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On weak-open $\pi$-images of metric spaces


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ON WEAK-OPEN $\pi$-IMAGES OF METRIC SPACES

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Abstract. In this paper, we give some characterizations of metric spaces under weak-open $\pi$-mappings, which prove that a space is $g$-developable (or Cauchy) if and only if it is a weak-open $\pi$-image of a metric space.

Keywords: weak-open mappings, $\pi$-mappings, $g$-developable spaces, Cauchy spaces, cs-covers, sn-covers, weak-developments, point-star networks

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1. INTRODUCTION AND DEFINITIONS

To find internal characterizations of certain images of metric spaces is one of the central problems in General Topology. Some characterization for certain quotient $\pi$-images (open $\pi$-images, pseudo-open $\pi$-images) of metric spaces are obtained in [5], [11], [12], [13], [14], [15], [18]. Recently, S. Xia [4] introduced the concept of weak-open mappings. By using it, certain $g$-first countable spaces are characterized as images of metric spaces under various weak-open mappings. Furthermore, we prove that a space is $g$-metrizable if and only if it is a weak-open $\sigma$-image of a metric space in [18].

The purpose of this paper is to give some characterizations of weak-open $\pi$-images of metric spaces. We prove that a space is $g$-developable (or Cauchy) if and only if it is a weak-open $\pi$-image of a metric space, and generalize the result of R. W. Heath in [12].

In this paper, all spaces are Hausdroff, all mappings are continuous and surjective. $\mathbb{N}$ denotes the set of all natural numbers. $\tau(X)$ denotes a topology on $X$. For the

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usual product space $\prod_{i \in \mathbb{N}} X_i$, $\pi_i$ denotes the projection from $\prod_{i \in \mathbb{N}} X_i$ onto $X_i$. For a sequence $\{x_n\}$ in $X$, denote $\langle x_n \rangle = \{x_n : n \in \mathbb{N}\}$.

**Definition 1.1.** Let $\mathcal{P} = \bigcup \{\mathcal{P}_x : x \in X\}$ be a collection of subsets of a space $X$. $\mathcal{P}$ is called a weak-base for $X$ if

1. for each $x \in X$, $\mathcal{P}_x$ is a network of $x$ in $X$,
2. if $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.
3. $G \subset X$ is open in $X$ if and only if for each $x \in G$, there exists $P \in \mathcal{P}_x$ such that $P \subset G$.

$\mathcal{P}_x$ is called a weak neighborhood base of $x$ in $X$, and every element of $\mathcal{P}$ is called a weak neighborhood of $x$ in $X$.

**Definition 1.2.** Let $f : X \rightarrow Y$ be a mapping.

1. $f$ is called a weak-open mapping [4], if there exists a weak-base $\mathcal{B} = \bigcup \{\mathcal{B}_y : y \in Y\}$ for $Y$, and for each $y \in Y$ there exists $x_y \in f^{-1}(y)$ satisfying the following condition: for each open neighborhood $U$ of $x_y$, $B_y \subset f(U)$ for some $B_y \in \mathcal{B}_y$.
2. $f$ is called a $\pi$-mapping [2], if $(X, d)$ is a metric space, and for each $y \in Y$ and its open neighborhood $V$ in $Y$, $d(f^{-1}(y), M \setminus f^{-1}(V)) > 0$.

It is easy to check that a weak-open mapping is quotient and a compact mapping on metric spaces is a $\pi$-mapping.

**Definition 1.3** [8]. Let $X$ be a space, and $P \subset X$. Then,

1. A sequence $\{x_n\}$ in $X$ is called eventually in $P$, if $\{x_n\}$ converges to $x$, and there exists $m \in \mathbb{N}$ such that $\{x\} \cup \{x_n : n \geq m\} \subset P$.
2. $P$ is called a sequential neighborhood of $x$ in $X$, if whenever a sequence $\{x_n\}$ in $X$ converges to $x$, then $\{x_n\}$ is eventually in $P$.
3. $P$ is called sequential open in $X$, if $P$ is a sequential neighborhood of each of its points.
4. $X$ is called a sequential space, if any sequential open subset of $X$ is open in $X$.

