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ON THE MAXIMAL SUBGROUP OF THE SANDWICH SEMIGROUP OF GENERALIZED CIRCULANT BOOLEAN MATRICES

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Abstract. Let n be a positive integer, and $C_n(r)$ the set of all $n \times n$ r-circulant matrices over the Boolean algebra $B = \{0, 1\}$, $G_n = \bigcup_{r=0}^{n-1} C_n(r)$. For any fixed r-circulant matrix C ($C \neq 0$) in G_n, we define an operation "$*$" in G_n as follows: $A * B = ACB$ for any A, B in G_n, where ACB is the usual product of Boolean matrices. Then $(G_n, *)$ is a semigroup. We denote this semigroup by $G_n(C)$ and call it the sandwich semigroup of generalized circulant Boolean matrices with sandwich matrix C. Let F be an idempotent element in $G_n(C)$ and $M(F)$ the maximal subgroup in $G_n(C)$ containing the idempotent element F. In this paper, the elements in $M(F)$ are characterized and an algorithm to determine all the elements in $M(F)$ is given.

Keywords: generalized ciculant Boolean matrix, sandwich semigroup, idempotent element, maximal subgroup

MSC 2000: 15A33

1. Introduction and preliminaries

Let $B = \{0, 1\}$ be the binary Boolean algebra. The matrices which we consider in this paper are $n \times n$ matrices over B, called *Boolean matrices.* Let r be a nonnegative integer. An $n \times n$ *r-circulant* (*generalized circulant*) *Boolean matrix* is an $n \times n$ matrix over B in which each row, except the first, is obtained from the preceding row by shifting the elements cyclically r columns to the right, i.e., $a_{ij} = a_{i-1,j-r}$ for $i, j = 0, 1, \ldots, n-1$, where the indices are reduced to their least nonnegative remainder modulo n.

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Let P be $n \times n$ 1-circulant with the first row $(0, 1, 0, \ldots, 0)$. Then an r-circulant matrix A with the first row $(a_0, a_1, \ldots, a_{n-1})$ can be written in the from

$$A = \sum_{i=0}^{n-1} a_i Q_r P^i,$$

where Q_r is the r-circulant matrix with the first row $(1, 0, 0, \ldots, 0)$. Let $\Delta(A) = \{i\colon\ a_i = 1,\ 0 \leqslant i \leqslant n-1\}$ for the matrix $A = \sum\limits_{i=0}^{n-1} a_i Q_r P^i$. Then A can be rewritten in the form

$$A = Q_r \sum_{i \in \Delta(A)} P^i.$$

It is very easy to verify that the matrix A satisfies

$$PA = AP^r. \tag{1.1}$$

Let $C_n(r)$ denote the set of all $n \times n$ r-circulant Boolean matrices, and $G_n = \bigcup\limits_{r=0}^{n-1} C_n(r)$. Then $C_n(1)$ and G_n form semigroups, called *the semigroup of circulant Boolean matrices* and *the semigroup of generalized circulant Boolean matrices*, respectively, under matrix multiplication and using Boolean operations for the entries of matrices. For an arbitrary but fixed element $C \in G_n$ $(C \neq 0)$, we can define an operation "$*$" in G_n as follows: For any $A, B \in G_n$, $A * B = ACB$, where ACB is the usual product of Boolean matrices. It can be easily proved that G_n is also a semigroup under the operation "$*$". We denote this semigroup by $G_n(C)$ and call it a *sandwich semigroup of generalized circulant Boolean matrices with sandwich matrix* C.

Let S be a semigroup. A subgroup M of S is called a maximal subgroup of S if it is not properly contained in any other subgroup of S. Clifford and Preston proved that a subgroup M of S containing an idempotent F is a maximal subgroup of S if and only if $M = M(F)$ and

$$M(F) = \{A \in S\colon\ FA = AF = A,\ XA = AY = F,\ \text{ for some } X, Y \in S\} \tag{1.2}$$

(see [4], pp. 22–23). Montague and Plemmons [5] dealt with maximal subgroups of the semigroup of relations. Kim and Schwarz [6] obtained the description of maximal subgroups of the semigroup of circulant Boolean matrices. Zhang [3] gave some necessary and sufficient conditions for an r-circulant Boolean matrix to be an element of a maximal subgroup of G_n and generalized the corresponding results in [6]. The purpose of this paper is to describe the maximal subgroups of $G_n(C)$. The main results obtained in this paper are generalizations of the corresponding results in [3].

The following notions and lemmas are used.

