mathcal{D}$  which has center either at  $v_i$  or at  $v_j$ , and distinct edges joining  $v_i$  and  $v_j$  belong to distinct stars. Thus  $\lambda \leq c(v_i) + c(v_j)$ . We distinguish two cases.

*Case 1.*  $c(v_i) \geq \frac{1}{2}(\lambda + 1)$  for  $i = 1, 2, \dots, n$ . Then

$$|\mathcal{D}| = c(v_1) + c(v_2) + \dots + c(v_n) \geq \frac{1}{2}n(\lambda + 1) > \frac{1}{2}(\lambda + 1)n - 1.$$

This completes Case 1.

*Case 2.*  $c(v_i) \leq \frac{1}{2}(\lambda - 1)$  for some  $i$ , say  $c(v_1) \leq \frac{1}{2}(\lambda - 1)$ . Then

$$\begin{aligned} |\mathcal{D}| &= c(v_1) + c(v_2) + \dots + c(v_n) \\ &= \sum_{i=2}^n (c(v_1) + c(v_i)) - (n - 2)c(v_1) \\ &\geq (n - 1)\lambda - \frac{1}{2}(n - 2)(\lambda - 1) \\ &= \frac{1}{2}(\lambda + 1)n - 1. \end{aligned}$$

This completes Case 2.

We have proved  $|\mathcal{D}| \geq \frac{1}{2}(\lambda + 1)n - 1$  for any star decomposition  $\mathcal{D}$  of  $\lambda K_n$ . Thus  $s(\lambda K_n) \geq \frac{1}{2}(\lambda + 1)n - 1$ . Now we prove the reverse inequality. Note that  $\lambda K_n$  can be decomposed into  $\frac{1}{2}(\lambda - 1)$  copies of  $2K_n$  and one copy of  $K_n$ . Since  $2K_n$  can be decomposed into  $n$  copies of  $S_{n-1}$ , and  $K_n$  can be decomposed into  $n - 1$  stars, namely  $S_{n-1}, S_{n-2}, \dots, S_1$ , we see that  $\lambda K_n$  can be decomposed into  $\frac{1}{2}(\lambda + 1)n - 1$  stars. Thus  $s(\lambda K_n) \leq \frac{1}{2}(\lambda + 1)n - 1$ . This completes the proof.  $\square$

Now we determine  $\text{sa}(\lambda K_n)$ . Due to Lemma 2, we only need to consider the case of  $\lambda$  odd. Note that  $\text{sa}(K_n)$  has been determined by J. Akiyama and M. Kano as follows.

**Proposition 4** ([1], [9]).

$$\text{sa}(K_n) = \begin{cases} \lceil \frac{1}{2}n \rceil, & n = 2, 3, \\ \lceil \frac{1}{2}n \rceil + 1, & n \geq 4. \end{cases}$$

The following lemma is helpful for our discussions.

**Lemma 5.** *Let  $\lambda$  be any odd integer and  $n$  be an integer at least 3. Suppose that  $\mathcal{F}$  is a family of edge-disjoint subgraphs of  $\lambda K_n$  such that each member in  $\mathcal{F}$  is isomorphic to  $S_{n-1}$ . Then  $|\mathcal{F}| \leq \frac{1}{2}(\lambda - 1)n + 1$ . Furthermore, if  $|\mathcal{F}| = \frac{1}{2}(\lambda - 1)n + 1$ , then there are  $\frac{1}{2}(\lambda + 1)$  stars in  $\mathcal{F}$  with centers at one specific vertex of  $\lambda K_n$ , and there are  $\frac{1}{2}(\lambda - 1)$  stars in  $\mathcal{F}$  with centers at each of the remaining vertices of  $\lambda K_n$ .*

**Proof.** Let  $v_1, v_2, \dots, v_n$  be the vertices of  $\lambda K_n$ . For  $i = 1, 2, \dots, n$ , let  $c(v_i)$  denote the number of stars in  $\mathcal{F}$  which have centers at  $v_i$ . Without loss of generality, assume that  $c(v_1) \geq c(v_i)$  for  $i = 2, 3, \dots, n$ . For each  $i$  with  $2 \leq i \leq n$ , every  $S_{n-1}$  in  $\mathcal{F}$  with center at  $v_1$  contributes an edge joining  $v_1$  and  $v_i$ ; so does every  $S_{n-1}$  in  $\mathcal{F}$  with center at  $v_i$ . Combining these with the fact that there are only  $\lambda$  edges joining  $v_1$  and  $v_i$  in  $\lambda K_n$ , we have  $c(v_1) + c(v_i) \leq \lambda$ . Now we show that  $|\mathcal{F}| \leq \frac{1}{2}(\lambda - 1)n + 1$ . We distinguish two cases.