**Definition 1.4.** Let $\mathcal{P}$ be a cover of a space $X$.

1. $\mathcal{P}$ is called a cs-cover for $X$, if every convergent sequence in $X$ is eventually in some element of $\mathcal{P}$.
2. $\mathcal{P}$ is called a sn-cover for $X$, if every element of $\mathcal{P}$ is a sequential neighborhood of some point in $X$, and for any $x \in X$ there exists a sequential neighborhood $P$ of $x$ in $X$ such that $P \in \mathcal{P}$.

**Definition 1.5.** Let $\{\mathcal{P}_n\}$ be a sequence of covers of a space $X$.

1. $\{\mathcal{P}_n\}$ is called a point-star network for $X$, if for each $x \in X$, $\langle \text{st}(x, \mathcal{P}_n) \rangle$ is a network of $x$ in $X$. 

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(2) \( \{ \mathcal{P}_n \} \) is called a weak-development for \( X \), if for each \( x \in X \), \( \langle \text{st}(x, \mathcal{P}_n) \rangle \) is a weak neighborhood base of \( x \) in \( X \).

If in a weak-development \( \{ \mathcal{P}_n \} \) for \( X \) each \( \mathcal{P}_n \) satisfies property \( C \), \( \{ \mathcal{P}_n \} \) is called a \( C \) weak-development for \( X \).

**Definition 1.6** ([6]). Let \( (X, d) \) is a symmetrizable space. Then,

1. A sequence \( \{ x_n \} \) in \( X \) is called \( d \)-Cauchy if, for each \( \varepsilon > 0 \), there exists \( k \in \mathbb{N} \) such that \( d(x_m, x_n) < \varepsilon \) for all \( n, m > k \).
2. \( X \) is called \( d \)-Cauchy (respectively weak \( d \)-Cauchy), if each convergent sequence is \( d \)-Cauchy (respectively each convergent sequence has a \( d \)-Cauchy subsequence).

For a space \( X \), let \( g \) be a mapping defined on \( \mathbb{N} \times X \) to the power-set of \( X \) such that \( x \in g(n, x) \) and \( g(n+1, x) \subset g(n, x) \) for each \( n \in \mathbb{N} \) and \( x \in X \), and a subset \( U \) of \( X \) is open if for each \( x \in U \), there exists \( n \in \mathbb{N} \) such that \( g(n, x) \subset U \). We call such a mapping a CWC-mapping (i.e., countable weakly-open covering mapping).

**Definition 1.7** ([7]). A space \( X \) is \( g \)-developable if \( X \) has a CWC-mapping \( g \) with the following property: If \( x, x_n \in g(n, y_n) \) for each \( n \in \mathbb{N} \), then sequence \( \{ x_n \} \) converges to \( x \).

2. Results

**Theorem 2.1.** The following are equivalent for a space \( X \):

1. \( X \) is a weak-open \( \pi \)-image of a metric space.
2. \( X \) has a cs-cover weak-development.
3. \( X \) has a sn-cover weak-development.
4. \( X \) is a Cauchy space.
5. \( X \) is a \( g \)-developable space.

**Proof.** (1) \( \Rightarrow \) (2): Suppose \( X \) is an image of a metric space \( (M, d) \) under a weak-open \( \pi \)-mapping \( f \). For each \( n \in \mathbb{N} \), put \( \mathcal{P}_n = \{ f(B(z, 1/n)) : z \in M \} \), where \( B(z, 1/n) = \{ y \in M : d(z, y) < 1/n \} \). Then \( \{ \mathcal{P}_n \} \) is a point-star network for \( X \). In fact, for each \( x \in X \), and its open neighborhood \( U \), since \( f \) is a \( \pi \)-mapping, there exists \( n \in \mathbb{N} \) such that \( d(f^{-1}(x), M \setminus f^{-1}(U)) > 1/n \). We can pick \( m \in \mathbb{N} \) such that \( m \geq 2n \). If \( z \in M \) with \( x \in f(B(z, 1/m)) \), then

\[ f^{-1}(x) \cap B(z, 1/m) \neq \emptyset. \]

If \( B(z, 1/m) \not\subset f^{-1}(U) \), then

\[ d(f^{-1}(x), M \setminus f^{-1}(U)) \leq 2/m \leq 1/n, \]

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a contradiction. Thus $B(z, 1/m) \subset f^{-1}(U)$, so $f(B(z, 1/m)) \subset U$. Hence 
$st(x, P_m) \subset U$. Therefore \{P_n\} is a point-star network for $X$.