Let $\mathbb{Z}$ denote the set of all integers, and $r \in \mathbb{Z}$, $U, V, W \subseteq \mathbb{Z}$. Let $U + V = \{u + v\colon\ u \in U, v \in V\}$, $rU = \{ru\colon\ u \in U\}$, $u + V = \{u\} + V$ and let $\sigma(U)$ be the greatest common divisor of the elements in U. Let n be a positive integer. U is said to be *included in V modulo n*, denoted by $U \subseteq V \pmod n$, if for each $u \in U$ there exists a $v \in V$ such that $u \equiv v \pmod n$. U is said to be *congruent to V modulo n*, denoted by $U \equiv V \pmod n$ if $U \subseteq V \pmod n$ and $V \subseteq U \pmod n$. Clearly, this congruence relation is reflexive, symmetric, and transitive (see [1]). Let $M = \{m_0, m_1, \ldots, m_{t-1}\}$ be a set of integers. $M \pmod n$ denotes the set $\{\overline{m_0}, \overline{m_1}, \ldots, \overline{m_{t-1}}\}$, where $0 \leqslant \overline{m_k} \leqslant n - 1$, and $\overline{m_k} \equiv m_k \pmod n$ for $k = 0, 1, \ldots, t-1$. A set $M = \{m_0, m_1, \ldots, m_{t-1}\}$ of integers is called an *arithmetic progression modulo n* with common difference d if the elements of $M \pmod n$ constitute an arithmetic progressions with the same common difference d and $dt = n$. We denote $\bigcup\limits_{u=0}^{e-1} \{i_u, i_u + d, \ldots, i_u + (m-1)d\}$, the union of e arithmetic progressions with the same common difference d, by $\bigcup\{i_0, i_1, \ldots, i_{e-1}, n, d, m\}$ (see [3]). (n, r) will denote the greatest common divisor of n and r.

Remark. Any integer set $U = \{j_0, j_1, \ldots, j_{t-1}\} \subseteq \{0, 1, \ldots, n-1\}$ can be represented as a union of some arithmetic progressions modulo n with the same common difference. The union of arithmetic progressions modulo n with the smallest common difference d is called the *final form of U* and we denote d by $d_n(U)$. For example, let $n = 12$ and $U = \{0, 1, 3, 4, 6, 7, 9, 10\}$. Then $U = \{0\} \cup \{1\} \cup \{3\} \cup \{6\} \cup \{7\} \cup \{9\} \cup \{10\}$, in this case, $d = 12$ and $m = 1$, and $U = \{0, 6\} \cup \{1, 7\} \cup \{3, 9\} \cup \{4, 10\}$, in this case, $d = 6$ and $m = 2$. Also $U = \{0, 3, 6, 9\} \cup \{1, 4, 7, 10\}$, in this case, $d = 3$ and $m = 4$. Obviously, $\{0, 3, 6, 9\} \cup \{1, 4, 7, 10\}$ is the final form of U and $d_{12}(U) = 3$. If $\bigcup\{i_0, i_1, \ldots, i_{e-1}, n, d, m\}$ is the final form of U and $\bigcup\{j_0, j_1, \ldots, j_{e_1-1}, n, d_1, m_1\}$ is not the final form of U, then it is easily verified that $d \mid d_1$.

Let c, r and d be positive integers. An integer s, $0 < s < d$, is called an *invertible integer relative to c, r and d* if s satisfies $s(cr - 1) \equiv 0 \pmod d$ and there exists an integer s' such that $s'cs \equiv r \pmod d$.

Lemma 1.1 ([8], Proposition 3.3.1)**.** *Let s, d and r be integers, and $s, d \neq 0$. Then the congruence $sx \equiv r \pmod d$ has solutions if and only if $(s, d) \mid r$.*

Lemma 1.2. *Let c, d and r be positive integers, and $r(cr - 1) \equiv 0 \pmod d$. If s_0 is the least positive integer such that $s_0(cr - 1) \equiv 0 \pmod d$, then each integer s which satisfies $s(cr - 1) \equiv 0 \pmod d$ is a multiple of s_0, and $s_0 = (r, d)$.*

P r o o f. Since $s_0(cr - 1) \equiv 0 \pmod d$ and $s(cr - 1) \equiv 0 \pmod d$, we have $s_0(cr - 1) = kd$ and $s(cr - 1) = k_1 d$ for some $k, k_1 \in \mathbb{Z}$, and so $skd = ss_0(cr - 1) =$

$s_0 s(cr-1) = s_0 k_1 d$. Therefore $sk = s_0 k_1$. Since s_0 is the least positive integer such that $s_0(cr-1) \equiv 0 \pmod{d}$, we have $(s_0, k) = 1$. Therefore s is a multiple of s_0. In the following we will prove that $s_0 = (r, d)$.

Let $d_1 = (r, d)$, $r = pd_1$ and $d = hd_1$. Then $(p, h) = 1$. Since $r(cr-1) \equiv 0 \pmod{d}$, we have $r(cr-1) = md$ for some $m \in \mathbb{Z}$, and so $pd_1(cr-1) = r(cr-1) = md = mhd_1$. Therefore we have $p(cr-1) = mh$. But $(p, h) = 1$, so we have $m = m_1 p$ for some $m_1 \in \mathbb{Z}$. Thus $phd_1(cr-1) = mhd = pm_1 hd$, and so $d_1(cr-1) = m_1 d$, i.e., $d_1(cr-1) \equiv 0 \pmod{d}$. Since s_0 is the least positive integer such that $s_0(cr-1) \equiv 0 \pmod{d}$, d_1 is a multiple of s_0. Now, suppose $d_1 = ts_0$, $t \in \mathbb{Z}$. Since $m_1 d = d_1(cr-1) = ts_0(cr-1) = tkd$, we have $m_1 = tk$. Hence t is a common divisor of d_1 and m_1. On the other hand, since $(r, cr-1) = 1$ and $r = pd_1$, we have $(d_1, cr-1) = 1$. Since $d_1(cr-1) = m_1 d = m_1 d_1 h$, we have $cr - 1 = m_1 h$. But $(d_1, cr-1) = 1$, so we have $(d_1, m_1) = 1$, and so $t = 1$. Therefore $s_0 = d_1 = (r, d)$. The proof is completed. □

