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NON-HOLONOMIC (r, s, q)-JETS

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Abstract. We generalize the concept of an (r, s, q)-jet to the concept of a non-holonomic (r, s, q)-jet. We define the composition of such objects and introduce a bundle functor $\tilde{J}^{r,s,q}\colon \mathcal{FM}_{k,l} \times \mathcal{FM}$ defined on the product category of (k, l)-dimensional fibered manifolds with local fibered isomorphisms and the category of fibered manifolds with fibered maps. We give the description of such functors from the point of view of the theory of Weil functors.

Further, we introduce a bundle functor $\tilde{J}_1^{r,s,q}\colon$ 2-$\mathcal{FM}_{k,l} \to \mathcal{FM}$ defined on the category of 2-fibered manifolds with $\mathcal{FM}_{k,l}$-underlying objects.

Keywords: bundle functor, jet, non-holonomic jet, Weil bundle

MSC 2000: 58A05, 58A20

Preliminaries

We present a contribution to the theory of jet functors. We come out from the classical concept of a non-holonomic r-jet, the introduction of which goes back to Ehresmann, [2]. The classical results on the theory of non-holonomic jets were achieved by Pradines, [12] and many problems related to them were studied by other authors, e.g. Kolář, Virsik, Kureš.

The present paper is devoted to the investigation of fibered non-holonomic jets from the point of view of bundle functors, [5]. We follow the basic terminology from [5]. The category of smooth manifolds with smooth maps is denoted by $\mathcal{M}f$ while the category of m-dimensional manifolds with local diffeomorphism is denoted by $\mathcal{M}f_m$. Further, denote by $\mathcal{FM}$ ($\mathcal{FM}_m$) the category of fibered manifolds with fibered maps (with m-dimensional bases and base maps formed by local diffeomorphisms).

The first starting point is the concept of an (r, s, q)-jet, introduced and studied by Kolář and Doupovec (e.g. in [1] and [3]). We essentially use the classical results

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on product preserving bundle functors. Bundle functors of this kind defined on the category $\mathcal{M}f$ coincide with Weil functors, [5], while those defined on the category $\mathcal{FM}$ are of the form T^μ for homomorphisms $\mu\colon A \to B$ of Weil algebras, [8]. Another starting point is the paper of Kolář, [3], devoted to the abstract definition of jet functors. Bundle functors of this kind are considered either on the product category $\mathcal{M}f_m \times \mathcal{M}f$ so that they preserve products in the second factor and staisfy the finiteness of order in the first factor, or they are considered as fiber-product preserving bundle functors on the category $\mathcal{FM}_m$.

In Section 1, we generalize the first concept to the concept of a fibered jet functor, which is defined on the product category $\mathcal{FM}_{k,l} \times \mathcal{FM}$, preserves products in the second factor and satisfies the assumption of finite order in the first factor. Its well-known example is the functor of (r,s,q)-jets. In Section 2, we define the concept of a non-holonomic (r,s,q)-jet and the composition of such objects. We prove some assertions related to them the character of which being rather technical. In Section 3, we define a bundle functor $\tilde{J}^{r,s,q}$ of non-holonomic (r,s,q)-jets and describe it from the point of view of the Weil theory. We also give a simple iteration formula for such functors. Moreover, we introduce a bundle functor $\tilde{J}^{r,s,q}_1\colon$ 2-$\mathcal{FM}_{k,l} \to \mathcal{FM}$ defined on the category of 2-fibered manifolds with $\mathcal{FM}_{k,l}$-underlying objects.

1. Bundle functors on $\mathcal{FM}_{k,l} \times \mathcal{FM}$

In [6], Kolář and Mikulski classified all bundle functors defined on the category $\mathcal{M}f_m \times \mathcal{M}f$, the product of the category of m-dimensional manifolds and the category of smooth manifolds, which are of order r in the first factor. They found a bijection between bundle functors of this kind and couples (G,H) of bundle functors $G\colon \mathcal{M}f \to \mathcal{FM}$ and group homomorphisms $H\colon G^r_m \to \mathcal{NE}(G)$ assigning natural equivalences over G to elements of G^r_m. The correspondence is given by $F \mapsto (G^F, H^F)$ for G^F defined by $G^F N = F_0(\mathbb{R}^m, N)$, $G^F f = F(\mathrm{id}_{\mathbb{R}^m}, f)$ and the action H^F of G^r_m on G^F defined by $H^F(j^r_0 g)(a) = F(g, \mathrm{id}_N)(a)$ for $a \in G^F N$. Conversely, any couple (G,H) is assigned a bundle functor $\mathcal{M}f_m \times \mathcal{M}f \to \mathcal{FM}$ defined by $F(M,N) = P^r M[GN, H_N]$, the associated bundle to the frame bundle $P^r M$ with the standard fiber GN and the action H_N of G^r_m on GN.

Moreover, the authors specified the result for the case of F preserving products in the second factor. Then any G is a Weil functor T^A and H is a homomorphism with values in $\mathrm{Aut}(A)$ since natural transformations of Weil functors are canonicaly identified with homomorphisms of the corresponding Weil algebras, ([4], [5]).

To generalize the result to the product category $\mathcal{FM}_{k,l} \times \mathcal{FM}$, we shall need the concept of an (r,s,q) jet, $s \geqslant r \leqslant q$, [3]. Let us recall it. For a fibered manifold $p\colon Y \to M$ and a manifold Z, two maps f, g are said to determine the same (r,s)-

jet ($r \leqslant s$) at $y \in M$ if $j^r_y f = j^r_y g$ and in addition $j^s_y(f|Y_x) = j^s_y(g|Y_x)$, where $x = p(y)$ and Y_x denotes the fiber of Y over x. In case of Z being a fibered manifold $q\colon Z \to N$, two (r,s)-jets $j^{r,s}_y f$ and $j^{r,s}_y g$ of fibered morphisms $f, g\colon Y \to Z$ are said to determine the same (r,s,q)-jet at $y \in Y$ ($q \geqslant r$) if $j^q_x \underline{f} = j^q_x \underline{g}$ for the base maps $\underline{f}$, $\underline{g}\colon M \to N$ associated to f, g.

In [1], Doupovec and Kolář studied bundle functors $T^{r,s,q}_{k,l}$, defined on objects by $T^{r,s,q}_{k,l} Y = J^{r,s,q}_{0,0}(\mathbb{R}^{k,l}, Y)$ and on morphisms by $T^{r,s,q}_{k,l} f(j^{r,s,q}_{0,0}\varphi) = j^{r,s,q}_{0,0}(f \circ \varphi)$ for any $\mathcal{FM}$-morphism $f\colon Y \to Z$. By $\mathbb{R}^{k,l}$, we denote the fibered manifold $pr_1\colon \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k$. In [1], there was also mentioned the differential group $G^{r,s,q}_{k,l} = \operatorname{inv} J^{r,s,q}_{0,0}(\mathbb{R}^{k,l}, \mathbb{R}^{k,l})_{0,0} = \operatorname{inv} T^{r,s,q}_{k,l}(\mathbb{R}^{k,l})_{0,0}$. Bundle functors $T^{r,s,q}_{k,l}$ preserve products and by [8] and [1] they can be considered as the functors T^ν for $\nu\colon \mathbb{D}^q_k \to \mathbb{D}^{r,s}_{k,l}$ defined by

$$(1.1) \qquad \nu = \pi^q_r \times O_l, \qquad \pi^q_r(j^q_0\varphi) = j^r_0\varphi, \quad 0_l(j^q_0\varphi) = 0 \in \mathbb{D}^s_l.$$

A bundle functor $F\colon \mathcal{FM}_{k,l} \times \mathcal{FM} \to \mathcal{FM}$ is said to be of order (r,s,q) in the first factor if for any local fibered isomorphisms $f, g\colon Y \to \overline{Y}$ satisfying $j^{r,s,q}_y f = j^{r,s,q}_y g$ it holds $F_{y,z}(f,h) = F_{y,z}(g,h)$ for any $y \in Y$, $z \in Z$ and any $\mathcal{FM}$-morphism $h\colon Z \to \overline{Z}$.