We shall prove that every $\mathcal{P}_k$ is a cs-cover for $X$. Since $f$ is weak-open, there
exists a weak-base $\mathcal{B} = \bigcup \{\mathcal{B}_x : x \in X\}$ for $X$, and for each $x \in X$ there exists
$m_x \in f^{-1}(x)$ satisfying the following condition: for each open neighborhood $U$
of $m_x$ in $M, B \subset f(U)$ for some $B \in \mathcal{B}_x$. For each $k \in \mathbb{N}$, if \{$x_n$\} converges to
$x \in X$ in $X$, there exists $B \in \mathcal{B}_x$ such that $B \subset f(B(m_x, 1/k))$ since $f$ is weak-
open. Since $B$ is a weak-neighborhood of $x$ in $X$, $B$ is a sequential neighborhood
of $x$ in $X$ by Corollary 1.6.18 in [9], so $f(B(m_x, 1/k))$ is too. Thus \{$x_n$\} is eventually
in $f(B(m_x, 1/k))$. This implies each $\mathcal{P}_k$ is a cs-cover for $X$.

For each $x \in X$ and $k \in \mathbb{N}$, since $f(B(m_x, 1/k))$ is a sequential neighborhood of $x$
in $X$, $st(x, \mathcal{P}_k)$ is too. Obviously, $X$ is a sequential space. So \{\text{st}(x, \mathcal{P}_k)\} is a weak
neighborhood base of $x$ in $X$.

In other words, \{\mathcal{P}_n\} is a cs-cover weak-development for $X$.

(2) $\Rightarrow$ (3): Suppose \{\mathcal{P}_n\} is a cs-cover weak-development for $X$. We can assume
that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for each $n \in \mathbb{N}$. For each $x, y \in X$, denoting
\[
t(x, y) = \min\{n : x \notin \text{st}(y, \mathcal{P}_n)\} \quad (x \neq y),
\]
we define
\[
d(x, y) = \begin{cases}
0, & x = y, \\
2^{-t(x, y)}, & x \neq y,
\end{cases}
\]
then $d : X \times X \to [0, +\infty)$ is a symmetric function on $X$.

**Claim.** For each $x, y \in X$, $x \in \text{st}(y, \mathcal{P}_n)$ if and only if $t(x, y) > n$.

In fact, the if part is obvious. For the only if part, suppose $x \in \text{st}(y, \mathcal{P}_n)$ but
$t(x, y) \leq n$. Since $\mathcal{P}_n$ refines $\mathcal{P}_t(x,y)$, $\text{st}(y, \mathcal{P}_n) \subset \text{st}(y, \mathcal{P}_{t(x,y)})$. Note that $x \notin \text{st}(y, \mathcal{P}_{t(x,y)})$, so $x \notin \text{st}(y, \mathcal{P}_n)$, a contradiction.

For each $x \in X$ and $n \in \mathbb{N}$, $\text{st}(x, \mathcal{P}_n) = B(x, 1/2^n)$ by the Claim. Because
\{\mathcal{P}_n\} is a point-star network for $X$, $(X, d)$ is symmetrizable. And $d$ has the
following property: for each $x \in X$ and $\varepsilon > 0$, there exists $\delta = \delta(x, \varepsilon) > 0$
such that $d(x, y) < \delta$ and $d(x, z) < \delta$ imply $d(y, z) < \varepsilon$. Otherwise, there exist $\varepsilon_0 > 0$ and two
sequences \{$y_n$\} and \{$z_n$\} in $X$ such that $d(y_n, z_n) \geq \varepsilon_0$ whenever $d(x, y_n) < 1/2^n$
and $d(x, z_n) < 1/2^n$. Since $\mathcal{P}_n$ is a point-star network for $X$, \{\mathcal{P}_n\} and \{\mathcal{P}_k\}
all converge to $x$. We choose $k \in \mathbb{N}$ such that $1/2^k < \varepsilon_0$. Since $\mathcal{P}_k$ is a cs-cover for $X$,
\{\mathcal{P}_m\} $\subset P$ for some $m \in \mathbb{N}$ and $x \in \mathcal{P}_k$. Thus $y_m \in \text{st}(z_m, \mathcal{P}_k)$. By the Claim,
$t(y_m, z_m) > k$. Thus, $d(y_m, z_m) = 1/2^t(y_m, z_m) < 1/2^k < \varepsilon_0$, a contradiction.