Lemma 1.3. *Let c, r and d be positive integers, and $r(cr-1) \equiv 0 \pmod{d}$. Let s be an invertible integer relative to c, r and d, and l be an integer such that $l(cr-1) \equiv 0 \pmod{d}$. Then the equation $scz \equiv l \pmod{d}$ has solutions.*

P r o o f. First we shall show that for any invertible integer s relative to c, r and d, the equation $scx \equiv r \pmod{d}$ has solutions if and only if the equation $scy \equiv s_0 \pmod{d}$ has solutions, where s_0 is the least positive integer such that $s_0(cr-1) \equiv 0 \pmod{d}$. From Lemma 1.2 we know that $s_0 \mid s$ and $s_0 = (r, d)$. Let now $r = r_1 s_0$, $d = d_1 s_0$, $s = s_1 s_0$, where r_1, d_1 and s_1 are positive integers. Then $(r_1, d_1) = 1$. If the equation $scx \equiv r \pmod{d}$ has some solution x_0, then $scx_0 \equiv r \pmod{d}$, and so $s_1 cx_0 \equiv r_1 \pmod{d_1}$, i.e., $s_1 cx_0 = r_1 + td_1$ for some $t \in \mathbb{Z}$. But $(r_1, d_1) = 1$, so we have $(s_1 c, d_1) = 1$. Then there exist $p, q \in \mathbb{Z}$ such that $ps_1 c + qd_1 = 1$. It follows that $(r_1 - 1)ps_1 c + (r_1 - 1)qd_1 = r_1 - 1$. So $r_1 - (r_1 - 1)ps_1 c - (r_1 - 1)qd_1 = r_1 - (r_1 - 1)$. By using $r_1 = s_1 cx_0 - td_1$, we have $s_1 cx_0 - td_1 - (r_1 - 1)ps_1 c - (r_1 - 1)qd_1 = 1$ and so $s_0 s_1 cx_0 - s_0 td_1 - (r_1 - 1)ps_0 s_1 c - (r_1 - 1)qd_1 s_0 = s_0$. Hence $scx_0 - td - (r_1 - 1)psc - (r_1 - 1)qd = s_0$, and so $sc(x_0 - (r_1 - 1)p) \equiv s_0 \pmod{d}$. Let $y_0 = x_0 - (r_1 - 1)p$. Then y_0 is one of solutions for the equation $scy \equiv s_0 \pmod{d}$. Conversely, if y_0 is a solution of the equation $scy \equiv s_0 \pmod{d}$, then, it is clear that $x = r_1 y_0$ is a solution of the equation $scx \equiv r \pmod{d}$.

In the following we show that the equation $scz \equiv l \pmod{d}$ has solutions. If $l = 0$, then 0 is a solution of this equation. We now suppose $l \neq 0$. Since s is an invertible integer relative to c, r and d, there exists an integer s' such that $scs' \equiv r \pmod{d}$, i.e., the equation $scx \equiv r \pmod{d}$ has solutions, and so the equation $scy \equiv s_0 \pmod{d}$ has solutions. Let y_0 be a solution of the equation $scy \equiv s_0 \pmod{d}$. Then $scy_0 \equiv s_0 \pmod{d}$. Since l satisfies $l(cr-1) \equiv 0 \pmod{d}$, by Lemma 1.2, we have

$l = qs_0$ for some $q \in \mathbb{Z}$. Let $z_0 = y_0 q$. Then $scy_0 q \equiv qs_0 \pmod{d}$, i.e., z_0 is a solution of the equation $scz \equiv l \pmod{d}$. This completes the proof. □

Lemma 1.4. *Let c, r and d be positive integers, and $r(cr-1) \equiv 0 \pmod{d}$, and s be an invertible integer relative to c, r and d. Then the equations $scx \equiv r \pmod{d}$ and $x(cr-1) \equiv 0 \pmod{d}$ have common solutions.*

