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± SIGN PATTERN MATRICES THAT ALLOW ORTHOGONALITY

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Abstract. A sign pattern A is a $\pm$ sign pattern if A has no zero entries. A allows orthogonality if there exists a real orthogonal matrix B whose sign pattern equals A. Some sufficient conditions are given for a sign pattern matrix to allow orthogonality, and a complete characterization is given for $\pm$ sign patterns with $n-1 \leqslant N_-(A) \leqslant n+1$ to allow orthogonality.

Keywords: sign pattern, orthogonality, orthogonal matrix

MSC 2000: 15A18, 15A48

1. Introduction

A *sign pattern* (*matrix*) A is a matrix whose entries are in the set $\{+,-,0\}$. Denote the set of all $n\times n$ sign patterns by Q_n. Associated with each $A\in Q_n$ is a class of real matrices, called the *qualitative class* of A, defined by

$$Q(A)=\{B\in M_n(R)\colon\ \operatorname{sign} B=A\}.$$

A sign pattern $P\in Q_n$ is called a *permutation pattern* if exactly one entry in each row and column is equal to $+$, and all other entries are 0. We call that sign patterns A and B are *permutation equivalent*, if there are permutation patterns P_1 and P_2 such that $B=P_1AP_2$. Let $A\in Q_n$. A *allows orthogonality* if there exists a real orthogonal matrix $B\in Q(A)$. Clearly, the following results hold.

Lemma 1.1. *Let $n\geqslant 2$.*

(1) *Every permutation pattern of order n allows orthogonality.*

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(2) *A sign pattern $A \in Q_n$ allows orthogonality if and only if $-A$ allows orthogonality.*
(3) *A sign pattern $A \in Q_n$ allows orthogonality if and only if the transpose, A^T, of A allows orthogonality.*
(4) *A sign pattern $A \in Q_n$ allows orthogonality if and only if P_1AP_2 allows orthogonality for any permutation patterns P_1 and P_2.*

A sign pattern whose entries belong to $\{+,-\}$ is called a $\pm$ sign pattern. A pair of sign pattern row vectors (or column vectors) allows orthogonality if the two vectors are the sign patterns for two real orthogonal row vectors (respectively, column vectors). A square sign pattern that does not have a zero row or zero column is sign potentially orthogonal (SPO) if every pair of rows and every pair of columns allows orthogonality. It is known that all SPO matrices of dimension $n < 5$ allow orthogonality and all $\pm$ SPO matrices of order $n < 6$ allow orthogonality ([3]).

The motivation of this paper is from Refs. [1]–[5].

Let $A \in Q_n$ and $n \geqslant 2$. We denote the number of negative entries in A by $N_-(A)$. Clearly, if a $\pm$ sign pattern A allows orthogonality, then $n-1 \leqslant N_-(A) \leqslant n^2-(n-1) = n^2-n+1$. In this paper, some sufficient conditions are given for a sign pattern matrix to allow orthogonality, and a complete characterization for $\pm$ sign patterns with $n-1 \leqslant N_-(A) \leqslant n+1$ to allow orthogonality is given. By negation, the combinatorial structure of orthogonal matrices A with $n^2-n-1 \leqslant N_-(A) \leqslant n^2-n+1$ is clear.

2. Preliminaries

Let $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{m\times m}$ be two real matrices (or sign patterns), $1 \leqslant s \leqslant n$ and $1 \leqslant t \leqslant m$. The following real matrix (or sign pattern) of order $n+m-1$

$$(2.1)\quad C = \begin{bmatrix} 0 & \ddots & 0 & b_{11} & b_{12} & \dots & b_{1m} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & b_{t-1,1} & b_{t-1,2} & \dots & b_{t-1,m} & 0 & \dots & 0 \\ a_{11} & \dots & a_{1,s-1} & b_{t1}a_{1s} & b_{t2}a_{1s} & \dots & b_{tm}a_{1s} & a_{1,s+1} & \dots & a_{1n} \\ a_{21} & \dots & a_{2,s-1} & b_{t1}a_{2s} & b_{t2}a_{2s} & \dots & b_{tm}a_{2s} & a_{2,s+1} & \dots & a_{2n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{n,s-1} & b_{t1}a_{ns} & b_{t2}a_{ns} & \dots & b_{tm}a_{ns} & a_{n,s+1} & \dots & a_{nn} \\ 0 & \dots & 0 & b_{t+1,1} & b_{t+1,2} & \dots & b_{t+1,m} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & b_{m1} & b_{m2} & \dots & b_{mm} & 0 & \dots & 0 \end{bmatrix}$$

is called the amalgamation of A and B according to the column s of A and the row t of B. Denote $U_{s,t}(A,B) = C$.