For each $x \in X$ and $n \in \mathbb{N}$, we can pick $\delta = \delta(x, n)$ such that $d(y, z) < 1/n$
whenever $d(x, y) < \delta$ and $d(x, z) < \delta$. Let $g(n, x) = B(x, \delta(x, n))$. Since $\mathcal{P}_n$ is a

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cs-cover for $X$, $\text{st}(x, \mathcal{P}_n)$ is a sequential neighborhood of $x$ in $X$, so $g(n, x)$ is too. Put
$$\mathcal{F}_n = \{g(n, x) : x \in X\},$$
then every $\mathcal{F}_n$ is an sn-cover for $X$.

If $\{\mathcal{F}_n\}$ is not a point-star network for $X$, then there exist $x \in G \in \tau(X)$ and two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $x \in g(n, y_n)$ and $x_n \in g(n, y_n) \setminus G$. So $\{x_n\}$ does not converge to $x$, and $d(y_n, x) < \delta(y_n, n)$, $d(y_n, x_n) < \delta(y_n, n)$. By the property above, $d(x, x_n) < 1/n$. This implies that $\{x_n\}$ converges to $x$, a contradiction. Hence $\{\mathcal{F}_n\}$ is a point-star network for $X$.

Since $X$ is symmetrizable, $X$ is a sequential space. For each $x \in X$ and $n \in \mathbb{N}$, by the above, $g(n, x)$ is a sequential neighborhood of $x$ in $X$. By $g(n, x) \subset \text{st}(x, \mathcal{F}_n)$, $\text{st}(x, \mathcal{F}_n)$ is too. So $\langle \text{st}(x, \mathcal{F}_n) \rangle$ is a weak neighborhood base of $x$ in $X$. Hence $\{\mathcal{F}_n\}$ is a sn-cover weak-development for $X$.

(3) $\Rightarrow$ (1): Suppose $\{\mathcal{P}_n\}$ is a sn-cover weak-development for $X$. For each $i \in \mathbb{N}$, let $\mathcal{P}_i = \{P_\alpha : \alpha \in \Lambda_i\}$. Endow $\Lambda_i$ with the discrete topology, then $\Lambda_i$ is a metric space. Put
$$M = \left\{\alpha = (\alpha_i) \in \prod_{i \in \mathbb{N}} \Lambda_i : \langle P_\alpha \rangle \text{ forms a network at some point } x_\alpha \text{ in } X \right\},$$
and endow $M$ with the subspace topology induced from the usual product topology of the collection $\{\Lambda_i : i \in \mathbb{N}\}$ of metric spaces, then $M$ is a metric space. Since $X$ is Hausdorff, $x_\alpha$ is unique in $X$. For each $\alpha \in M$, we define $f : M \rightarrow X$ by $f(\alpha) = x_\alpha$. For each $x \in X$ and $i \in \mathbb{N}$, there exists $\alpha_i \in \Lambda_i$ such that $x \in P_\alpha$. Since $\{\mathcal{P}_i\}$ is a point-star network for $X$, $\{P_\alpha : i \in \mathbb{N}\}$ is a network of $x$ in $X$. Put $\alpha = (\alpha_i)$, then $\alpha \in M$ and $f(\alpha) = x$. Thus $f$ is surjective. Suppose $\alpha = (\alpha_i) \in M$ and $f(\alpha) = x \in U \in \tau(X)$, then there exists $n \in \mathbb{N}$ such that $P_{\alpha_n} \subset U$. Put
$$V = \{\beta \in M : \text{the nth coordinate of } \beta \text{ is } \alpha_n\},$$
then $\alpha \in V \in \tau(M)$, and $f(V) \subset P_{\alpha_n} \subset U$. Hence $f$ is continuous.