Analogously to [6], define a bundle functor $G^F\colon \mathcal{FM} \to \mathcal{FM}$ by $G^F Z = F_{0,0}(\mathbb{R}^{k,l}, Z)$ and $G^F f = F(\mathrm{id}_{\mathbb{R}^{k,l}}, f)$ for any $\mathcal{FM}$-morphism $f\colon Z \to \overline{Z}$. Further, define an action H^F_Z of $G^{r,s,q}_{k,l}$ on $G^F Z$ by $H^F_Z(j^{r,s,q}_{0,0} g)(s) = F(g, \mathrm{id}_Z)(s)$ for any $s \in G^F Z$. It holds $G^F f \circ H^F_Z(g)(s) = F(\mathrm{id}_{\mathbb{R}^{k,l}}, f) \circ F(g, \mathrm{id}_Z)(s) = F(g, \mathrm{id}_{\overline{Z}}) \circ F(\mathrm{id}_{\mathbb{R}^{k,l}}, f)(s) = H^F_{\overline{Z}}(j^{r,s,q}_{0,0} g) \circ G^F f(s)$. It follows that $H^F(j^{r,s,q}_{0,0} g)$ is a natural equivalence. Since $H^F(j^{r,s,q}_{0,0} g_1 \circ j^{r,s,q}_{0,0} g_2) = H^F(j^{r,s,q}_{0,0} g_1) \circ H^F(j^{r,s,q}_{0,0} g_2)$ we deduce that $H^F\colon G^{r,s,q}_{k,l} \to \mathcal{NE}(G^F)$ is a homomorphism assigning to elements of $G^{r,s,q}_{k,l}$ natural equivalences on G^F.

Conversely, having a bundle functor G on $\mathcal{FM}$ and a homomorphism $G^{r,s,q}_{k,l} \to \mathcal{NE}(G)$, we construct a bundle functor $(G,H)\colon \mathcal{FM}_{k,l} \times \mathcal{FM} \to \mathcal{FM}$ assigning to any couple of objects Y, Z the associated bundle $(G,H)(Y,Z) = P^{r,s,q}Y[GZ, H_Z]$ to the frame bundle $P^{r,s,q}Y = \operatorname{inv} J^{r,s,q}(\mathbb{R}^{k,l}, Y)$ of order (r,s,q) with standard fiber GZ and the left action H_Z on GZ.

The following assertion is the modification of that in [6] to bundle functors under discussion.

Proposition 1. *Let $F\colon \mathcal{FM}_{k,l} \times \mathcal{FM} \to \mathcal{FM}$ be a bundle functor of order (r,s,q) in the first factor. Then $F = (G^F, H^F)$.*

Proof. For any $j^{r,s,q}_{0,0}\varphi \in P^{r,s,q}_y Y$ we define the frame map $\widetilde{j^{r,s,q}_{0,0}\varphi}\colon G^F_z Z \to F_{y,z}(Y,Z)$ by $\widetilde{j^{r,s,q}_{0,0}\varphi}(s) = F(\varphi, \mathrm{id}_Z)(s)$. The proof is accomplished by the verification of its correctness and bijectivity which is easy and analogous to the situation in [6]. □

Let $\overline{G}\colon \mathcal{FM} \to \mathcal{FM}$ be another bundle functor of order (r,s,q) in the first factor and $\overline{H}\colon G^{r,s,q}_{k,l} \to \mathcal{NE}(\overline{G})$ a group homomorphism. Further, let us have a natural transformation $t\colon G \to \overline{G}$ that is $G^{r,s,q}_{k,l}$-invariant, i.e. $\overline{H}(j^{r,s,q}_{0,0}g)\circ t_Z = t_Z \circ H(j^{r,s,q}_{0,0}g)$. Then we have an induced natural transformation $\tilde{\tau}\colon (G,H) \to (\overline{G},\overline{H})$ defined by $\tilde{\tau}_{Y,Z} = \mathrm{id}_{P^{r,s,q}Y}[t_Z]\colon P^{r,s,q}Y[GZ,H_Z] \to P^{r,s,q}Y[\overline{G}Z,\overline{H}_Z]$. Conversely, let F and $\overline{F}$ be two bundle functors on $\mathcal{FM}_{k,l} \times \mathcal{FM}$ of order (r,s,q) in the first factor and $\tau\colon F \to \overline{F}$ be a natural transformation. Analogously as in Lemma 14.11 of [5] one deduces that $\tau_{Y,Z}$ is over identity on $Y \times Z$ which induces a natural transformation $t\colon G^F \to G^{\overline{F}}$ defined by $t_Z = \tau_{\mathbb{R}^{k,l},Z|F_0(\mathbb{R}^{k,l}Z)}$. Analogously as in [6] one can show that $H^{\overline{F}}_Z(j^{r,s,q}_{0,0}g) \circ t_Z = t_Z \circ H^F_Z(j^{r,s,q}_{0,0}g)$. This follows from $\tau_{\mathbb{R}^{k,l},Z} \circ F(g,\mathrm{id}_Z) = \overline{F}(g,\mathrm{id}_Z) \circ \tau_{\mathbb{R}^{k,l},Z}$ upon restricting over $(0,0) \in \mathbb{R}^{k,l}$.

The following assertion is the modification of Proposition 6 in [6] to the investigated bundle functors preserving products in the second factor, i.e. those bundle functors on $\mathcal{FM}_{k,l} \times \mathcal{FM}$ which satisfy $F(Y, Z_1 \times Z_2) = F(Y,Z_1) \times_Y F(Y,Z_2)$.

Proposition 2. *(G,H) preseves products in the second factor if and only if $G = T^\mu$ for some Weil algebra homomorphism $\mu\colon A \to B$ and $H\colon G^{r,s,q} \to \mathrm{Aut}(A,\mu,B)$ is a homomorphism of Lie groups. In this case it holds $(G,H)(g, f\times h) = (G,H)(g,f) \times_g (G,H)(g,h)$ for every local fibered isomorphism $g\colon Y \to \overline{Y}$ and every fibered maps $f\colon Z_1 \to \overline{Z}_1$ and $h\colon Z_2 \to \overline{Z}_2$.*

2. Non-holonomic (r,s,q)-jets

In the present section, we introduce the concept of a non-holonomic (r,s,q)-jet as a generalization of an (r,s,q) jet, introduced and studied by Doupovec and Kolář in [1]. Further, we define the composition of such objects. We start from the concept of a non-holonomic r-jet introduced by Ehresmann, [2] and studied by many authors, e.g. Kolář ([3]), Pradines ([12]), Kureš ([6]) and in [13].