P r o o f. Let s_0 be the least positive integer such that $s_0(cr-1) \equiv 0 \pmod{d}$. Since s is an invertible integer relative to c, r and d, we have $s(cr-1) \equiv 0 \pmod{d}$. From Lemma 1.2, we have $s_0 = (r, d)$ and $s_0 \mid s$. Let $s = s_1 s_0$, $r = r_1 s_0$ and $d = d_1 s_0$, where $s_1, r_1, d_1 \in \mathbb{Z}$. Then $(r_1, d_1) = 1$. In the following we shall show that the equation $scz \equiv r_1 \pmod{d_1}$ has solutions. Since s is an invertible integer relative to c, r and d, there exists an integer s' such that $scs' \equiv r \pmod{d}$. Then we have $s_1 s_0 cs' - r_1 s_0 = pd_1 s_0$ for some $p \in \mathbb{Z}$, i.e., $s_1 cs' - r_1 = pd_1$. But $(r_1, d_1) = 1$, so we have $(s_1 c, d_1) = 1$. It follows that $(s_1, d_1) = 1$, and so $(s, d) = s_0$. Since $(r, cr-1) = 1$ and $s_0 \mid r$, we have $(s_0, cr-1) = 1$. Since $s(cr-1) \equiv 0 \pmod{d}$, we have $s(cr-1) = qd$ for some $q \in \mathbb{Z}$. Then $s_1 s_0(cr-1) = qd_1 s_0$, and so $s_1(cr-1) = qd_1$. But $(s_1, d_1) = 1$, so we have $d_1 \mid (cr-1)$. Since $(s_0, cr-1) = 1$, we have $(s_0, d_1) = 1$. Since $(s_1 c, d_1) = 1$, we have $(s_1 s_0 c, d_1) = 1$, i.e., $(sc, d_1) = 1$. By Lemma 1.1, we have that the equation $scz \equiv r_1 \pmod{d_1}$ has solutions. Suppose that k ($k \in \mathbb{Z}$) is a solution of the equation $scz \equiv r_1 \pmod{d_1}$, i.e., $sck \equiv r_1 \pmod{d_1}$. Then $scks_0 \equiv r \pmod{d}$, and so ks_0 is a solution of the equation $scx \equiv r \pmod{d}$.

In the following we shall show that the equations $scx \equiv r \pmod{d}$ and $y(cr-1) \equiv 0 \pmod{d}$ have common solutions. Since s_0 is the least positive integer such that $s_0(cr-1) \equiv 0 \pmod{d}$, we have $ks_0(cr-1) \equiv 0 \pmod{d}$. Then ks_0 is a solution of the equation $x(cr-1) \equiv 0 \pmod{d}$. Therefore the equations $scx \equiv r \pmod{d}$ and $x(cr-1) \equiv 0 \pmod{d}$ have common solutions. This completes the proof. □

Lemma 1.5. *Let n, r, s and c be positive integers such that $r(cr-1) \equiv 0 \pmod{d}$ and $s(cr-1) \equiv 0 \pmod{d}$. Let U, V be two subsets of $\mathbb{Z}$. If $(\sigma(crU + rV), n) = d$, then $d \mid (\sigma(csU + sV), n)$.*

P r o o f. Let s_0 be the least positive integer such that $s_0(cr-1) \equiv 0 \pmod{d}$. Then, by Lemma 1.2, we have $s = qs_0$ and $s_0 = (r, d)$, where $q \in \mathbb{Z}$. Let $r = r_1 s_0$, $d = d_1 s_0$, where $r_1, d_1 \in \mathbb{Z}$. Then $(r_1, d_1) = 1$. Since $(\sigma(crU + rV), n) = d$, we have $d \mid (cru + rv)$ for any $u \in U$ and $v \in V$. And so $cru + rv = m_{uv} d$ for some $m_{uv} \in \mathbb{Z}$. It follows that $cru + rv = cr_1 s_0 u + r_1 s_0 v = m_{uv} d_1 s_0$. Then we have $cr_1 u + r_1 v = m_{uv} d_1$. But $(r_1, d_1) = 1$, so we have $r_1 \mid m_{uv}$. Let $m_{uv} = m'_{uv} r_1$. Then $cu + v = m'_{uv} d_1$, and so $s_0(cu + v) = s_0 m'_{uv} d_1 = m'_{uv} d$. Since $s = qs_0$, we have $s(cu+v) = qs_0(cu+v) = qm'_{uv} d$. Hence for any $u \in U$, $v \in V$, we have $d \mid (scu + sv)$, and so $d \mid (\sigma(csU + sV), n)$ (because $d \mid n$). The proof is completed. □

Lemma 1.6. *Let U be a union of some arithmetic progressions modulo n with common difference and $d = d_n(U)$. Let $k \in \mathbb{Z}$. Then $k + U \equiv U \pmod{n}$ if and only if $d \mid k$.*

P r o o f. The proof is omitted. □

Lemma 1.7 ([7], Lemma 1.2)**.** *Let n be a positive integer. Let $M = \{m_0, m_1, \ldots, m_{t-1}\} \subseteq \mathbb{Z}$, where $0 \leqslant m_0 \leqslant m_1 \leqslant \ldots \leqslant m_{t-1} \leqslant n-1$, and let $N = \{n_0, n_1, \ldots, n_{l-1}\}$ be a set of nonnegative integers. Let $d = (\sigma(N), n)$, $s = n/d$. Then the following are equivalent:*

(1) $N + M \equiv M \pmod{n}$.