*Case 1.*  $c(v_1) \leq \frac{1}{2}(\lambda - 1)$ . Then

$$\begin{aligned} |\mathcal{F}| &= c(v_1) + c(v_2) + \dots + c(v_n) \\ &\leq \frac{1}{2}n(\lambda - 1) < \frac{1}{2}(\lambda - 1)n + 1. \end{aligned}$$

This completes Case 1.

*Case 2.*  $c(v_1) \geq \frac{1}{2}(\lambda + 1)$ .

Let  $c(v_1) = \frac{1}{2}(\lambda + 1) + s$  where  $s$  is a nonnegative integer. Then  $c(v_i) \leq \frac{1}{2}(\lambda - 1) - s$ , for  $2 \leq i \leq n$ . Hence

$$\begin{aligned} (1) \quad |\mathcal{F}| &= c(v_1) + c(v_2) + \dots + c(v_n) \\ &\leq \left(\frac{1}{2}(\lambda + 1) + s\right) + (n - 1)\left(\frac{1}{2}(\lambda - 1) - s\right) \\ &= \frac{1}{2}(\lambda - 1)n + 1 + (2 - n)s \leq \frac{1}{2}(\lambda - 1)n + 1. \end{aligned}$$

The last inequality is due to  $n \geq 3$  and  $s$  being nonnegative. This completes Case 2.

The required inequality that  $|\mathcal{F}| \leq \frac{1}{2}(\lambda - 1)n + 1$  has thus been established. Now we prove the “Furthermore” part. Since  $|\mathcal{F}| = \frac{1}{2}(\lambda - 1)n + 1$ , only Case 2 in the above discussion is possible and the inequalities in (1) become equalities; from the last inequality, we have  $s = 0$  since  $n \geq 3$ , and from the first inequality, we have

$$\begin{aligned} c(v_2) &= c(v_3) = \dots = c(v_n) = \frac{1}{2}(\lambda - 1) - s = \frac{1}{2}(\lambda - 1), \\ c(v_1) &= \frac{1}{2}(\lambda + 1) + s = \frac{1}{2}(\lambda + 1). \end{aligned}$$

Thus the required conclusion holds. □

The above lemma is used in the following.

**Lemma 6.** For an odd integer  $\lambda \geq 3$ ,  $\text{sa}(\lambda K_n) = \frac{1}{2}(\lambda - 1)n + \text{sa}(K_n)$ .

*Proof.* It is easy to see that  $\text{sa}(\lambda K_2) = \lambda$  for any  $\lambda \geq 1$ . Thus the required equality holds for  $n = 2$ . So we let  $n \geq 3$ .

By the definition of star arboricity,  $\text{sa}(\lambda K_n) \leq \text{sa}((\lambda - 1)K_n) + \text{sa}(K_n)$  for  $\lambda \geq 2$ . Now  $\lambda$  is odd. By Lemma 2,  $\text{sa}((\lambda - 1)K_n) = \frac{1}{2}(\lambda - 1)n$ . Thus  $\text{sa}(\lambda K_n) \leq \frac{1}{2}(\lambda - 1)n + \text{sa}(K_n)$ . We now prove the reverse inequality.

Let  $\mathcal{D}$  be an arbitrary star forest decomposition of  $\lambda K_n$ . We need to show that  $|\mathcal{D}| \geq \frac{1}{2}(\lambda - 1)n + \text{sa}(K_n)$ . Let  $\mathcal{D}'$  be a subfamily of  $\mathcal{D}$  consisting of members which are isomorphic to  $S_{n-1}$ . By Lemma 5,  $|\mathcal{D}'| \leq \frac{1}{2}(\lambda - 1)n + 1$ . Consider two cases: Case 1:  $|\mathcal{D}'| = \frac{1}{2}(\lambda - 1)n + 1$ , Case 2:  $|\mathcal{D}'| \leq \frac{1}{2}(\lambda - 1)n$ .