For $r = 1$, non-holonomic r-jets coincide with the holonomic jets and so does their composition. By induction in respect to r, we remind the definition and the composition of non-holonomic r-jets. For manifolds M, N and $x \in M$, $y \in N$ denote by $\tilde{J}^r_x(M,N)_y$ the space of non-holonomic r-jets with the source x and the target y. By α and β, denote the source and target projections. To define non-holonomic $(r+1)$-jets, consider local sections $\sigma\colon M \to \tilde{J}^r(M,N)$ in a neighbourhood of x. Then non-holonomic $(r+1)$-jets X satisfying $\alpha(X) = x$ are just elements of the form

$j^1_x\sigma$. Further, let $X_1 = j^1_x\sigma_1 \in \tilde{J}^r_x(M,N)_y$ and $X_2 = j^1_x\sigma_2 \in \tilde{J}^r_y(N,P)$. Then the composition $X_2 \circ_{r+1} X_1$ is defined by

$$X_2 \circ_{r+1} X_1 = j^1_x(\sigma_2(\beta(\sigma_1(u)) \circ_r \sigma_1(u)) \tag{2.1}$$

where $\circ_r$ denotes the composition of non-holonomic r-jets. It will be useful to consider the formula (2.1) in the form

$$X_2 \circ_{r+1} X_1 = j^1_x s \quad \text{for } s = (\circ_r) \circ (\mathrm{id} \times (\sigma_2 \circ \beta)) \circ (\sigma_1 \times \sigma_1). \tag{2.2}$$

Consider $\mathscr{FM}$-objects $p_Y\colon Y \to M$, $q_Z\colon Z \to N$. In the following definition, we introduce the concept of a fibered non-holonomic r-jet. It is defined by induction as follows

Definition 1. 1-jets $j^1_y f \in J^1_y(Y,Z)_z$ of $\mathscr{FM}$-morphisms $f\colon Y \to Z$ are said to be fibered non-holonomic 1-jets with the source $y \in Y$ and the target $z \in Z$. The space of such objects considered as a fibered manifold $p^1_{Y,Z}\colon J^1(Y,Z) \to J^1(M,N)$ is said to be the space of fibered non-holonomic 1-jets for $p^1_{Y,Z}$ defined by $p^1_{Y,Z}(j^1_y f) = j^1_x \underline{f}$, $x = p_Y(y)$. We write $\tilde{J}^1(Y,Z)$.

Suppose we have defined fibered non-holonomic r-jets and the surjective submersion $p^r_{Y,Z}\colon \tilde{J}^r(Y,Z) \to \tilde{J}^r(M,N)$ covering (p_Y, q_Z) with respect to the product (α,β) of the source and target projections. Let $\sigma\colon Y \to \tilde{J}^r(Y,Z)$ be fibered local sections, i.e. commutative diagrams of the form

$$\begin{array}{ccc} Y & \xrightarrow{\sigma} & \tilde{J}^r(Y,Z) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p^r_{Y,Z}} \\ M & \xrightarrow{\underline{\sigma}} & \tilde{J}^r(M,N) \end{array}$$

where $\underline{\sigma}$ denotes the base map associated to σ. Then 1-jets $j^1_y\sigma$ of such elements are said to be fibered non-holonomic $(r+1)$-jets. Define the fibered manifold $p^{r+1}_{Y,Z}\colon \tilde{J}^{r+1}(Y,Z) \to \tilde{J}^{r+1}(M,N)$ by $p^{r+1}_{Y,Z}(j^1_y\sigma) = j^1_x\underline{\sigma} = j^1_x(p(u) \mapsto \underline{\sigma}(p(u)) = j^1_x(p(u) \mapsto p^r_{Y,Z}(\sigma(u))$. Then $p^{r+1}_{Y,Z}$ is said to be the space of fibered non-holonomic $(r+1)$-jets.

In the last definition, one should verify that $p^r_{Y,Z}$ is a surjective submersion. This is done by induction. For $r = 1$, the assertion is obvious. In the induction step, suppose having a local section $s^r_{Y,Z}$ with respect to the diagram from Definition 1, and a local section $\nu\colon M \to \tilde{J}^r(M,N)$ with repect to (α,β). We define the local section $s^{r+1}_{Y,Z}\colon \tilde{J}^{r+1}(M,N) \to \tilde{J}^{r+1}(Y,Z)$ by

$$j^1_x\nu \mapsto j^1_{\alpha(s^r_{Y,Z}\circ\nu(x))} s^r_{Y,Z}(\nu) j^1_{s^0_{Y,Z}(x)}(s^r_{Y,Z} \circ \nu \circ p_Y),$$

where $s^0_{Y,Z}$ is the local section $M \to Y$ covered by every $s^r_{Y,Z}$. Now one can immediately verify that $p^{r+1}_{Y,Z} \circ s^{r+1}_{Y,Z}(j^1_x\nu) = j^1_x\nu$.

Remark. The formula 2.2 implies that $p^r_{Y,Z}$ preserves the composition of fibered non-holonomic r-jets. In formulas this reads

$$p^r_{Z,W}(j^1_z\sigma_2) \circ_r p^r_{Y,Z}(j^1_y\sigma_1) = p^r_{Y,W}(j^1_z\sigma_2 \circ_r j^1_y\sigma_1) \tag{2.3}$$

where $W \to P$ is another fibered manifold.

For any fibered manifolds $p_Y\colon Y \to M$ and $p_Z\colon Z \to N$, elements of $J^1(Y,Z)$ coincide with tangent maps Tf of $\mathscr{FM}$-morphisms $f\colon Y \to Z$. Consider elements $j^1_y(f|Y_x)$, $p_Y(y) = x$ and let Y_x denote the fiber of Y over x. Elements of this kind are identified with V_yf where V denotes the vertical tangent bundle. Let us denote by $J^{1,V}(Y,Z)$ the space of all elements of this kind. Further, consider the maps $\pi^{1,V}_{Y,Z}\colon J^1(Y,Z) \to J^{1,V}(Y,Z)$ defined by

$$\pi^{1,V}_{Y,Z}(j^1_yf) = j^1_y(f|Y_x) \tag{2.4}$$

Then the following holds

Lemma 1. *The maps $\pi^{1,V}$ from* (2.4) *are surjective submersions invariant with respect to the composition, i.e.*

$$\pi^{1,V}_{Y,W}(j^1_y(g \circ f)) = \pi^{1,V}_{Z,W}(j^1_zg) \circ \pi^{1,V}_{Y,Z}(j^1_yf) \tag{2.5}$$

for arbitrary $\mathscr{FM}$-morphism $g\colon (Z \to N) \to (W \to P)$.

Proof. The proof is immediately obtained from the definition of $\pi^{1,V}$. □

For arbitrary manifolds M, N, the space of non-holonomic r-jets can be considered as the r-th iteration of J^1 in the following sense. Put $J^1_MN = J^1(M,N)$. We can identify $J^1(M,N)$ with the space $J^1_1(M \times N)$ of local sections with respect to the canonical projection $M \times N \to M$, [5]. Then $\tilde{J}^r(M,N)$ can be defined by

$$\tilde{J}^r(M,N) = J^1_M \circ \ldots \circ J^1_MN. \tag{2.6}$$

For fibered manifolds $p_Y\colon Y \to M$ and $q_Z\colon Z \to N$ we can express the space of fibered 1-jets $J^1(Y,Z)$ by $J^1(Y,Z) \simeq J^1_YZ \simeq p\colon J^1_YZ \to J^1_MN$ for p defined by $p(j^1_yf) = j^1_x\underline{f}$, $x = p_Y(y)$. We can analogously obtain the space of fibered non-holonomic r-jets $\tilde{J}^r(Y,Z)$ as the r-th iteration of J^1_Y to Z, i.e. $\tilde{J}^r(Y,Z) \simeq J^1_Y \circ \ldots \circ J^1_YZ$.

We can also define an arbitrary composition of J^1_Y and $J^{1,V}_Y$. By induction we can define the composition of elements from $\hat{J}^r(Y,Z)$ and $\hat{J}^r(Z,W)$ where $\hat{J}^r$ denotes the space of generalized fibered non-holonomic r-jets. Such a space is obtained as an arbitrary r-th iteration of J^1_Y and $J^{1,V}_Y$ to Z. The composition is defined as follows. Let $\sigma\colon Y\to \hat J^r(Y,Z)$ and $\varrho\colon \hat J^r(Z,W)$ be local fibered sections satisfying $\beta(\sigma(y))=z\in Z$, $y\in Y$. Then $j^1_y\sigma$ and $j^{1,V}_y\sigma$ are elements of $\hat J^{r+1}(Y,Z)$ and so are $j^1_z\varrho$ and $j^{1,V}_z\varrho$. Their composition is given by

$$\begin{aligned} j^1_z\varrho\hat\circ_{r+1}j^1_y\sigma &= j^1_y(\varrho(\beta(\sigma(u)))\hat\circ_r\sigma(u)) \quad \text{or}\\ j^{1,V}_z\varrho\hat\circ_{r+1}j^{1,V}_y\sigma &= j^{1,V}_y(\varrho(\beta(\sigma(u)))\hat\circ_r\sigma(u)) \end{aligned} \tag{2.7}$$

for the obvious source and target projections $\alpha\colon \hat J^r(Y,Z)\to Y$ and $\beta\colon \hat J^r(Y,Z)\to Z$.