(2) *For all $f \in \{0, 1, \ldots, l-1\}$, we have $n_f + M \equiv M \pmod{n}$.*

(3) *M is a union of arithmetic progressions modulo n with common difference d.*

Lemma 1.8. *Let $U \subseteq \{0, 1, 2, \ldots, n-1\}$ be a union of some arithmetic progressions modulo n with common difference and $d = d_n(U)$. Let $A = Q_r\Big(\sum\limits_{i \in U} P^i\Big)$, $B = Q_s\Big(\sum\limits_{i \in U} P^i\Big) \in G_n$. Then $A = B$ if and only if $r \equiv s \pmod{d}$.*

P r o o f. Necessity: Let $A = B$. Obviously, we have $\Delta(A) = \Delta(B) = U$. We know that the second row of A is obtained from the first row of A by shifting the elements cyclically r columns to the right and the second row of B is obtained from the first row of B by shifting the elements cyclically s columns to the right. Since the second row of A is equal to that of B, we have $\Delta(A) + r \equiv \Delta(B) + s \pmod{n}$, i.e., $U + r \equiv U + s \pmod{n}$. It follows that $(r - s) + U \equiv U \pmod{n}$. By Lemma 1.7, U is an union of some arithmetic progressions modulo n with the same common difference $d_1 = (r - s, n)$. Since $d = d_n(U)$, we have $d \mid d_1$. Hence $d \mid (r - s)$, i.e., $r \equiv s \pmod{d}$.

Sufficiency: Let $r \equiv s \pmod{d}$, i.e., $d \mid (r-s)$. By Lemma 1.6, we have $(r-s)+U \equiv U \pmod{n}$. Since $\Delta(A) = \Delta(B) = U$, it follows that $\Delta(A)+r \equiv \Delta(B)+s \pmod{n}$. That is, the second row of A is the same as the second row of B. Similarly, we can prove that the ith row of A is the same as the ith row of B for $i = 3, 4, \ldots, n$. Therefore $A = B$. This proves the lemma. □

2. A characterization

In order to characterize the elements in $M(F)$, we need the following lemmas.

Lemma 2.1. *Let* $C = Q_c\Big(\sum\limits_{k\in\Delta(C)} P^k\Big) \in G_n$. *Then* F *is an idempotent of* $G_n(C)$ *if and only if* F *can be written in the form:*

$$F = Q_r\bigg(\sum_{u=0}^{e_1-1}(P^{i_u} + P^{i_u+d_1} + \ldots + P^{i_u+(m_1-1)d_1})\bigg),$$

where $d_1 = (\sigma(cr\Delta(F) + r\Delta(C)), n)$, $d_1 \mid (r^2c - r)$, *and* $n = m_1d_1$.

P r o o f. By Theorem 2.2 and Lemma 3.4 in [7], we can obtain the lemma. □

Lemma 2.2. *Let* n, c *and* r *be positive integers. Let* $C = Q_c\Big(\sum\limits_{k\in\Delta(C)} P^k\Big) \in G_n$ *and* $F = Q_r\Big(\sum\limits_{i\in\Delta(F)} P^i\Big)$ *be an idempotent of* $G_n(C)$. *If* $A = Q_s\Big(\sum\limits_{j\in\Delta(A)} P^j\Big)$ *is an element in the maximal subgroup* $M(F)$ *of* $G_n(C)$ *containing* F, *then* $\Delta(A) \equiv \Delta(F) + l \pmod n$ *for some integer* l.

P r o o f. Since A is an element of $M(F)$, by (1.2), we have $A * F = A$, and there exists an element in $G_n(C)$ such that $X * A = F$, i.e. $ACF = A$ and $XCA = F$. Let now $X = Q_t\Big(\sum\limits_{f\in\Delta(X)} P^f\Big)$. Then, by (1.1), we have

$$ACF = Q_{scr}\bigg(\sum_{l\in cr\Delta(A)+r\Delta(C)+\Delta(F)} P^l\bigg) = Q_s\bigg(\sum_{j\in\Delta(A)} P^j\bigg)$$

and

$$XCA = Q_{tcs}\bigg(\sum_{l\in cs\Delta(X)+s\Delta(C)+\Delta(A)} P^l\bigg) = Q_r\bigg(\sum_{i\in\Delta(F)} P^i\bigg).$$

Therefore, $cr\Delta(A)+r\Delta(C)+\Delta(F) \equiv \Delta(A) \pmod n$ and $cs\Delta(X)+s\Delta(C)+\Delta(A) \equiv \Delta(F) \pmod n$. Thus, $\forall\, l \in cr\Delta(A)+r\Delta(C)$ and $l' \in cs\Delta(X)+s\Delta(C)$, and we have

$$l + \Delta(F) \subseteq \Delta(A) \pmod n \quad \text{and} \quad l' + \Delta(A) \subseteq \Delta(F) \pmod n.$$

It follows that

$$l + l' + \Delta(A) \subseteq l + \Delta(F) \pmod n,$$

and so

$$l' + l + \Delta(A) \subseteq \Delta(A) \pmod n \quad (\text{because } l + \Delta(F) \subseteq \Delta(A) \pmod n).$$

Hence

$$l + l' + \Delta(A) \equiv \Delta(A) \pmod n.$$

Thus we have

$$\Delta(A) \equiv l + (l' + \Delta(A)) \pmod{n} \subseteq l + \Delta(F) \pmod{n} \subseteq \Delta(A) \pmod{n}.$$

Therefore $\Delta(A) \equiv \Delta(F) + l \pmod{n}$ for some integer l. This completes the proof. □

Theorem 2.1. *Let* $C = Q_c\Big(\sum\limits_{k\in\Delta(C)} P^k\Big) \in G_n$. *Let* $F = Q_0\Big(\sum\limits_{i\in\Delta(F)} P^i\Big)$ *be an idempotent element in* $G_n(C)$. *Then the maximal subgroup in* $G_n(C)$ *containing the idempotent element* F *is* $\{F\}$.

