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PASTING TOPOLOGICAL SPACES AT ONE POINT

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Abstract. Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be a family of topological spaces and $x_\alpha \in X_\alpha$, for every $\alpha \in \Lambda$. Suppose X is the quotient space of the disjoint union of X_α's by identifying x_α's as one point σ. We try to characterize ideals of $C(X)$ according to the same ideals of $C(X_\alpha)$'s. In addition we generalize the concept of rank of a point, see [9], and then answer the following two algebraic questions. Let m be an infinite cardinal.

(1) Is there any ring R and I an ideal in R such that I is an irreducible intersection of m prime ideals?

(2) Is there any set of prime ideals of cardinality m in a ring R such that the intersection of these prime ideals can not be obtained as an intersection of fewer than m prime ideals in R?

Finally, we answer an open question in [11].

Keywords: pasting topological spaces at one point, rings of continuous (bounded) real functions on X, z-ideal, z°-ideal, C-embedded, P-space, F-space.

MSC 2000: 54C40, 54C45, 54G05, 54G10

1. Introduction

In this paper, by R we mean a commutative ring with unity, and every ideal is proper. The ring of continuous (bounded) real functions on X is denoted by $C(X)$ ($C^*(X)$). Suppose $f \in C(X)$, we denote $f^{-1}\{0\}$ by $Z(f)$ and the complement of $Z(f)$ by $\operatorname{co}Z(f)$. An ideal I in $C(X)$ is a *z-ideal* (*z°-ideal*) if whenever $Z(f) \subseteq Z(g)$ ($\operatorname{int}_X Z(f) \subseteq \operatorname{int}_X Z(g)$) and $f \in I$, then $g \in I$ (for more information about z°-ideals see [2] and [7]). Also an ideal I in $C(X)$ is *convex* (*absolutely convex*) if whenever $0 \leqslant g \leqslant f$ ($|g| \leqslant |f|$) and $f \in I$, then $g \in I$. Let $p \in \beta X$, we say p is a *P-point* (*F-point*) with respect to X if $O^p(X) = \{f \in C(X)\colon p \in \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} Z(f)\}$ is a maximal ideal (prime ideal) in $C(X)$. In addition, p is an *almost P-point* if for every $f \in C(X)$, whenever $p \in \operatorname{cl}_{\beta X} Z(f)$, then $\operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} Z(f) \neq \emptyset$. Clearly $p \in \beta X$ is an *almost P-point* if and only if $M^p(X)$ is a z°-ideal, see [1]. For undefined terms and notions, see [7].

Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be a family of topological spaces and $x_\alpha \in X_\alpha$, for every $\alpha \in \Lambda$. Suppose X is the quotient space of the disjoint union of the X_α's by identifying x_α's as one point σ. We denote X by $\mathcal{J}(X_\alpha, x_\alpha)_{\alpha\in\Lambda}$. We can easily see that for any $x \neq \sigma$, the neighborhood base of x is the same in X_α (which contains x) and open sets containing σ are of the form $\bigcup_{\alpha\in\Lambda} U_\alpha$, where U_α is an open set in X_α containing σ. So, by renaming x_α to σ, we can consider X_α as a closed subspace of $X = \mathcal{J}(X_\alpha, x_\alpha)_{\alpha\in\Lambda}$.

The "pasting of topological spaces at one point" is very useful when regarding $C(X)$ and owing to the simplicity of structure allows us to easily construct many examples for different purposes. One of these applications can be found in [3] and some others in this paper.

Our aim is to characterize the ideals of $C(X)$, by using the same ideal structures of $C(X_\alpha)$'s. In $C(X)$ the ideals, in particular prime ideals, containing $O^p(X)$, where $p \in \beta(X)$, are important and hence we focus on these ideals. To do this, we divide these ideals into three classes. The first class is the set of ideals over $O^p(X)$, where $p \in \bigcup_{\alpha\in\Lambda} \operatorname{cl}_{\beta X} X_\alpha \setminus \{\sigma\}$, and it will be characterized in Section 2. The second class is the set of ideals over $O_\sigma(X)$. In Section 3 we first determine a new class of ideals and then we deal with the second class of ideals. The third class is the set of ideals over $O^p(X)$, where $p \in \beta X \setminus \bigcup_{\alpha \operatorname{int} \Lambda} \operatorname{cl}_{\beta X} X_\alpha$. Dealing with this class is not easy and needs more future work. Finally in Section 4 we will give some applications and answer two questions that we already mentioned. We also apply this method to answer the open question that appeared in [11] following Lemma 3.3.