Definition 2. Let Y, Z be fibered manifolds. Then the space $\tilde J^{r,s}(Y,Z)$ identified with $\underbrace{J^{1,V}_Y\circ\ldots\circ J^{1,V}_Y}_{(s-r)\text{-times}}\circ\underbrace{J^1_Y\circ\ldots\circ J^1_Y}_{r\text{-times}}Z$ is said to be the space of (r,s)-non-holonomic jets between Y and Z. The composition of (r,s)-non-holonomic jets is defined by the formula (2.7).

In what follows, we are going to define the projection $\pi^{r,s}_{Y,Z}\colon \tilde J^s(Y,Z)\to\tilde J^{r,s}(Y,Z)$ and prove that it preserves compositions. Thus we prove that non-holonomic (r,s)-jets are in fact equivalence classes of fibered non-holonomic s-jets corresponding to $\pi^{r,s}$. Moreover, their composition will be given by the composition of any representative fibered non-holonomic s-jets.

Let us define the projection $\pi^{r,s}_{Y,Z}\colon \tilde J^s(Y,Z)\to\tilde J^{r,s}(Y,Z)$ by induction with respect to $s-r$ as follows. Put $\pi^{r,r+1}_{Y,Z}:=\pi^{1,V}_{Y,\tilde J^r(Y,Z)}$ for all $r\in\mathbb N$. Suppose we have defined $\pi^{r,s}_{Y,Z}$. Then $\pi^{r,s+1}_{Y,Z}\colon \tilde J^{s+1}(Y,Z)\to\tilde J^{r,s+1}(Y,Z)$ is defined by

$$j^1_y\sigma\mapsto \pi^{1,V}_{Y,\tilde J^{r,s}(Y,Z)}(j^1_y(\pi^{r,s}_{Y,Z}\circ\sigma)) \tag{2.8}$$

for any local fibered section $\sigma\colon Y\to\tilde J^s(Y,Z)$.

Proposition 3. *Let $X_1\in\tilde J^s(Y,Z)$ and $X_2\in\tilde J^s(Z,W)$ be fibered non-holon-omic s-jets satisfying $\beta(X_1)=\alpha(X_2)$. Then the $\pi^{r,s}$ defined by* (2.8) *satisfies*

$$\pi^{r,s}_{Y,W}(X_2\circ_s X_1)=\pi^{r,s}_{Z,W}(X_2)\circ_{r,s}\pi^{r,s}_{Y,Z}(X_1) \tag{2.9}$$

for the composition $\circ_{r,s}$ defined in (2.7).

P r o o f. We prove the assertion by induction in respect to $s-r$. For $s=r+1$ the assertion is immediately deduced from (2.2) and Lemma 1. Suppose (2.9) be valid for r and s. For $s+1$, we have $\pi^{r,s+1}_{Y,W}(j^1_z\varrho\circ_{s+1}j^1_y\sigma)=\pi^{1,V}_{Y,\tilde J^{r,s}(Y,W)}(j^1_y(u\mapsto \pi^{r,s}_{Y,W}(\varrho(\beta(\sigma(u)))\circ_s\sigma(u))))=\pi^{1,V}_{Y,\tilde J^{r,s}(Y,W)}(j^1_y(u\mapsto\pi^{r,s}_{Y,W}(\varrho(\beta(\sigma(u))))\circ_{r,s}\pi^{r,s}_{Y,Z}(\sigma(u))))$ by the induction step. By Lemma 2.2 and (2.7) we have finally $\pi^{r,s+1}_{Z,W}(j^1_z\varrho)\circ_{r,s+1}\pi^{r,s+1}_{Y,Z}(j^1_y\sigma)$ which proves the assertion. □

Let $s_1 \geqslant s_2$. Denote by $\pi^{s_1}_{s_2}$ the $(s_1 - s_2) - th$ iteration of the projection β^1 defined as follows. For $j^1_y s \in \tilde{J}^r(Y,Z)$ put $\beta^1(j^1_y s) = s(y)$ where s is a local fibered section $Y \to \tilde{J}^{r-1}(Y,Z)$ for arbitrary r. Thus we have $\pi^{s_1}_{s_2,Y,Z}\colon \tilde{J}^{s_1}(Y,Z) \to \tilde{J}^{s_2}(Y,Z)$. Clearly, $\pi^{s_1}_{s_2}$ preserves compositions of non-holonomic jets. By definition 2.1. there is a surjective submersion $p^r_{Y,Z}\colon \tilde{J}^r(Y,Z) \to \tilde{J}^r(M,N)$ defining the structure of a fibered manifold on $\tilde{J}^r(Y,Z)$. In the following definition we give the concept of a non-holonomic (r,s,q)-jet. The composition of such objects will be given by Proposition 3 and Definition 1.

Definition 3. Let $q \geqslant r \leqslant s$ be integers, $t = \max\{q,s\}$ and $p_Y\colon Y \to M$, $p_Z\colon Z \to N$ be fibered manifolds. Define the concept of a non-holonomic (r,s,q)-jet and the projection $\pi^{r,s,q}_{Y,Z}\colon \tilde{J}^t(Y,Z) \to \tilde{J}^{r,s,q}(Y,Z)$ as follows. Two fibered non-holonomic t-jets $X_1,\ X_2 \in \tilde{J}^t(Y,Z)$ are said to determine the same non-holonomic (r,s,q)-jet, i.e. $\pi^{r,s,q}_{Y,Z}(X_1) = \pi^{r,s,q}_{Y,Z}(X_2)$ if the following conditions are satisfied

(i) $\pi^t_r(X_1)$ and $\pi^t_r(X_2)$ determine the same fibered non-holonomic r jet.

(ii) $\pi^{r,s}_{Y,Z} \circ \pi^t_s(X_1)$ and $\pi^{r,s}_{Y,Z} \circ \pi^t_s(X_2)$ determine the same non-holonomic (r,s)-jet.

(iii) $p^q_{Y,Z} \circ \pi^t_q(X_1)$ and $p^q_{Y,Z} \circ \pi^t_q(X_2)$ determine the same non-holonomic q-jet over manifolds M, N.

The composition $\circ_{r,s,q}$ of non-holonomic (r,s,q)-jets $\xi_1 \in \tilde{J}^{r,s,q}(Y,Z)$ and $\xi_2 \in \tilde{J}^{r,s,q}(Z,W)$ is defined as the equivalence class corresponding to the composition of any representatives X_1 and X_2, (i.e. elements satisfying $\pi^{r,s,q}_{Y,Z}(X_1) = \xi_1$ and $\pi^{r,s,q}_{Z,W}(X_1) = \xi_2$). In formulas, it reads

$$\xi_2 \circ_{r,s,q} \xi_1 = \pi^{r,s,q}_{Y,W}(X_2 \circ_t X_1) = \pi^{r,s,q}_{Z,W}(X_2) \circ_{r,s,q} \pi^{r,s,q}_{Y,Z}(X_1) \tag{2.10}$$

It follows from the definition of $\circ_{r,s}$ and (2.3) that the composition $\circ_{r,s,q}$ is well defined. The following assertion shall be used in Section 3 showing that all of the so called generalized r,s-jets or (r,s,q)-jets are naturally equivalent.