P r o o f. If $F = 0$, clearly $M(F) = \{F\}$. Now suppose that $F \neq 0$. Let $A = Q_s\Big(\sum\limits_{j\in\Delta(A)} P^j\Big) \in M(F)$. By (1.2), we know that $A \neq 0$ and $A * F = ACF = A$. Hence

$$\begin{aligned} A = ACF &= Q_s\Big(\sum_{j\in\Delta(A)} P^j\Big) Q_c\Big(\sum_{k\in\Delta(C)} P^k\Big) Q_0\Big(\sum_{i\in\Delta(F)} P^i\Big) \\ &= Q_{s\cdot c\cdot 0}\Big(\sum_{j\in\Delta(A),\, k\in\Delta(C),\, i\in\Delta(F)} P^{0\cdot c\cdot j + 0\cdot k + i}\Big) \\ &= Q_0\Big(\sum_{i\in\Delta(F)} P^i\Big) = F. \end{aligned}$$

Conversely, if $A = F$, then clearly $A \in M(F)$. Therefore $M(F) = \{F\}$. This proves the theorem. □

Theorem 2.2. *Let* $C = Q_c\Big(\sum\limits_{k\in\Delta(C)} P^k\Big) \in G_n$, *and let* $F = Q_r\Big(\sum\limits_{i\in\Delta(F)} P^i\Big)$ $(r \neq 0)$ *be an idempotent element of* $G_n(C)$. *Then* $A = Q_s\Big(\sum\limits_{j\in\Delta(A)} P_j\Big)$ *is an element of the maximal subgroup* $M(F)$ *of* $G_n(C)$ *containing* F *if and only if*

(2.1) s *is an invertible integer relative to* c, r *and* d;

(2.2) $\Delta(A) \equiv \Delta(F) + l \pmod{n}$ *for some* l *with* $0 \leqslant l \leqslant n-1$;

(2.3) *the integer* l *in* (2.2) *satisfies* $l(cr-1) \equiv 0 \pmod{d}$, *where* d *is the common difference of* $\Delta(F)$ *in the final form. i.e.,* $d = d_n(\Delta(F))$.

P r o o f. Sufficiency: Suppose that the conditions (2.1), (2.2) and (2.3) hold.

Since F is an idempotent in $G_n(C)$, by Lemma 2.1, we know that F can be written in the form: $F = Q_r\Big(\sum\limits_{u=0}^{e_1-1}(P^{i_u} + P^{i_u+d_1} + \ldots + P^{i_u+(m_1-1)d_1})\Big)$, where $d_1 = (\sigma(cr\Delta(F) + r\Delta(C)), n)$, and $r^2c \equiv r \pmod{d_1}$, and $n = m_1d_1$.

Let $\bigcup\{i_0, i_1, \ldots, i_{e-1}, n, d, m\}$ be the final form of $\Delta(F)$. Then we have $d \mid d_1$. Hence $r^2c \equiv r \pmod{d}$ and $n = md$ for some m. In the following we will prove that (1.2) holds.

First, we show that $A * F = A$.

Since $F * F = F$, we have

$$F * F = FCF = Q_{r^2c}\Big(\sum_{t\in cr\Delta(F)+r\Delta(C)+\Delta(F)} P^t\Big) = F,$$

and so

$$rc\Delta(F) + r\Delta(C) + \Delta(F) \equiv \Delta(F) \pmod{n}. \tag{2.4}$$

Since $\Delta(F)$ can be represented as the union of some arithmetic progression modulo n with common difference d and $d = d_n(\Delta(F))$, by the condition (2.3) and Lemma 1.6, we have $crl + \Delta(F) \equiv l + \Delta(F) \pmod{n}$. By the condition (2.2), we have

$$\begin{aligned} rc\Delta(A) + r\Delta(C) + \Delta(F) &\equiv rc(\Delta(F) + l) + r\Delta(C) + \Delta(F) \pmod{n} \\ &\equiv rc\Delta(F) + r\Delta(C) + (crl + \Delta(F)) \pmod{n} \\ &\equiv rc\Delta(F) + r\Delta(C) + \Delta(F) + l \pmod{n} \\ &\equiv \Delta(F) + l \pmod{n} \quad \text{(by (2.4))}. \end{aligned}$$

Hence

$$A * F = ACF = Q_{scr}\Big(\sum_{t\in cr\Delta(A)+r\Delta(C)+\Delta(F)} P^t\Big) = Q_{scr}\Big(\sum_{t\in\Delta(F)+l} P^t\Big).$$

Since s is an invertible integer relative to c, r and d, we have $s(cr-1) \equiv 0 \pmod{d}$. By Lemma 1.8, we have

$$A * F = ACF = Q_{scr}\Big(\sum_{t\in\Delta(F)+l} P^t\Big) = Q_s\Big(\sum_{t\in\Delta(F)+l} P^t\Big) = Q_s\Big(\sum_{t\in\Delta(A)} P^t\Big) = A.$$

Secondly, we prove that $F * A = A$.