Remark 1.1. For every $B \subseteq \Lambda$ and every $f \in C\Big(\bigcup_{\alpha\in B} X_\alpha\Big)$ there is a continuous extension $\overline{f}$ to $X = \mathcal{J}(X_\alpha, x_\alpha)_{\alpha\in\Lambda}$ defined as follows

$$\overline{f}(x) = f(x) \text{ where } x \in \bigcup_{\alpha\in B} X_\alpha \quad \text{and} \quad \overline{f}(x) = f(\sigma) \text{ where } x \in X \setminus \bigcup_{\alpha\in B} X_\alpha.$$

Henceforth, we use $\overline{f}$ with this meaning. Also, if $B \subseteq \Lambda$, $f_\alpha \in C(X_\alpha)$ and $f_\alpha(\sigma) = r$, for every $\alpha \in B$, then the function $f\colon \bigcup_{\alpha\in B} X_\alpha \to \mathbb{R}$ with $f(x) = f_\alpha(x)$, where $x \in X_\alpha$, is continuous and we denote it by $\bigcup_{\alpha\in B} f_\alpha$.

Considering what we have already explained about the quotient space $X = \mathcal{J}(X_\alpha, x_\alpha)_{\alpha\in\Lambda}$, without loss of generality, we can suppose that

$$X = \bigcup_{\alpha\in\Lambda} X_\alpha, \qquad \bigcap_{\alpha\in\Lambda} X_\alpha = \{\sigma\}$$

and every $X_\alpha \setminus \{\sigma\}$ is open. Hence, by hypothesis every X_α is a C-embedded closed subset of X for every $\alpha \in \Lambda$ and clearly the map

$$\varphi_\alpha\colon\ C(X) \to C(X_\alpha), \quad \varphi_\alpha(f) = f|_{X_\alpha}$$

is an onto homomorphism for every $\alpha \in \Lambda$. Also we can easily see that

$$O_\sigma(X) = \bigcap_{\alpha\in\Lambda} \varphi_\alpha^{-1}\left(O_\sigma(X_\alpha)\right) \tag{1}$$

and one can show that X is T_3 $(T_{3\frac{1}{2}}, T_4)$ if and only if X_α is such, for every $\alpha \in \Lambda$.

From now on, we assume that X_α's are $T_{3\frac{1}{2}}$ and by X we mean $X = \mathcal{J}(X_\alpha, x_\alpha)_{\alpha\in\Lambda}$ unless we emphasis otherwise. First we consider the connections between βX and the βX_α's.

Proposition 1.2.

(a) $\bigcup\limits_{\alpha\in\Lambda} \beta X_\alpha \subseteq \beta X$.

(b) *If X_α's are not empty or singleton, then $\bigcup\limits_{\alpha\in\Lambda} \beta X_\alpha$ is compact (equivalently $\bigcup\limits_{\alpha\in\Lambda} \beta X_\alpha = \beta X$) if and only if Λ is finite.*

(c) *$\beta X_{\alpha_1} \cap \beta X_{\alpha_2} = \{\sigma\}$ for every $\alpha_1 \neq \alpha_2$.*

(d) *$\beta X_\alpha \setminus \{\sigma\}$ is an open set in $\beta(X)$.*

Proof. Since the X_α's are C-embedded in $C(X)$, (a) and (b) follow clearly.

(c) Suppose $x \neq \sigma$, then there exists an open set U in X such that $\sigma \in U$ and $x \notin \operatorname{cl}_{\beta X} U$. Define

$$A = (X - U) \cap X_{\alpha_1}, \quad B = (X - U) \cap X_{\alpha_2}.$$

It is easy to see that A and B are completely separated and so $x \notin \operatorname{cl}_{\beta X} A \cap \operatorname{cl}_{\beta X} B$. Hence $x \notin \operatorname{cl}_{\beta X} A$ or $x \notin \operatorname{cl}_{\beta X} B$. For instance if $x \notin \operatorname{cl}_{\beta X} A$, then

$$x \notin \operatorname{cl}_{\beta X} U \cup \operatorname{cl}_{\beta X} A = \operatorname{cl}_{\beta X}(U \cup A) \supseteq \operatorname{cl}_{\beta X} X_{\alpha_1} \Rightarrow x \notin \beta X_{\alpha_1} \Rightarrow x \notin \beta X_{\alpha_1} \cap \beta X_{\alpha_2}.$$

(d) Without loss of generality we can suppose X_α's are compact and then prove that $X_\alpha \setminus \{\sigma\}$ is an open set in $\beta(X)$. Suppose $p \in X_\alpha \setminus \{\sigma\}$, then there exists $f_\alpha \in C^*(X_\alpha)$ such that $p \in \operatorname{co}Z(f_\alpha)$ and $\sigma \in Z(f_\alpha)$. Suppose g is the extension of $\overline{f_\alpha}$ to βX, then obviously $p \in \operatorname{co}Z(g) = \operatorname{co}Z(f_\alpha) \subseteq X_\alpha \setminus \{\sigma\}$. □

2. First class of ideals

In what follows, we consider the class of ideals of $C(X)$ containing $O^p(X)$, where $p \in \bigcup_{\alpha\in\Lambda} \beta X_\alpha \setminus \{\sigma\}$. We first need the following lemma.

Definition 2.1. Let X be a topological space and $p \in \beta X$, an X-neighborhood of p is a subset A of X such that there exists a neighborhood B of p in βX such that $A = B \cap X$.