Proposition 4. *There is a canonical bijection $s_{Y,Z}\colon J^{1,V} \circ J^1(Y,Z) \to J^1 \circ J^{1,V}(Y,Z)$ defined by*

$$j^{1,V}_y \sigma \mapsto j^1_y(u \mapsto \pi^{1,V}_{Y,Z}(\sigma(u))) \tag{2.11}$$

for any local fibered section $\sigma\colon Y \to J^1(Y,Z)$. Moreover, $s_{Y,Z}$ preserves compositions, i.e.

$$s_{Y,W}(j^1_z \varrho \circ_{1V,1} j^{1,V}_y \sigma) = s_{Z,W}(j^{1,V}_z \varrho) \circ_{1,1V} s_{Y,Z}(j^{1,V}_y \sigma) \tag{2.12}$$

for the jet composition $\circ_{1V,1}$ *on* $J^{1,V} \circ J^1$ *and* $\circ_{1,1V}$ *on* $J^1 \circ J^{1,V}$ *and a fibered local section* $\varrho\colon Z \to J^1(Z,W)$. *Further,* $s_{Y,Z}$ *satisfies*

$$s_{Y,Z} \circ \pi^{1,V}_{Y,J^1(Y,Z)} = T\pi^{1,V}_{Y,Z} \tag{2.13}$$

Proof. We use coordinates. Let x^i be base and y^p the fiber coordinates of the fibered manifold Y over M. Similarly, let u^ι be the base and Z^a the fiber coordinates of the fibered manifold Z over N. Consider the additional coordinates u^ι_i, Z^a_i, Z^a_p of the fibered 1-jets from $J^1(Y,Z)$ and further, the additional coordinates Z^a_{0p}, Z^a_{ip}, Z^a_{pq} on $J^{1,V} \circ J^1_{Y,Z}$. Thus on $J^{1,V}(Y,Z)$ we have the induced coordinates x^i,y^p,u^ι , Z^a. Define the additional coordinates u^ι_{0i}, Z^a_{0i}, Z^a_{0p}, Z^a_{pi} , Z^a_{pq} on $J^1 \circ J^{1,V}(Y,Z)$. Then $s_{Y,Z}$ satisfies in coordinates $u^\iota_{0i} = u^\iota_i$, $Z^a_{0i} = Z^a_i$ and $Z^a_{pi} = Z^a_{ip}$. This proves that s is a bijection. From the formulas for the composition of fibered non-holonomic 2-jets, using the fact that they are fibered and the definition of $\circ_{1,1V}$ and $\circ_{1V,1}$, one can immediately verify that s preserves compositions. Using coordinates one can also directly prove (2.13). □

Remark. Let us consider the space $\hat{J}^s$ of the so called generalized non-holonomic fibered s-jets with their composition defined in (2.7). Further, denote by $\hat{J}^{r,s}$ the space of generalized fibered non-holonomic s-jets obtained by r iterations of J^1 and $s-r$ iterations of $J^{1,V}$. Further, denote by $\hat{J}^{s;r_0}$ or $\hat{J}^{r,s;r_0}$ such a space $\hat{J}^s$ or $\hat{J}^{r,s}$ that is obtained by making first r_0 iterations of J^1. Then we can define $\hat{\pi}^{r,s}_{Y,Z}\colon\ \tilde{J}^s(Y,Z) \to \hat{J}^{r,s}(Y,Z)$ modifying the construction of $\pi^{r,s}_{Y,Z}$ before Proposition 3 as follows. We construct $\hat{\pi}^{s;r_0}_{Y,Z}\colon\ \tilde{J}^s_{Y,Z} \to \hat{J}^{s;r_0}$ by induction in respect to $s-r_0$ analogously but modifying (2.8) to

$$j^1_y\sigma \mapsto j^1_y(\hat{\pi}^{r,s}_{Y,Z} \circ \sigma) \tag{2.14}$$

in case the current iteration is J^1. We can formulate an assertion which is the modification of Proposition 3 to the generalized projection $\hat{\pi}^{r,s}$ and the generalized composition (2.7). Its proof is only a modification of that of Proposition 3 and so we omit it.

Proposition 5. *Let* $X_1 \in \hat{J}^s(Y,Z)$ *and* $X_2 \in \hat{J}^s(Z,W)$ *be generalized fibered non-holonomic s-jets of the same type satisfying* $\beta(X_1) = \alpha(X_2)$. *Then* $\hat{\pi}^{r,s}$ *satisfies*

$$\hat{\pi}^{r,s}_{Y,W}(X_2 \circ_s X_1) = \hat{\pi}^{r,s}_{Z,W}(X_2)\hat{\circ}_{r,s}\hat{\pi}^{r,s}_{Y,Z}(X_1) \tag{2.15}$$

for the composition $\hat{\circ}_{r,s}$ *defined in* (2.7).

In the very end we remark that that the bijection $s_{Y,Z}$ from Proposition 4 yields the identification of all $\hat{J}^{r,s}$ no matter what is the order of J^1 and $J^{1,V}$.

3. Bundle functors of non-holonomic (r,s,q)-jets

In the present section, we are going to introduce a bundle functor $\tilde{J}^{r,s,q}\colon \mathcal{FM}_{k,l}\times\mathcal{FM}\to\mathcal{FM}$ and a bundle functor functor $\tilde{J}_1^{r,s,q}\colon 2\text{-}\mathcal{FM}_{k,l}\to\mathcal{FM}$. In this way we generalize the functors J^r and $\tilde{J}^r$ from $\mathcal{M}f_m\times\mathcal{M}f$ and their modifications J_1^r and $\tilde{J}_1^r$ to $\mathcal{FM}_m$. Further, we generalize the bundle functor $J^{r,s,q}$ on $\mathcal{FM}_m\times\mathcal{FM}$ defined and studied by Doupovec and Kolář in [1]. We also describe $\tilde{J}^{r,s,q}$ in the form of $P^{r,s,q}[T^\mu, H^{\tilde{J}^{r,s,q}}]$ according to Section 1.

Let us recall that for a local diffeomorphism $g\colon M\to\overline{M}$, a smooth map $h\colon N\to\overline{N}$ and a non-holonomic r-jet $X\in\tilde{J}^r(M,N)$, the element $\tilde{J}^r(g,h)(X)$ is defined as $j^r_{\beta(X)}h\circ_r X\circ_r j^r_{g(\alpha(X))}g^{-1}$. In the last formula, holonomic r-jets $j^r_{\beta(X)}h$ and $j^r_{g(\alpha(X))}g^{-1}$ are considered as non-holonomic ones applying the canonical inclusion j of holonomic into non-holonomic r-jets defined by the formula

$$(3.1)\quad j^r_x f\mapsto j^1_x(u_{r-1}\mapsto j^1_{u_{r-1}}(u_{r-2}\mapsto j^1_{u_{r-2}}(\dots(u_2\mapsto j^1_{u_2}(u_1\mapsto\tilde{f}(u_1,\dots,u_{r-1})))\dots))),$$

where $\tilde{f}$ is the constant map assingning every element of the form $(u_1,\dots,u_{r-1})$ constantly the Taylor polynomial $T^r_x f$ of order r in x.