By the condition (2.1), we have $s(cr-1) \equiv 0 \pmod{d}$. Since $r^2c \equiv r \pmod{d}$ and $d \mid d_1 = (\sigma(cr\Delta(F)+r\Delta(C)), n)$, by Lemma 1.5, we have $d \mid (\sigma(cs\Delta(F)+s\Delta(C)), n)$. By Lemma 1.6, we have

$$sc\Delta(F) + s\Delta(C) + \Delta(F) \equiv \Delta(F) \pmod{n}. \tag{2.5}$$

Hence

$$\begin{aligned}
F * A = FCA &= Q_r\Big(\sum_{i\in\Delta(F)} P^i\Big)Q_c\Big(\sum_{k\in\Delta(C)} P^k\Big)Q_s\Big(\sum_{j\in\Delta(A)} P^j\Big)\\
&= Q_{scr}\Big(\sum_{t\in sc\Delta(F)+s\Delta(C)+\Delta(A)} P^t\Big)\\
&= Q_{scr}\Big(\sum_{t\in sc\Delta(F)+s\Delta(C)+\Delta(F)} P^t\Big)P^l \quad \text{(by the condition (2.2))}\\
&= Q_{scr}\Big(\sum_{t\in\Delta(F)} P^t\Big)P^l \quad \text{(by (2.5))}\\
&= Q_s\Big(\sum_{t\in\Delta(F)} P^t\Big)P^l \quad \text{(by Lemma 1.8)}\\
&= A.
\end{aligned}$$

Finally, we shall show that there exist $X, Y \in G_n(C)$ such that $X * A = F$ and $A * Y = F$.

Since s is an invertible integer relative to c, r and d, and $r^2c \equiv r \pmod d$, the equations $scx \equiv r \pmod d$ and $x(cr-1) \equiv 0 \pmod d$ have common solutions by Lemma 1.4. Let s' be a common solution of them, i.e., $s'cs \equiv r \pmod d$ and $s'(cr-1) \equiv 0 \pmod d$. By Lemma 1.3, we can find an integer w such that $scw \equiv l \pmod d$ and l satisfies the condition (2.3). Hence

$$\begin{aligned}
&cs\Delta(F) + s\Delta(C) + \Delta(F) + l + csd - csw\\
&\quad\equiv \Delta(F) + l - csw \pmod n \quad \text{(by (2.5))}\\
&\quad\equiv \Delta(F) \pmod n \quad \text{(because } csw \equiv l \pmod d\text{)}.
\end{aligned}$$

Also, by Lemmas 1.5 and 1.6, we have

$$cs'\Delta(F) + s'\Delta(C) + \Delta(F) \equiv \Delta(F) \pmod n. \tag{2.6}$$

Let now $X = Q_{s'}\Big(\sum_{x\in\Delta(F)+d-w} P^x\Big)$ and $Y = Q_{s'}\Big(\sum_{y\in\Delta(F)-ls'c} P^y\Big) \in G_n(C)$. Then we have

$$\begin{aligned}
X * A = XCA &= Q_{s'}\Big(\sum_{x\in\Delta(F)+d-w} P^x\Big)Q_c\Big(\sum_{k\in\Delta(C)} P^k\Big)Q_s\Big(\sum_{j\in\Delta(A)} P^j\Big)\\
&= Q_{s'cs}\Big(\sum_{t\in cs\Delta(F)+s\Delta(C)+\Delta(A)+csd-csw} P^t\Big)
\end{aligned}$$

$$= Q_{s'cs}\Big(\sum_{t\in cs\Delta(F)+s\Delta(C)+\Delta(F)+l+csd-csw} P^t\Big) \quad \text{(by the condition (2.2))}$$
$$= Q_{s'cs}\Big(\sum_{t\in\Delta(F)} P^t\Big)$$
(by the fact that $cs\Delta(F)+s\Delta(C)+\Delta(F)+l+csd-csw \equiv \Delta(F) \pmod n$)
$$= Q_r\Big(\sum_{t\in\Delta(F)} P^t\Big) \quad \text{(by Lemma 1.8 and the fact that } s'cs\equiv r \pmod d)$$
$$= F,$$

and

$$A*Y = ACY = Q_s\Big(\sum_{j\in\Delta(A)} P^j\Big)Q_c\Big(\sum_{k\in\Delta(C)} P^k\Big)Q_{s'}\Big(\sum_{y\in\Delta(F)-ls'c} P^y\Big)$$
$$= Q_{scs'}\Big(\sum_{t\in cs'\Delta(A)+s'\Delta(C)+\Delta(F)-ls'c} P^t\Big)$$
$$= Q_{scs'}\Big(\sum_{t\in cs'\Delta(F)+s'\Delta(C)+\Delta(F)} P^t\Big) \quad \text{(by the condition (2.2))}$$
$$= Q_{scs'}\Big(\sum_{t\in\Delta(F)} P^t\Big) \quad \text{(by (2.6))}$$
$$= Q_r\Big(\sum_{t\in\Delta(F)} P^t\Big) \quad \text{(by Lemma 1.8 and the fact that } s'cs\equiv r \pmod d)$$
$$= F.$$

Therefore A is an element of the maximal subgroup $M(F)$ of $G_n(C)$.