*Case 1.*  $|\mathcal{D}'| = \frac{1}{2}(\lambda - 1)n + 1$ .

Let  $v_1, v_2, \dots, v_n$  be the vertices of  $\lambda K_n$ . From the “Furthermore” part of Lemma 5,  $\mathcal{D}'$  is a family consisting of the following stars:  $\frac{1}{2}(\lambda + 1) S_{n-1}$ ’s with centers at a specific vertex, say  $v_1$ , and  $\frac{1}{2}(\lambda - 1) S_{n-1}$ ’s with centers at each of the remaining vertices. Thus  $\bigcup_{G \in \mathcal{D}'} E(G)$  is an edge set consisting of the following edges:  $\lambda$  edges joining  $v_1$  and  $v_i$  for every  $i$  with  $2 \leq i \leq n$ , and  $\lambda - 1$  edges joining  $v_i$  and  $v_j$  for every pair  $i, j$  with  $2 \leq i < j \leq n$ . We then see that  $\lambda K_n - \bigcup_{G \in \mathcal{D}'} E(G)$  is a disjoint union of  $K_{n-1}$  and  $K_1$  (to be specific, the complete graph on the vertices  $v_2, v_3, \dots, v_n$  and the trivial graph on the vertex  $v_1$ ). Thus  $\mathcal{D} - \mathcal{D}'$  is a star forest decomposition of  $K_{n-1}$ , which implies  $|\mathcal{D} - \mathcal{D}'| \geq \text{sa}(K_{n-1})$ . Hence

$$\begin{aligned} |\mathcal{D}| &= |\mathcal{D}'| + |\mathcal{D} - \mathcal{D}'| \\ &\geq \frac{1}{2}(\lambda - 1)n + 1 + \text{sa}(K_{n-1}) \geq \frac{1}{2}(\lambda - 1)n + \text{sa}(K_n). \end{aligned}$$

The last inequality is due to the fact  $\text{sa}(K_n) \leq \text{sa}(K_{n-1}) + 1$ , which follows from the definition of star arboricity. This completes Case 1.

Case 2.  $|\mathcal{D}'| \leq \frac{1}{2}(\lambda - 1)n$ .

Note that each member in  $\mathcal{D}'$  has exactly  $n - 1$  edges and each member in  $\mathcal{D} - \mathcal{D}'$  has at most  $n - 2$  edges. Hence

$$\begin{aligned} |E(\lambda K_n)| &\leq |\mathcal{D}'|(n - 1) + |\mathcal{D} - \mathcal{D}'|(n - 2) \\ &= |\mathcal{D}|(n - 2) + |\mathcal{D}'| \leq |\mathcal{D}|(n - 2) + \frac{1}{2}(\lambda - 1)n. \end{aligned}$$

Thus  $\lambda \binom{n}{2} \leq |\mathcal{D}|(n - 2) + \frac{1}{2}(\lambda - 1)n$ , which implies that

$$(n - 2)|\mathcal{D}| \geq \lambda \binom{n}{2} - \frac{1}{2}(\lambda - 1)n = \frac{1}{2}\lambda n(n - 2) + \frac{1}{2}n.$$

Hence

$$\begin{aligned} |\mathcal{D}| &\geq \frac{1}{2}\lambda n + \frac{n}{2(n-2)} = \frac{1}{2}(\lambda - 1)n + \frac{n(n-1)}{2(n-2)} \\ &> \frac{1}{2}(\lambda - 1)n + \frac{1}{2}(n + 1) \geq \frac{1}{2}(\lambda - 1)n + \lceil \frac{1}{2}n \rceil. \end{aligned}$$

Thus  $|\mathcal{D}| \geq \frac{1}{2}(\lambda - 1)n + \lceil \frac{1}{2}n \rceil + 1$ .

Combining this with Proposition 4, we obtain

$$|\mathcal{D}| \geq \frac{1}{2}(\lambda - 1)n + \text{sa}(K_n).$$

This completes Case 2.

Since we have proved that  $|\mathcal{D}| \geq \frac{1}{2}(\lambda - 1)n + \text{sa}(K_n)$  for any star forest decomposition  $\mathcal{D}$  of  $\lambda K_n$ , we obtain  $\text{sa}(\lambda K_n) \geq \frac{1}{2}(\lambda - 1)n + \text{sa}(K_n)$ . This completes the proof.  $\square$

Now we have the star arboricity of  $\lambda K_n$  as follows.