For each $\alpha, \beta \in M$, we define
$$d(\alpha, \beta) = \begin{cases} 0, & \alpha = \beta \\
\max\{1/k : \pi_k(\alpha) \neq \pi_k(\beta)\}, & \alpha \neq \beta,
\end{cases}$$
then $d$ is a distance in $M$. Because the topology of $M$ is the subspace topology induced from the usual product topology of the collection $\{\Lambda_i : i \in \mathbb{N}\}$ of discrete spaces, $d$ is metric in $M$. For each $x \in U \in \tau(X)$, note that $\{\mathcal{P}_n\}$ is a point-star
network for $X$, hence there exists $n \in \mathbb{N}$ such that $st(x, \mathcal{P}_n) \subset U$. For $\alpha \in f^{-1}(x)$, $\beta \in M$, if $d(\alpha, \beta) < 1/n$, then $\pi_i(\alpha) = \pi_i(\beta)$ for all $i \leq n$. So $x \in P_{\pi_n(\alpha)} = P_{\pi_n(\beta)}$. Thus

$$f(\beta) \in \bigcap_{i \in \mathbb{N}} P_{\pi_i(\beta)} \subset P_{\pi_n(\beta)} \subset U.$$ 

Hence

$$d(f^{-1}(x), M \setminus f^{-1}(U)) \geq 1/n.$$ 

Therefore $f$ is a $\pi$-mapping.

We shall prove that $f$ is weak-open. For each $x \in X$, since every $\mathcal{P}_i$ is an sn-cover for $X$, there exists $\alpha_i \in \Lambda_i$ such that $P_{\alpha_i}$ is a sequential neighborhood of $x$ in $X$. Since $\{\mathcal{P}_i\}$ is a point-star network for $X$, $\langle P_{\alpha_i} \rangle$ is a network of $x$ in $X$. Put $\beta_x = (\alpha_i) \in \prod_{i \in \mathbb{N}} \Lambda_i$, then $\beta_x \in f^{-1}(x)$.

Let $\{U_{m\beta_x}\}$ be a decreasing neighborhood base of $\beta_x$ in $M$, and put

$$\mathcal{B}_x = \{f(U_{m\beta_x}): m \in \mathbb{N}\},$$

$$\mathcal{B} = \bigcup \{\mathcal{B}_x: x \in X\},$$

then $\mathcal{B}$ satisfies (1), (2) in Definition 1.1. Suppose $G$ is open in $X$. For each $x \in G$, from $\beta_x \in f^{-1}(x)$, we see that $f^{-1}(G)$ is an open neighborhood of $\beta_x$ in $M$. Thus $U_{m\beta_x} \subset f^{-1}(G)$ for some $m \in \mathbb{N}$, so $f(U_{m\beta_x}) \subset G$ and $f(U_{m\beta_x}) \in \mathcal{B}_x$. On the other hand, suppose that $G \subset X$ and for $x \in G$, there exists $B \in \mathcal{B}_x$ such that $B \subset G$. Denote $B = f(U_{m\beta_x})$ for some $m \in \mathbb{N}$. Let $\{x_n\}$ be a sequence converging to $x$ in $X$. Since $P_{\alpha_i}$ is a sequential neighborhood of $x$ in $X$ for each $i \in \mathbb{N}$, $\{x_n\}$ is eventually in $P_{\alpha_i}$. For each $n \in \mathbb{N}$, if $x_n \in P_{\alpha_i}$, let $\alpha_{in} = \alpha_i$; if $x_n \notin P_{\alpha_i}$, pick $\alpha_{in} \in \Lambda_i$ such that $x_n \in P_{\alpha_{in}}$. Thus there exists $n_i \in \mathbb{N}$ such that $\alpha_{in} = \alpha_i$ for all $n > n_i$. So $\{\alpha_{in}\}$ converges to $\alpha_i$. For each $n \in \mathbb{N}$, put

$$\beta_n = (\alpha_{in}) \in \prod_{i \in \mathbb{N}} \Lambda_i,$$

then $f(\beta_n) = x_n$ and $\{\beta_n\}$ converges to $\beta_x$. Since $U_{m\beta_x}$ is an open neighborhood of $\beta_x$ in $M$, $\{\beta_n\}$ is eventually in $U_{m\beta_x}$, so $\{x_n\}$ is eventually in $G$. Hence $G$ is a sequential neighborhood of $x$. So $G$ is sequential open in $X$. Since $X$ is a sequential space, $G$ is open in $X$. This implies that $\mathcal{B}$ is a weak-base for $X$.

By the definition of $\mathcal{B}$, $f$ is weak-open.