For a local $\mathcal{FM}_{k,l}$-isomorphism $g\colon Y_1\to Y_2$ and an $\mathcal{FM}$-morphism $Z_1\to Z_2$ define $\tilde{J}^{r,s,q}(g,f)\colon \tilde{J}^{r,s,q}(Y_1,Z_1)\to\tilde{J}^{r,s,q}(Y_2,Z_2)$ by

$$\tilde{J}^{r,s,q}(g,f)(X)=j^{r,s,q}_{\beta(X)}f\circ_{r,s,q}X\circ_{r,s,q}j^{r,s,q}_{g(\alpha(X))}g^{-1}\tag{3.2}$$

for any $X\in\tilde{J}^{r,s,q}(Y_1,Z_1)$. The composition of non-holonomic (r,s,q)-jets $\circ_{r,s,q}$ with the holonomic ones $j^{r,s,q}_{\beta(X)}f$ and $j^{r,s,q}_{g(\alpha(X))}g^{-1}$ is considered as the composition of non-holonomic (r,s,q)-jets applying the embedding $\pi^{r,s,q}\circ i$ with the canonical inclusion i of $\max\{s,q\}$-holonomic into $\max\{s,q\}$-non-holonomic jets. The correctness of (3.2) is a consequence of (2.10).

It follows from the construction of $\tilde{J}^{r,s,q}$ that it is a bundle functor on $\mathcal{FM}_{k,l}\times\mathcal{FM}$ of order (r,s,q) in the first factor and that it preserves products in the second factor. Analogously one deduces the same for the bundle functor $J^{1,V}$ defined on $\mathcal{FM}k,l\times\mathcal{FM}$. It is easy to see that $\pi^{r,s}_{Y,Z}\colon \tilde{J}^s(Y,Z)\to\tilde{J}^{r,s}(Y,Z)$, $\pi^{r,s,q}_{Y,Z}\colon \tilde{J}^t(Y,Z)\to\tilde{J}^{r,s,q}(Y,Z)$ and $\pi^{s_1}_{s_2,Y,Z}\colon \tilde{J}^{s_1}(Y,Z)\to\tilde{J}^{s_2}(Y,Z)$ form natural transformations.

In the following proposition, we find step by step $G^{\tilde{J}^{r,s,q}}$ in the form of T^μ, ([8]) and $H^{\tilde{J}^{r,s,q}}$ applying essentially Proposition 1 and Proposition 2. Further, we find the couples of Weil algebra homomorphisms associated to the recently disscussed natural transformations π^r_s and $\pi^{r,s,q}$, ([8], [14]). Let us put $\tilde{T}^s_{k,l}=\tilde{J}^s_{0,0}(\mathbb{R}^{k,l},\cdot)$ and analogously do with $\tilde{T}^{r,s}_{k,l}$ and $\tilde{T}^{r,s,q}_{k,l}$.

Proposition 6. (i) *The bundle functor $\tilde{J}^s$ of fibered non-holonomic jets is naturally equivalent to $(G^{\tilde{J}^s}, H^{\tilde{J}^s})$ where $G^{\tilde{J}^s}$ is of the form T^ν for a Weil algebra homomorphism ν defined by*

$$\nu = \underbrace{\iota \otimes \ldots \otimes \iota}_{s\text{-times}} : \widetilde{\mathbb{D}}^s_k = \mathbb{D}^1_k \otimes \ldots \otimes \mathbb{D}^1_k \to \mathbb{D}^1_{k+l} \otimes \ldots \otimes \mathbb{D}^1_{k+l} = \widetilde{\mathbb{D}}^s_{k+l}. \tag{3.3}$$

The map $\iota\colon \mathbb{D}^1_k \to \mathbb{D}^1_{k+l}$ is the canonical injection homomorphism. Further, $H^{\tilde{J}^s}\colon G^{s,s,s}_{k,l} \to \mathrm{Aut}(T^\nu)$ is given by the formula (3.2) if we substitute the identity map for f. Moreover, H can be decomposed into the couple of homomorphisms $H_1^{\tilde{J}^s}\colon G^s_k \to \mathrm{Aut}(\widetilde{\mathbb{D}}^s_k)$ and $H_2^{\tilde{J}^s}\colon G^{s,s}_{k,l} \to \mathrm{Aut}(\widetilde{\mathbb{D}}^s_{k,l})$, both of $H_1^{\tilde{J}^s}$ and $H_2^{\tilde{J}^s}$ being the inclusion j from (3.1).

(ii) *The bundle functor $\tilde{J}^{r,s}$ of non-holonomic (r,s)-jets is naturally equivalent to $(T^\mu, H^{\tilde{J}^{r,s}})$ where μ is a Weil algebra homomorphism defined by*

$$\mu = \underbrace{\iota \otimes \ldots \otimes \iota}_{r\text{-times}} \otimes \underbrace{i \otimes \ldots \otimes i}_{(r-s)\text{-times}} : \widetilde{\mathbb{D}}^r_k \to \widetilde{\mathbb{D}}^r_{k+l} \otimes \widetilde{\mathbb{D}}^{s-r}_l \tag{3.4}$$

for $\widetilde{\mathbb{D}}^1_k = \mathbb{D}^1_k \otimes \ldots \otimes \mathbb{D}^1_k \otimes \mathbb{R} \otimes \ldots \otimes \mathbb{R}$ and the canonical inclusion $i\colon \mathbb{R} \to \mathbb{D}^1_l$. The homomorphism $H^{\tilde{J}^{r,s}}\colon G^{r,s,r}_{k,l} \to \mathrm{Aut}(T^\mu)$ is given by the formula (3.2) substituting the identity map for f.

(iii) *The natural transformation $\pi^{r,s}$ is represented by a couple of (ν,μ)-related Weil algebra homomorphisms (ϱ,σ), i.e. commutative diagrams of the form*

$$\begin{array}{ccc} \widetilde{\mathbb{D}}^s_k & \xrightarrow{\ \nu\ } & \widetilde{\mathbb{D}}^s_{k+l} \\ \downarrow{\scriptstyle\varrho} & & \downarrow{\scriptstyle\sigma} \\ \widetilde{\mathbb{D}}^r_k & \xrightarrow{\ \mu\ } & \widetilde{\mathbb{D}}^r_{k+l} \otimes \widetilde{\mathbb{D}}^{s-r}_l \end{array} \tag{3.5}$$

where $\varrho = \pi^s_r = \underbrace{\mathrm{id} \otimes \ldots \mathrm{id}}_{r\text{-times}} \otimes \underbrace{p_0 \otimes \ldots \otimes p_0}_{(s-r)\text{-times}}$ and $\sigma = \underbrace{\mathrm{id} \otimes \ldots \mathrm{id}}_{r\text{-times}} \otimes \underbrace{p_{k,l} \otimes \ldots p_{k,l}}_{(s-r)\text{-times}}$. The projection p_0 is the Weil algebra homomorphism $\mathbb{D}^1_k \to \mathbb{R}$ annihilating nilpotent parts and $p_{k,l}\colon \mathbb{D}^1_{k+l} \to \mathbb{D}^1_l$ annihilates first k variables. Moreover, H can be decomposed into the couple of group homomorphisms $H^1\colon G^r_k \to \mathrm{Aut}(\widetilde{\mathbb{D}}^r_k)$ and $H^2\colon G^{r,s}_{k,l} \to \mathrm{Aut}(\widetilde{\mathbb{D}}^{r,s}_{k,l})$ where H^1 is the inclusion j from (3.1) and H^2 is defined by $H^2(\pi^{r,s}_{\mathbb{R}^{k,l},(\mathbb{R}\to\mathrm{pt})}(j^s_{(0,0)}\varphi)) = \pi^{r,s}_{\mathbb{R}^{k,l},(\mathbb{R}\to\mathrm{pt})}(j(j^s_{(0,0)}\varphi))$.

Proof. (i) It is easy to verify that $J^1_{0,0}((\mathbb{R}^{k,l} \to \mathbb{R}^k), (\mathbb{R} \to \mathbb{R})) = \mathbb{D}^1_k$ and $J^1_{0,0}((\mathbb{R}^{k,l} \to \mathbb{R}^k), (\mathbb{R} \to \mathrm{pt})) = \mathbb{D}^1_{k+l}$. Further, one can immediately deduce that $J^1_{0,0}(\mathrm{id}_{\mathbb{R}^{k,l}}, \tau) = \tilde{T}^1_{k,l}(\tau) = \iota$, where $\tau\colon (\mathbb{R} \to \mathbb{R}) \to (\mathbb{R} \to \mathrm{pt})$ is the identity map, [8].