Theorem 2.1. *Let $A \in Q_n$ allow orthogonality and $B \in Q_m$ allow orthogonality, $1 \leqslant s \leqslant n$ and $1 \leqslant t \leqslant m$. Then the sign pattern $U_{s,t}(A,B)$ allows orthogonality.*

Proof. The proof is easy, and we omit it.

Theorem 2.2. *Let $1 \leqslant s, t \leqslant n$, $s \neq t$, and $A = (a_{ij}) \in Q_n$ with $a_{it} \neq 0$ if $a_{is} \neq 0$, $i = 1, 2, \ldots, n$. If A allows orthogonality, then the sign pattern $\tilde{A} = (\tilde{a}_{ij}) \in Q_n$, where*

$$\tilde{a}_{ij} = \begin{cases} a_{it}, & \text{if } j = s \text{ and } a_{ij} = 0; \\ a_{ij}, & \text{otherwise,} \end{cases}$$

also allows orthogonality.

Proof. Since A allows orthogonality, there is a real orthogonal matrix $B = (b_{ij}) \in Q(A)$, where $b_{it} \neq 0$ if $b_{is} \neq 0$, $i = 1, 2, \ldots, n$. Let $\theta > 0$ be a sufficiently small real number. We consider the real matrix $C = (c_{ij})$ of order n, where

$$c_{ij} = \begin{cases} 1, & \text{if } i = j \neq s, \text{ or } i = j \neq t; \\ \cos\theta, & \text{if } i = j = s, \text{ or } i = j = t; \\ -\sin\theta, & \text{if } i = s, \ j = t; \\ \sin\theta, & \text{if } i = t, \ j = s; \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, for any real number $0 < \theta < 1$, C is an orthogonal matrix. Thus BC is an orthogonal matrix.

Note that B and BC are entrywise equal except for the sth and the tth columns. The sth and the tth columns of BC are

$$\begin{bmatrix} b_{1s}\cos\theta + b_{1t}\sin\theta \\ b_{2s}\cos\theta + b_{2t}\sin\theta \\ \vdots \\ b_{ns}\cos\theta + b_{nt}\sin\theta \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -b_{1s}\sin\theta + b_{1t}\cos\theta \\ -b_{2s}\sin\theta + b_{2t}\cos\theta \\ \vdots \\ -b_{ns}\sin\theta + b_{nt}\cos\theta \end{bmatrix},$$

respectively. Now we can choose some $\theta > 0$ sufficiently close to 0 so that $BC \in Q(\tilde{A})$. Thus the sign pattern $\tilde{A}$ allows orthogonality.

By Theorem 2.2 and Lemma 1.1 (3), we obviously have the following corollary.

Corollary 2.3. *Let* $1 \leqslant s,t \leqslant n$, $s \neq t$, *and* $A = (a_{ij}) \in Q_n$ *with* $a_{tj} \neq 0$ *if* $a_{sj} \neq 0$, $j = 1, 2, \ldots, n$. *If* A *allows orthogonality, then the sign pattern* $\tilde{A} = (\tilde{a}_{ij}) \in Q_n$, *where*

$$\tilde{a}_{ij} = \begin{cases} a_{tj}, & \text{if } i = s \text{ and } a_{ij} = 0; \\ a_{ij}, & \text{otherwise,} \end{cases}$$

also allows orthogonality.

3. Sufficient conditions

In this section, we give some sufficient conditions for a sign pattern to allow orthogonality. We need the following notation. We denote the negative entries distribution of rows (columns) of A as $d_r(A) = (r_1, r_2, \ldots, r_n)$ ($d_l(A) = (l_1, l_2, \ldots, l_n)$), if there are exactly r_i (l_i) negative entries in the ith row (column) of A for $i = 1, 2, \ldots, n$. Note that changing the order of $r_1, r_2, \ldots, r_n$ ($l_1, l_2, \ldots, l_n$) means changing the rows' (columns') order of A. If $d_r(B) = (r'_1, r'_2, \ldots, r'_n)$, $d_l(B) = (l'_1, l'_2, \ldots, l'_n)$, and $\{r_1, r_2, \ldots, r_n\} = \{r'_1, r'_2, \ldots, r'_n\}$, $\{l_1, l_2, \ldots, l_n\} = \{l'_1, l'_2, \ldots, l'_n\}$, then there are permutation matrices P_1 and P_2 such that $d_r(P_1BP_2) = d_r(A)$, $d_l(P_1BP_2) = d_l(A)$.