(2) $\Rightarrow$ (4): Suppose $\{\mathcal{P}_i\}$ is a cs-cover weak-development for $X$. We can assume that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for each $n \in \mathbb{N}$. Similarly as in the proof of (2) $\Rightarrow$ (3), we can define a symmetric distance function $d$ on $X$ such that $st(x, \mathcal{P}_n) = B(x, 1/2^n)$ for each $x \in X$ and $n \in \mathbb{N}$. So $(X, d)$ is symmetrizable. For each sequence $\{x_n\}$ in $X$
converging to \( x \in X \) and \( \varepsilon > 0 \), there exists \( k \in \mathbb{N} \) such that \( 1/2^k < \varepsilon \). Since \( \mathcal{P}_k \) is a cs-cover for \( X \), there exist \( P \in \mathcal{P}_k \) and \( l \in \mathbb{N} \) such that \( \{x\} \cup \{x_n: n \geq l\} \subset P \). If \( n, m \geq l \), then \( x_n, x_m \in P \), so \( x_n \in \text{st}(x_m, \mathcal{P}_k) \). Thus \( t(x_n, x_m) > k \) by the Claim in (2) \( \Rightarrow \) (3). Hence \( d(x_n, x_m) = 1/2^t(x_n, x_m) < 1/2^k < \varepsilon \) whenever \( n, m \geq l \). Therefore \( \{x_n\} \) is \( d \)-Cauchy. This implies that \( X \) is a Cauchy space.

(4) \( \Rightarrow \) (2): Suppose \( X \) is a Cauchy space. For each \( n \in \mathbb{N} \), put

\[
\mathcal{P}_n = \{ A \subset X: \sup\{d(x, y): x, y \in A\} < 1/n \}
\]

then \( \text{st}(x, \mathcal{P}_n) = B(x, 1/n) \) for each \( x \in X \), so \( \{\mathcal{P}_n\} \) is a point-star network for \( X \). It is clear that \( X \) is a sequential space. We need only prove that each \( \mathcal{P}_n \) is a cs-cover for \( X \). For each sequence \( \{x_n\} \) converging to \( x \) in \( X \), since \( \{x_n\} \) is \( d \)-Cauchy and \( X \) is symmetrizable, there exists \( m \in \mathbb{N} \) such that \( d(x, x_i) < 1/(n + 1) \) and \( d(x_i, x_j) < 1/(n + 1) \) for all \( i, j \geq m \) by Lemma 9.3 in [16]. Put

\[
P = \{x\} \cup \{x_i: i \geq m\}
\]

then \( P \in \mathcal{P}_n \). Hence each \( \mathcal{P}_n \) is a cs-cover for \( X \).

(4) \( \Leftrightarrow \) (5) follows from Theorem 2.3 in [7]. \( \square \)

By Theorem 2.1, Proposition 2.2 in [7], Proposition 2.1.16(3) in [9] and Proposition 2.1.16 in [9], we have

**Proposition 2.2.** A space is developable if and only if it is a weak-open, \( \pi \), pseudo-open image of a metric space.

**Corollary 2.3** ([12]). A space is developable if and only if it is an open \( \pi \)-image of a metric space.

We give examples illustrating Theorem 2.1 of this paper.

**Example 2.4.** Let \( X \) be the Arens space \( S_2 \) (see [9, Example 1.8.6]). Since \( X \) is Cauchy, \( X \) is a weak-open \( \pi \)-image of a metric space by Theorem 2.1. But \( X \) is not an open \( \pi \)-image of a metric space because \( X \) is not first countable. Thus the following holds:

A weak-open \( \pi \)-image of a metric space needn’t be an open \( \pi \)-image of a metric space.

**Example 2.5.** Let \( X \) be the weak Cauchy space in [5, Example 2.14(3)]. By Theorem 12 in [15], \( X \) is a quotient \( \pi \)-image of a metric space. But \( X \) is not Cauchy, \( X \) is not a weak-open \( \pi \)-image of a metric space by Theorem 2.1. Thus the following holds:

A quotient \( \pi \)-image of a metric space needn’t be a weak-open \( \pi \)-image of a metric space.
**Example 2.6.** Let $X$ be the Mrowka space $\psi(\mathbb{N})$ (see [9, Example 1.8.4]). Since $X$ is developable, $X$ is an open $\pi$-image of a metric space. But $X$ has no point-countable $c^*$-networks. Thus $X$ is not a quotient $s$-image of a metric space by Corollary 2.7.6 in [9]. Thus the following holds:

1. A weak-open $\pi$-image of a metric space needn’t be a weak-open compact image of a metric space.
2. A weak-open $\pi$-image of a metric space needn’t be a weak-open $s$-image of a metric space.

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