**Theorem 7.**

$$\text{sa}(\lambda K_n) = \begin{cases} \lceil \frac{1}{2}\lambda n \rceil & \text{if } \lambda \text{ is odd, } n = 2, 3 \text{ or } \lambda \text{ is even,} \\ \lceil \frac{1}{2}\lambda n \rceil + 1 & \text{if } \lambda \text{ is odd, } n \geq 4. \end{cases}$$

*Proof.* By Lemma 2, the formula holds for even  $\lambda$ . By Proposition 4, the formula holds for  $\lambda = 1$ . As to odd  $\lambda \geq 3$ , by Lemma 6 and Proposition 4,

$$\begin{aligned} \text{sa}(\lambda K_n) &= \frac{1}{2}(\lambda - 1)n + \text{sa}(K_n) = \begin{cases} \frac{1}{2}(\lambda - 1)n + \lceil \frac{1}{2}n \rceil, & n = 2, 3, \\ \frac{1}{2}(\lambda - 1)n + \lceil \frac{1}{2}n \rceil + 1, & n \geq 4 \end{cases} \\ &= \begin{cases} \lceil \frac{1}{2}\lambda n \rceil, & n = 2, 3, \\ \lceil \frac{1}{2}\lambda n \rceil + 1, & n \geq 4. \end{cases} \end{aligned}$$

This completes the proof.  $\square$

## References

- [1] *J. Akiyama, M. Kano*: Path factors of a graph. In: Graphs and Applications. Proceedings of the first Colorado Symposium on Graph Theory (F. Harary, J. Maybee, eds.). 1982, pp. 1–21.
- [2] *I. Algor, N. Alon*: The star arboricity of graphs. *Discrete Math.* 75 (1989), 11–22.  
[Zbl 0684.05033](#)
- [3] *Y. Aoki*: The star-arboricity of the complete regular multipartite graphs. *Discrete Math.* 81 (1990), 115–122.  
[Zbl 0737.05038](#)
- [4] *B. Bollobás*: *Extremal Graph Theory*. Academic Press, New York, 1978.  
[Zbl 0419.05031](#)
- [5] *J. A. Bondy, U. S. R. Murty*: *Graph Theory with Applications*. North Holland, New York, 1976.
- [6] *Y. Egawa, T. Fukuda, S. Nagoya, M. Urabe*: A decomposition of complete bipartite graphs into edge-disjoint subgraphs with star components. *Discrete Math.* 58 (1986), 93–95.  
[Zbl 0596.05050](#)
- [7] *H. Enomoto, Y. Usami*: The star arboricity of complete bipartite graphs. *Graph Theory, Combinatorics and Applications 1* (1988), 389–396.  
[Zbl 0841.05047](#)
- [8] *J. Hagauer, S. Klavžar*: On independence numbers of the Cartesian product of graphs. *ARS Combin.* 43 (1996), 149–157.  
[Zbl 0881.05061](#)
- [9] *S. L. Hakimi, J. Mitchem, E. Schmeichel*: Star arboricity of graphs. *Discrete Math.* 149 (1996), 93–98.  
[Zbl 0843.05037](#)
- [10] *P. K. Jha, S. Klavžar*: Independence in direct-product graphs. *ARS Combin.* 50 (1998), 53–63.  
[Zbl 0963.05069](#)
- [11] *C. Lin, J.-J. Lin, and H.-C. Lee*: Some decomposition invariants of crowns. *ARS Combin.* 66 (2003), 33–48.
- [12] *C. Lin, J.-J. Lin, T.-W. Shyu*: Isomorphic star decomposition of multicrowns and the power of cycles. *ARS Combin.* 53 (1999), 249–256.  
[Zbl 0994.05113](#)
- [13] *K.-W. Lih, D.-F. Liu, and X. Zhu*: Star extremal circulant graphs. *SIAM J. Discrete Math.* 12 (1999), 491–499.  
[Zbl 0935.05042](#)
- [14] *C. St. J. A. Nash-Williams*: Decomposition of finite graphs into forests. *J. London Math. Soc.* 39 (1964), 12.  
[Zbl 0119.38805](#)
- [15] *M. Truszczynski*: Decomposing graphs into forests of stars. *Congr. Numer.* 54 (1986), 73–86.  
[Zbl 0646.05022](#)

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