Lemma 3.1. *The sign pattern*

$$A = \begin{bmatrix} - & + & + & \ldots & + \\ 0 & - & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & + \\ 0 & \ldots & 0 & - & + \\ + & \ldots & + & + & + \end{bmatrix} \in Q_n \quad (n \geqslant 3) \tag{3.1}$$

allows orthogonality.

Take

$$A_2 = \begin{bmatrix} + & + \\ - & + \end{bmatrix}.$$

It is clear that A_2 allows orthogonality.

Take

$$A_3 = U_{2,1}(A_2, A_2) = \begin{bmatrix} + & + & + \\ - & + & + \\ 0 & - & + \end{bmatrix}, \ A_4 = U_{3,1}(A_3, A_2) = \begin{bmatrix} + & + & + & + \\ - & + & + & + \\ 0 & - & + & + \\ 0 & 0 & - & + \end{bmatrix}, \ \ldots,$$

$$A_n = U_{n-1,1}(A_{n-1}, A_2) = \begin{bmatrix} + & + & \ldots & + & + \\ - & + & \ldots & + & + \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & + & + \\ 0 & 0 & \ldots & - & + \end{bmatrix}.$$

Then, for $k = 3, 4, \ldots, n$, it is easy to see that A_k allows orthogonality by Theorem 2.1. By interchanging the ith row and the $(i+1)$th row of A_n, $i = 1, 2, \ldots, n-1$, respectively, we see that the lemma holds. □

Lemma 3.2. *The sign pattern*

$$A = \begin{bmatrix} + & - & - & \ldots & - & - \\ + & - & - & \ldots & - & + \\ + & - & \ldots & - & + & 0 \\ \vdots & \vdots & \iddots & \iddots & \iddots & \vdots \\ + & - & + & 0 & \ldots & 0 \\ + & + & 0 & 0 & \ldots & 0 \end{bmatrix} \in Q_n \quad (n \geqslant 3) \tag{3.2}$$

allows orthogonality.

P r o o f. Clearly, the sign pattern

$$A_2 = \begin{bmatrix} + & - \\ + & + \end{bmatrix}$$

allows orthogonality.

Take

$$A_3 = U_{1,1}(A_2, A_2) = \begin{bmatrix} + & - & - \\ + & - & + \\ + & + & 0 \end{bmatrix}, \ A_4 = U_{1,1}(A_3, A_2) = \begin{bmatrix} + & - & - & - \\ + & - & - & + \\ + & - & + & 0 \\ + & + & 0 & 0 \end{bmatrix}, \ \ldots,$$

$$A_n = U_{1,1}(A_{n-1}, A_2) = \begin{bmatrix} + & - & \ldots & \ldots & - & - \\ + & - & \ldots & \ldots & - & + \\ + & - & \ldots & - & + & 0 \\ \vdots & \vdots & \iddots & \iddots & \iddots & \vdots \\ + & - & + & 0 & \ldots & 0 \\ + & + & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

Then, for $k = 3, 4, \ldots, n$, A_k allows orthogonality by Theorem 2.1. □

Lemma 3.3. *Let $A = (a_{ij}) \in Q_n$ $(n \geqslant 3)$, where*

$$a_{ij} = \begin{cases} -, & i = j; \\ +, & \text{otherwise.} \end{cases}$$

Then A allows orthogonality.

Proof. Let

$$B = \begin{bmatrix} -\frac{n-2}{n} & \frac{2}{n} & \frac{2}{n} & \ldots & \frac{2}{n} \\ \frac{2}{n} & -\frac{n-2}{n} & \frac{2}{n} & \ldots & \frac{2}{n} \\ \frac{2}{n} & \frac{2}{n} & -\frac{n-2}{n} & \ldots & \frac{2}{n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{2}{n} & \frac{2}{n} & \frac{2}{n} & \ldots & -\frac{n-2}{n} \end{bmatrix}.$$

Then $B \in Q(A)$ and B is a real orthogonal matrix. Thus the lemma holds. □

Lemma 3.4. *Let $3 \leqslant t \leqslant n-1$ and $A = (a_{ij}) \in Q_n$, where*

$$a_{ij} = \begin{cases} -, & i = j; \\ -, & j = 1, \ 1 \leqslant i \leqslant n-t+1; \\ 0, & i > j, \ j = 2, 3, \ldots, n-t+1; \\ +, & \text{otherwise.} \end{cases}$$

Then A allows orthogonality.