By the iteration property of functors T^{μ}, i.e. $T^{\mu_1} \circ T^{\mu_2} = T^{\mu_1 \otimes \mu_2}$ presented in [14] and [11] we obtain the first assertion. The second assertion is directly obtained by the formula (3.2), Remark following Definition 2.1 and Proposition 3.

(ii) It is easy to see that $J^{1,V}_{0,0}((\mathbb{R}^{k,l} \to \mathbb{R}^k), (\mathbb{R} \to \mathbb{R})) = \mathbb{R}$ and $J^{1,V}_{0,0}((\mathbb{R}^{k,l} \to \mathbb{R}^k), (\mathbb{R} \to \mathrm{pt})) = \widetilde{\mathbb{D}}^1_l$. One can easily verify that $J^{1,V}_{0,0}(\mathrm{id}_{\mathbb{R}^{k,l}}, \tau)$ is the canonical injection of $\mathbb{R}$ into $\mathbb{D}^1_l$. The iteration property of functors T^{μ} is deduced analogously as in (i) of the first assertion. The second assertion follows immediately from Proposition 3 and the formula (3.2).

(iii) We procced by induction following exactly the formula (2.8). The first assertion is obtained if we consider the fact that every natural transformation $T^{\nu} \to T^{\mu}$ corresponds to the couple of (ν, μ)-related Weil algebra homomorphisms, [8] and [14].

The second assertion is obtained from the formula (3.2), (2.10) and the composition of non-holonomic fibered jets taking into account that every element of $G^{r,s}_{k,l}$ can be considered in the form $(j^r_0 g_1, j^{r,s}_{0,0} g_2) = \pi^{r,s}_{\mathbb{R}^{k,l}, \mathbb{R}^{k,l}}(j^s_0 g_1, j^s_{0,0} g_2)$. □

The following Proposition gives the main result

Proposition 7. (i) *$\tilde{J}^{r,s,q}$ is a bundle functor on $\mathscr{FM}_{k,l} \times \mathscr{FM}$ identified with $P^{r,s,q}[\tilde{T}^{r,s,q}_{k,l}, H^{\tilde{J}^{r,s,q}_{k,l}}]$, where $\tilde{T}^{r,s,q}_{k,l}$ and $H^{\tilde{J}^{r,s,q}_{k,l}}$ are as follows.*

The bundle functor $\tilde{T}^{r,s,q}_{k,l} \colon \mathscr{FM}_{k,l} \to \mathscr{FM}$ is of the form T^{ϑ} for a Weil algebra homomorphism $\vartheta \colon \widetilde{\mathbb{D}}^q_k \to \widetilde{\mathbb{D}}^{r,s}_{k,l}$ defined by

$$\vartheta = (\underbrace{\iota \otimes \ldots \otimes \iota}_{r\text{-times}} \otimes \underbrace{i \otimes \ldots \otimes i}_{(s-r)\text{-times}}) \circ \pi^q_r \tag{3.6}$$

where $\iota \colon \mathbb{D}^1_k \to \mathbb{D}^1_{k+l}$ is the canonical inclusion, $i \colon \mathbb{R} \to \mathbb{D}^1_l$ is the canonical inclusion of reals and $p_0 \colon \mathbb{D}^1_k \to \mathbb{R}$ is the projection annihilating the nilpotent parts.

Further, $H^{\tilde{J}^{r,s,q}_{k,l}}$ is defined by $H^{\tilde{J}^{r,s,q}_{k,l}}(j^{r,s,q}_{0,0} g)(X) = X \circ_{r,s,q} j^{r,s,q}_{0,0} g^{-1}$. Moreover, $H^{\tilde{J}^{r,s,q}_{k,l}}$ can be considered as a couple of homomorphisms $H_1 \colon G^q_k \to \mathrm{Aut}\,(\widetilde{\mathbb{D}}^q_k)$ and $H_2 \colon G^{r,s}_{k,l} \to \mathrm{Aut}(\widetilde{\mathbb{D}}^{r,s}_{k,l})$ defined as follows. $H_1(j^q_0 g_1)(j^{\widetilde{\mathbb{D}}^q_k} \varphi) = j^{\widetilde{\mathbb{D}}^q_k}(\varphi \circ g^{-1})$ and $H_2(j^{r,s}_{0,0} g_2)(j^{\widetilde{\mathbb{D}}^{r,s}_{k,l}} \psi) = j^{\widetilde{\mathbb{D}}^{r,s}_{k,l}}(\psi \circ g_2^{-1})$, i.e. H^1 and H^2 are as in Proposition 6, part (iii).

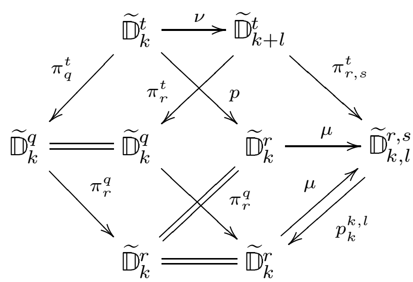
(ii) Let $t = \max\{s, q\}$. Then the natural transformation $\pi^{r,s,q}$ corresponds to a couple of (ν, ϑ)-related Weil algebra homomorphisms (ϱ, σ), where $\varrho = \pi^t_q \colon \widetilde{\mathbb{D}}^t_k \to \widetilde{\mathbb{D}}^q_k$ is the canonical projection and $\sigma \colon \widetilde{\mathbb{D}}^t_{k,l} \to \widetilde{\mathbb{D}}^{r,s}_{k,l}$ is the composition $\pi^{r,s} \circ \pi^t_s$.

It is easy to see that a bundle functor T^{id_A} can be identified with the Weil functor T^A. Further, $\tilde{T}^{r,s,q}_{k,l}$ is identified with the couple of bundle functors $\tilde{T}^q_k \colon \mathscr{M}f \to \mathscr{FM}$ and the functor T^{μ} from Proposition 6., (ii). Thus $\tilde{T}^{r,s,q}_{k,l}$ is identified with the couple of Weil algebra homomorphisms $\mathrm{id}_{\widetilde{\mathbb{D}}_k^q}$ and $\mu\colon \widetilde{\mathbb{D}}_k^r \to \widetilde{\mathbb{D}}_{k,l}^{r,s}$ which is due to the form of μ obviously identified with ϑ. This proves the first claim.

As for the second assertion, it is easy to see that for two fibered t jets $j_{0,0}^t g$, $j_{0,0}^t h$ determining the same (r,s) jet and for two fibered non-holonomic t-jets X_1, X_2 determining the same fibered non-holonomic (r,s)-jet the compositions $X_1 \circ (j_{0,0}^t g)^{-1}$ and $X_2 \circ (j_{0,0}^t h)^{-1}$ determine the same non-holonomic (r,s)-jet by the second assertion of Proposition 6, (iii). This yields the required reducibility with respect to the (r,s)-components of the non-holonomic (r,s,q) jet. This reducibility with respect to the first components follows from the formula projection property of the projection p^t from Definition 1.