Necessity: Suppose that A is an element in $M(F)$. By Lemma 2.2, we have $\Delta(A)\equiv\Delta(F)+l \pmod n$. This means that the condition (2.2) holds.

Since $A*F=A$ (by (1.2)) and $A*F=ACF=Q_{scr}\Big(\sum_{t\in cr\Delta(A)+r\Delta(C)+\Delta(F)} P^t\Big) = Q_{scr}\Big(\sum_{t\in cr\Delta(F)+r\Delta(C)+\Delta(F)+crl} P^t\Big)$ and $A=Q_s\Big(\sum_{j\in\Delta(A)} P^j\Big)=Q_s\Big(\sum_{j\in\Delta(F)+l} P^j\Big)$, we have $cr\Delta(F)+r\Delta(C)+\Delta(F)+crl\equiv\Delta(F)+l \pmod n$. By (2.4), we have $\Delta(F)+crl\equiv\Delta(F)+l \pmod n$. By Lemma 1.8, we have $scr\equiv s \pmod d$, i.e., $s(cr-1)\equiv 0 \pmod d$. By Lemma 1.6, we have $l(cr-1)\equiv 0 \pmod d$.

There exists an $X\in G_n(C)$ such that $X*A=F$. Let $X=Q_{s'}\Big(\sum_{x\in\Delta(X)} P^x\Big)$. Then $X*A=XCA=Q_{s'cs}\Big(\sum_{t\in sc\Delta(X)+s\Delta(C)+\Delta(A)} P^t\Big)$. Since $F=Q_r\Big(\sum_{i\in\Delta(F)} P^i\Big)$,

we have $Q_{s'cs}\Big(\sum\limits_{t\in sc\Delta(X)+s\Delta(C)+\Delta(A)} P^t\Big) = Q_r\Big(\sum\limits_{i\in\Delta(F)} P^i\Big)$. Hence $scs' \equiv r \pmod d$ by Lemma 1.8. These mean that the condition (2.1) and the condition (2.3) hold. This proves Theorem 2.2. □

In Theorem 2.2, if $C = E$ (E is the identity matrix), then $G_n(C) = G_n$. In this case, we have the following corollary.

Corollary 2.1 ([3], Theorem 2). *Let* $F = Q_r\Big(\sum\limits_{u=0}^{e-1}(P^{i_u} + P^{i_u+d} + \ldots + P^{i_u+(m-1)d})\Big)$ *be an idempotent in* G_n. *Then* $A = Q_s\Big(\sum\limits_{j\in\Delta(A)} P^j\Big)$ *is an element of the maximal subgroup* $M(F)$ *of* G_n *containing* F *if and only if*

(1) s *is an invertible integer relative to* r *and* d;
(2) *there exists an integer* l, $0 \leqslant l \leqslant n-1$, *such that* $\Delta(A) \equiv \Delta(F) + l \pmod n$;
(3) *the integer* l *in* (2) *satisfies* $l(cr-1) \equiv 0 \pmod d$.

3. Algorithm and example

For any fixed r-circulant matrix $C \in G_n$ we will present an algorithm to find all the elements in the maximal subgroup $M(F)$ in the semigroup $G_n(C)$ containing F.

Let

$$C = Q_c\Big(\sum_{k\in\Delta(C)} P^k\Big),$$
$$F = Q_r\Big(\sum_{u=0}^{e-1}(P^{i_u} + P^{i_u+d} + \ldots + P^{i_u+(m-1)d})\Big), \quad n = md.$$

Step 1. Compute all the invertible integers relative to c, r and d, say $s_0, s_1, \ldots, s_{g-1}$.
Step 2. Compute all integers l such that $l(cr-1) \equiv 0 \pmod d$, say $l_0, l_1, \ldots, l_{h-1}$.
Step 3. Form all elements of

$$M(F) = \Big\{A_{pq} = Q_{s_p}\Big(\sum_{u=0}^{e-1}(P^{i_u} + P^{i_u+d} + \ldots + P^{i_u+(m-1)d})\Big)P^{l_q} : \\ p = 0, 1, \ldots, g-1, \;\; q = 0, 1, \ldots, h-1\Big\}.$$

Example. Let $n = 144$, $C = Q_2(P^3 + P^6) \in G_{144}$, we can verify that $F = Q_8\Big(\sum\limits_{u=0}^{5} P^{3+24u}\Big)$ is one of the idempotent elements in $G_{144}(C)$.

Step 1. The invertible integers relative to $2, 8$ and 24 are $\{8, 16\}$;

Step 2. All integers which satisfy $l(2 \cdot 8 - 1) \equiv 0 \pmod{24}$ are $\{0, 8, 16\}$;

Step 3. $M(F) = \Big\{ Q_{s_p} \sum_{u=0}^{5} (P^{3+24u}) P^{l_q} : \ s_p = 8, 16, \ \ l_q = 0, 8, 16 \Big\}$.

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