Proof. Let

$$B_t = \begin{bmatrix} - & + & \ldots & + \\ + & - & \ldots & + \\ \vdots & \vdots & \ddots & \vdots \\ + & + & \ldots & - \end{bmatrix}$$

be a sign pattern of order t, and

$$A_2 = \begin{bmatrix} + & + \\ - & + \end{bmatrix}.$$

Take

$$C_{t+1} = U_{2,1}(A_2, B_t) = \begin{bmatrix} + & - & + & + & \ldots & + \\ - & - & + & + & \ldots & + \\ 0 & + & - & + & \ldots & + \\ 0 & + & + & - & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & + \\ 0 & + & + & \ldots & + & - \end{bmatrix}.$$

A_{t+1} is a sign pattern obtained by interchanging the first column and the second column of C_{t+1}. Taking $C_{k+1} = U_{2,1}(A_2, A_k)$, then A_{k+1} is a sign pattern obtained by interchanging the first column and the second column of C_{k+1} for $k = t+1, \ldots, n-1$. It is easy to see that $A_n = A$, and A_n allows orthogonality by Theorem 2.1. □

Lemma 3.5. *Let $n > t \geqslant 4$, and let the $\pm$ sign pattern*

$$A_t = \begin{bmatrix} B & D \\ C & + \end{bmatrix} \in Q_t$$

allow orthogonality, where the entries of D are all "+". Then the $n \times n$ sign pattern

$$F_n = \begin{bmatrix} B & D & \ldots & D & D \\ + & - & + & \ldots & + \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ + & \ldots & + & - & + \\ C & + & \ldots & + & + \end{bmatrix}$$

allows orthogonality.

Proof. Let

$$B_2 = \begin{bmatrix} + & + \\ - & + \end{bmatrix}.$$

Take $A_{k+1} = U_{k,1}(A_k, B_2)$ for $k = t, t+1, \ldots, n-1$. C_n is a sign pattern obtained by interchanging the kth row and the $(k+1)$th row of A_n for $k = t, t+1, \ldots, n-1$. Then

$$C_n = \begin{bmatrix} B & D & \ldots & D & D \\ 0 & - & + & \ldots & + \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & - & + \\ C & + & \ldots & + & + \end{bmatrix}.$$

So F_n allows orthogonality by Theorem 2.2. □

Lemma 3.6. *Let a $\pm$ sign pattern $A = (a_{ij}) \in Q_n$ allow orthogonality, $d_r(A) = (r_1, r_2, \ldots, r_n)$, $d_l(A) = (l_1, l_2, \ldots, l_n)$, and let there be $l_i > 1$ in $d_l(A)$.*

(1) *If $a_{si} = a_{ti} = -$, then either $r_s > 1$ or $r_t > 1$.*

(2) *There are at least $l_i - 1$ entries in $d_r(A)$ whose values are larger than 1.*

Proof. (1) If $r_s = r_t = 1$, then the sign patterns of the sth row and the tth row are the same. This is a contradiction.

(2) By (1), this is clear. □

Corollary 3.7. *Let $\alpha = (r_1, r_2, \ldots, r_n), \beta = (l_1, l_2, \ldots, l_n)$. If there are m entries altogether in α whose values are larger than 1, and there is l_i in β with $l_i \geqslant m+2$, then there is no $\pm$ sign pattern A allowing orthogonality with $d_r(A) = \alpha$ and $d_l(A) = \beta$ (or $d_l(A) = \alpha$ and $d_r(A) = \beta$).*

4. Main results

In this section, a complete characterization for $\pm$ sign patterns with $n-1 \leqslant N_-(A) \leqslant n+1$ to allow orthogonality is given.