(ii) Let $p_k^{k,l}$ and p denote projection homomorphisms annihilating the last l variables. Then the assertion follows from the following diagram

(3.7)

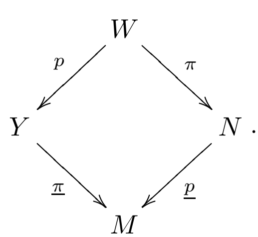


the commutativity of which is easy to verify. □

Remark. Obviously, the $s_{Y,Z}$ from Proposition 4 is a natural equivalence $J^{1,V} \circ J^1 \to J^1 \circ J^{1,V}$ and, consquentely, all of the so called generalized fibered non-holonomic (r,s)-jets and (r,s,q)-jets are mutualy identified in the sense of a natural equivalence. The natural equivalence $s\colon J^{1,V} \circ J^1 \to J^1 \circ J^{1,V}$ corresponds to the couple of Weil algebra exchange isomorphisms $\varrho = \mathrm{id}_{\mathbb{D}^{1,k}} : \mathbb{D}_k^1 \simeq \mathbb{D}_k^1 \otimes \mathbb{R} \to \mathbb{R} \otimes \mathbb{D}_k^1 \simeq \mathbb{D}_k^1$ and $\sigma\colon \mathbb{D}_{k+l}^1 \otimes \mathbb{D}_l^1 \to \mathbb{D}_l^1 \otimes \mathbb{D}_{k+l}^1$.

In what follows, we introduce a bundle functor $\tilde{J}_1^{r,s,q}\colon 2\text{-}\mathscr{FM}_{k,l} \to \mathscr{FM}$ where $2\text{-}\mathscr{FM}_{k,l}$ denotes the category of 2-fibered manifolds with $\mathscr{FM}_{k,l}$ base objects. We recall that the objects of the category $2\text{-}\mathscr{FM}$, ([9], [10]) are fibered surjective submersions between fibered manifolds, i.e. commutative squares as follows

(3.8)



All arrows in (3.8) are surjective submersions. 2-$\mathscr{FM}$-morphisms are quadruples $f\colon W \to W'$, $\underline{f}\colon Y \to Y'$, $f_0\colon N \to N'$ and $\underline{f}_0\colon M \to M'$ such that $\pi' \circ f = f_0 \circ \pi$, $p' \circ f = \underline{f} \circ p$, $\underline{\pi}' \circ \underline{f} = \underline{f}_0 \circ \underline{\pi}$ and $\underline{p}' \circ f_0 = \underline{f}_0 \circ \bar{p}$ if we have considered another 2-$\mathscr{FM}$-object W' of the form (3.8). If in addition to that $\underline{\pi}\colon Y \to M$ are $\mathscr{FM}_{k,l}$-objects and $\underline{f}\colon Y \to Y'$ are $\mathscr{FM}_{k,l}$-local isomorphisms, we obtain the category 2-$\mathscr{FM}_{k,l}$. For an 2-$\mathscr{FM}_{k,l}$-object W define

$$\tilde{J}_1^{r,s,q}(W) := \{X \in \tilde{J}^{r,s,q}((Y \to M),(W \to N));\, \tilde{J}^{r,s,q}(\underset{y}{\mathrm{id}}, p)(X) = j^{r,s,q}_{\alpha(X)} \underset{Y}{\mathrm{id}}\} \tag{3.9}$$

and for an 2-$\mathscr{FM}_{k,l}$-morphism $f\colon W \to W'$ define $\tilde{J}^{r,s,q}(f) := j^{r,s,q}(f, f_0) \circ_{r,s,q} X \circ_{r,s,q} (j^{r,s,q}(\underline{f}, \underline{f}_0))^{-1}$. This defines a bundle functor on the category 2-$\mathscr{FM}_{k,l}$ since $\tilde{J}^{r,s,q}(\mathrm{id}_Y, p)(j^{r,s,q}(f, f_0) \circ_{r,s,q} X \circ_{r,s,q} (j^{r,s,q}(\underline{f}, \underline{f}_0))^{-1}) = j^{r,s,q}_{\alpha(X)} \mathrm{id}_Y$. One can easily verify that $\tilde{J}^{r,s,q}$ preserves fiber products $W_1 \times_Y W_2$. Thus we have obtained the following Proposition

Proposition 8. *The $\tilde{J}_1^{r,s,q}$ defined by (3.9) is a bundle functor on the category 2-$\mathscr{FM}_{k,l}$ preserving fibered products.*

In the very end we prove that the projection $\pi^{r,s}$ defined by induction in (2.8) can be obtained as the composition of $\tilde{J}^p(\mathrm{id}, \pi^{1,V})$ and $\tilde{J}^p(\mathrm{id}, s)$ only. It reads as follows

Proposition 9. *Let $Y \in \mathrm{Obj}(\mathscr{FM}_{k,l})$, $Z \in \mathscr{FM}$, $s \geqslant r$. Then it holds*

$$\begin{aligned}\pi^{r,s+1}_{Y,Z} = \pi^{1,V}_{Y,\tilde{J}^{r,s}(Y,Z)} \circ \tilde{J}^1(\mathrm{id}_Y, \pi^{1,V}_{Y,\tilde{J}^{r,s-1}(Y,Z)}) \circ \tilde{J}^2(\mathrm{id}_Y, \pi^{1,V}_{Y,\tilde{J}^{r,s-2}(Y,Z)}) \circ \dots \\ \circ \tilde{J}^{s-r}(\mathrm{id}_Y, \pi^{1,V}_{Y,\tilde{J}^{r}(Y,Z)})\end{aligned} \tag{3.10}$$

Moreover,

$$\begin{aligned}\tilde{J}^l \pi^{1,V}_{Y,\tilde{J}^{r,s+1-l}(Y,Z)} &= \tilde{J}^{l-1}(\underset{Y}{\mathrm{id}}, s_{Y,\tilde{J}^{r,s+1-l}(Y,Z)}) \\ &\quad \circ \dots \circ \tilde{J}^{l-i}(\underset{Y}{\mathrm{id}}, s_{Y,J^{i-1}_Y \circ \tilde{J}^{r,s+1-l}(Y,Z)}) \circ \dots \\ &\quad \circ \tilde{J}^1(\underset{Y}{\mathrm{id}}, s_{Y,J^{l-2}_Y \circ \tilde{J}^{r,s+1-l}(Y,Z)}) \\ &\quad \circ s_{Y,J^{l-1}_Y \circ \tilde{J}^{r,s+1-l}(Y,Z)} \circ \pi^{1,V}_{Y,J^{l}_Y \circ \tilde{J}^{r,s+1-l}(Y,Z)}\end{aligned}$$

Proof. The proof is done by induction with respect to $s - r$. For $s = r$, the right-hand side of (3.10) is $\pi^{1,V}_{Y,\tilde{J}^r(Y,Z)}$ which is obviously $\pi^{r,r+1}_{Y,Z}$. The induction step will be proved if we show that $T\tilde{J}^r(\mathrm{id}_Y, f) = \tilde{J}^{r+1}(\mathrm{id}_Y, f)$ since in this case the rigt-hand side of (3.10) is exactly (2.8). The verification is given by $T\tilde{J}^r(\mathrm{id}_Y, f)(j^1_y\sigma) = j^1_y(\tilde{J}^r(\mathrm{id}_Y, f)(\sigma(u)) = j^1_y(j^r_y f \circ_r \sigma(u)) = j^{r+1}_y f \circ_{r+1} j^1_y(\sigma(u))$ since $j^r f$ is in fact a constant local section on $\tilde{J}^r$. The second assertion is verified applying $(l-1)$-times (2.13). □

Remark. In the very end we remark that the iteration of non-holonomic (r, s, q)-jets is given by the formula

$$\tilde{J}^{r_1,s_1,q_1} \circ \tilde{J}^{r_2,s_2,q_2} = \tilde{J}^{r_1+r_2,s_1+s_2,q_1+q_2} \tag{3.11}$$

for any $s_1 \geqslant r_1 \leqslant q_1$ and $s_2 \geqslant r_2 \leqslant q_2$. It can be directly obtained from the formula for the iteration of Weil bundles, i.e. $T^A \circ T^B = T^{A\otimes B}$ for any Weil algebras A and B.

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