Theorem 4.1. *Up to the transpose, and permutation equivalence, an $n \times n$ ($n \geqslant 3$) $\pm$ sign pattern A with $N_-(A) = n-1$ allows orthogonality if and only if*

$$A = \begin{bmatrix} - & + & \ldots & + & + \\ + & - & \ddots & & + \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ + & & \ddots & - & + \\ + & + & \ldots & + & + \end{bmatrix}. \tag{4.1}$$

P r o o f. We see that the A in (4.1) allows orthogonality by Lemma 3.1 and Theorem 2.2.

Conversely, let A allow orthogonality and $N_-(A) = n-1$. Note that A can not have two rows (columns) which have the same patterns. The negative entries must be in different rows and columns. So (4.1) holds. $\square$

Theorem 4.2. *Up to the transpose, and permutation equivalence, an $n \times n$ ($n \geqslant 3$) $\pm$ sign pattern A with $N_-(A) = n$ allows orthogonality if and only if*

$$A = \begin{bmatrix} - & + & + & \ldots & + & + \\ * & - & + & \ldots & + & + \\ + & + & - & \ldots & + & + \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ + & + & + & \ldots & - & + \\ + & + & + & \ldots & + & * \end{bmatrix}, \tag{4.2}$$

where $* \in \{+, -\}$, and exactly one $*$ can be "$-$".

P r o o f. When $a_{21} = -$, we see that the A in (4.2) allows orthogonality by Lemma 3.1 and Theorem 2.2. When $a_{nn} = -$, we see that the A in (4.2) allows orthogonality by Lemma 3.3.

Conversely, let A allow orthogonality and $N_-(A) = n$. Note that A can not have two rows (columns) which have the same patterns, and $d_r(A)$ ($d_l(A)$) must be one of $(1, 1, \ldots, 1)$, $(2, 1, \ldots, 1, 0)$.

Case 1. $d_r(A) = (1, 1, \ldots, 1)$.

By Corollary 3.7, $d_l(A) = (1, 1, \ldots, 1)$. So (4.2) holds ($a_{nn} = -$).

Case 2. $d_r(A) = (2, 1, \ldots, 1, 0)$.

We only need to consider $d_l(A) = (2, 1, \ldots, 1, 0)$. Let $a_{11} = a_{21} = -$. By Lemma 3.6 we can let $r_2 = 2$, $a_{22} = -$, and $a_{ii} = -$, $3 \leqslant i \leqslant n-1$. So (4.2) holds ($a_{21} = -$). $\square$

Theorem 4.3. *Up to the transpose, and permutation equivalence, an $n \times n$ ($n \geqslant 5$) $\pm$ sign pattern A with $N_-(A) = n+1$ allows orthogonality if and only if*

$$A = \begin{bmatrix} - & + & + & + & + & \ldots & + & + \\ - & - & + & + & + & \ldots & + & + \\ * & * & - & + & + & \ldots & + & + \\ + & + & * & - & + & \ldots & + & + \\ + & + & + & + & - & \ldots & + & + \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ + & + & + & + & + & \ldots & - & + \\ + & * & + & + & + & \ldots & + & * \end{bmatrix}, \tag{4.3}$$

where $* \in \{+, -\}$, and exactly one $*$ can be "$-$".

P r o o f. First, we prove that the A in (4.3) allows orthogonality.

When there is "$-$" in $\{a_{31}, a_{32}, a_{43}\}$, we see that the A in (4.3) allows orthogonality by Lemma 3.1 and Theorem 2.2.

When $a_{nn} = -$, we see that the A in (4.3) allows orthogonality by Lemma 3.4 ($t = n-1$) and Theorem 2.2.

When $a_{n2} = -$, note that

$$C = \begin{bmatrix} - & + & + & + \\ - & - & + & + \\ + & + & - & + \\ + & - & + & + \end{bmatrix}$$

is a $\pm$ SPO matrix and all $\pm$ SPO matrices of order $n < 6$ allow orthogonality. C allows orthogonality. By Lemma 3.5, we see that the A in (4.3) allows orthogonality.

Conversely, let A allow orthogonality and $N_-(A) = n+1$. Note that A can not have two rows (columns) which have the same patterns, and $d_r(A)$ ($d_l(A)$) must be one of $(2, 1, \ldots, 1)$, $(3, 1, \ldots, 1, 0)$, or $(2, 2, 1, \ldots, 1, 0)$.

Case 1. $d_r(A) = (2, 1, \ldots, 1)$.

We only need to consider $d_l(A) = (2, 1, \ldots, 1)$ or $d_l(A) = (2, 2, 1, \ldots, 1, 0)$ by Corollary 3.7.

(1) $d_l(A) = (2, 1, \ldots, 1)$. Let $a_{11} = a_{21} = -$. By Lemma 3.6, we can let $r_2 = 2$, $a_{22} = -$, and $a_{ii} = -$, $3 \leqslant i \leqslant n$. So (4.3) holds ($a_{nn} = -$).

(2) $d_l(A) = (2, 2, 1, \ldots, 1, 0)$. Let $a_{11} = a_{21} = -$. By Lemma 3.6, we can let $r_2 = 2$, and $a_{22} = -$. Let $l_i = 2$. If $i \geqslant 3$, and $a_{si} = a_{ti} = -$, it is clear that $s > 2$, $t > 2$ and the sign patterns of the sth row and the tth row are the same. This is a contradiction. So $l_2 = 2$. Let $a_{n2} = -$, and $a_{ii} = -$, $3 \leqslant i \leqslant n - 1$. So (4.3) holds ($a_{n2} = -$).

Case 2. $d_r(A) = (3, 1, \ldots, 1, 0)$.

We only need to consider $d_l(A) = (2, 2, 1, \ldots, 1, 0)$ by Corollary 3.7. Taking the transpose, we can consider $d_l(A) = (3, 1, \ldots, 1, 0)$, $d_r(A) = (2, 2, 1, \ldots, 1, 0)$. Let $a_{11} = a_{21} = a_{31} = -$. By Lemma 3.6, we can let $r_2 = r_3 = 2$, and $a_{22} = -$. If $a_{32} = -$, then the sign patterns of the second row and the third row are the same. This is a contradiction. So let $a_{33} = -$ and $a_{ii} = -$, $4 \leqslant i \leqslant n - 1$. Thus (4.3) holds ($a_{31} = -$).

Case 3. $d_r(A) = (2, 2, 1, \ldots, 1, 0)$.

We only need to consider $d_l(A) = (2, 2, 1, \ldots, 1, 0)$. Let $a_{11} = a_{21} = -$. By Lemma 3.6, let $r_2 = 2$, and $a_{22} = -$.

(1) $l_2 = 2$.

It is clear that $a_{12} = +$. Let $a_{32} = -$. If there is $i > 3$ such that $r_i = 2$, and $a_{is} = a_{it} = -$, we have $s > 2$, $t > 2$ and the sign patterns of the sth column and the tth column are the same. This is a contradiction. So either $r_1 = 2$ or $r_3 = 2$. Let either $a_{13} = -$ or $a_{33} = -$, and $a_{ii} = -$, $4 \leqslant i \leqslant n - 1$. If $a_{13} = -$, note that

$$\begin{bmatrix} 0 & 0 & + \\ 0 & + & 0 \\ + & 0 & 0 \end{bmatrix} \begin{bmatrix} - & + & - \\ - & - & + \\ + & - & + \end{bmatrix} \begin{bmatrix} 0 & + & 0 \\ + & 0 & 0 \\ 0 & 0 & + \end{bmatrix} = \begin{bmatrix} - & + & + \\ - & - & + \\ + & - & - \end{bmatrix}.$$

We see that (4.3) holds ($a_{32} = -$). If $a_{33} = -$, then (4.3) holds ($a_{32} = -$).

(2) $l_2 = 1$.

Let $l_3 = 2$. If $a_{13} = -$, let $a_{33} = -$. Note that

$$\begin{bmatrix} 0 & 0 & + \\ + & 0 & 0 \\ 0 & + & 0 \end{bmatrix} \begin{bmatrix} - & + & - \\ - & - & + \\ + & + & - \end{bmatrix} \begin{bmatrix} 0 & + & 0 \\ 0 & 0 & + \\ + & 0 & 0 \end{bmatrix} = \begin{bmatrix} - & + & + \\ - & - & + \\ + & - & - \end{bmatrix}.$$

We see that (4.3) holds ($a_{32} = -$). If $a_{13} = +$, letting $a_{33} = a_{43} = -$, then either $r_3 = 2$ or $r_4 = 2$. Let $r_4 = 2$, $a_{44} = -$, and $a_{ii} = -$, $5 \leqslant i \leqslant n - 1$. We see that (4.3) holds ($a_{43} = -$). □

Consequently, we have characterized the $\pm$ sign patterns with $n-1 \leqslant N_-(A) \leqslant n+1$ allowing orthogonality completely.

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