Lemma 2.2. *Let X be a topological space, $p \in \beta X$ and A be a C-embedded X-neighborhood of p, then there exists a one to one correspondence preserving inclusion between the ideals in $C(X)$ containing $O^p(X)$ and the ideals in $C(A)$ containing $O^p(A)$.*

Proof. Note that, since A is a C-embedded X-neighborhood of p, $p \in \mathrm{cl}_{\beta X} A = \beta A$ and consequently $O^p(A)$ is well-defined. Suppose φ is the restriction homomorphism from $C(X)$ to $C(A)$. Clearly φ is onto. It only remains to prove that $\ker(\varphi) \subseteq O^p(X)$ and this is easily seen. □

Proposition 2.3. *Let $p \neq \sigma$ and $p \in \beta X_\alpha$ for an $\alpha \in \Lambda$, then the map $I \to \varphi_\alpha(I)$ is a one to one correspondence preserving inclusion between the ideals in $C(X)$ containing $O^p(X)$ and the ideals in $C(X_\alpha)$ containing $O^p(X_\alpha)$.*

Proof. By Lemma 2.2, this is clear. □

Clearly the map $I \to \varphi_\alpha(I)$ preserves prime ideals, z-ideals, z°-ideals, convex ideals and absolutely convex ideals.

It is obvious that for every prime ideal P in $C(X_\alpha)$, $\varphi_\alpha^{-1}(P)$ is a prime ideal in $C(X)$ and $\varphi_\alpha^{-1}(P)$ is a maximal ideal whenever P is a maximal ideal in $C(X_\alpha)$. Now a natural question is this, "Is every prime (maximal) ideal Q in $C(X)$ of the form $\varphi_\alpha^{-1}(P)$?" By Proposition 2.3 if the ideal Q is maximal or in the first class of ideals, then it is of the form $\varphi_\alpha^{-1}(P)$. So it is enough to consider nonmaximal prime ideals that are not in the first class of ideals. But, before answering this in general, let us consider some special cases.

Proposition 2.4. *Let Λ be finite and Q be an ideal in $C(X)$. Q is a prime ideal in $C(X)$ if and only if there is a prime ideal P in $C(X_\alpha)$ such that $Q = \varphi_\alpha^{-1}(P)$ for some $\alpha \in \Lambda$.*

Proof. Clearly in this case $\beta X = \bigcup_{\alpha\in\Lambda} \beta X_\alpha$. Therefore, by Proposition 2.3, without loss of generality we can assume that Q is a prime ideal over $O_\sigma(X)$. Hence by (1)

$$\bigcap_{\alpha\in\Lambda}\varphi_\alpha^{-1}(O_\sigma(X_\alpha))=O_\sigma(X)\subseteq Q.$$

Since Λ is finite, there exists an $\alpha\in\Lambda$ such that

$$\varphi_\alpha^{-1}(O_\sigma(X_\alpha))\subseteq Q\Rightarrow\ker(\varphi_\alpha)\subseteq Q\Rightarrow\varphi_\alpha^{-1}\varphi_\alpha(Q)=Q$$

where $P=\varphi_\alpha(Q)$ is a prime ideal in $C(X_\alpha)$. The converse is trivially true. □

Later, we will find that the above theorem is true in general if and only if σ is a P-point with respect to all but finitely many of the X_α's. Therefore, if we choose X_α's such that σ is not a P-point with respect to infinitely many of them, then there exist prime ideals which are not of the form $\varphi_\alpha^{-1}(P)$. Most important is that if we choose X_α's such that σ is not a P-point with respect to X_α, for every $\alpha\in\Lambda$, then the number of ideals that are not of this form is $\geqslant 2^{2^{|\Lambda|}}$, see Proposition 4.2.

Proposition 2.5. *Let Q be a nonmaximal prime ideal in $C(X)$. Q is of the form $\varphi_\alpha^{-1}(P)$ (where P is a prime ideal in $C(X_\alpha)$) if and only if there exists an $\alpha\in\Lambda$ such that $\varphi_\alpha(Q)$ is a nonmaximal ideal in $C(X_\alpha)$.*

Proof. Suppose $\varphi_\alpha(Q)\subsetneq M_\alpha$, where M_α is a maximal ideal in $C(X_\alpha)$. It is enough to show $\ker(\varphi_\alpha)\subseteq Q$. Since $\varphi_\alpha(Q)\subsetneq M_\alpha$, there exists an $f_\alpha\in M_\alpha\setminus\varphi_\alpha(Q)$ such that $f_\alpha(\sigma)=0$. Now, if $g\in\ker(\varphi_\alpha)$, then $g\overline{f_\alpha}=0\in Q$ and so $g\in Q$. Therefore $\ker(\varphi_\alpha)\subseteq Q$ and we are done. The converse is obvious. □

Corollary 2.6. *If Q is a prime ideal in $C(X)$, then $\varphi_\alpha(Q)$ is a prime ideal, for every $\alpha\in\Lambda$. Moreover $\varphi_\alpha(Q)$ is a maximal ideal for every $\alpha\in\Lambda$, except at most one.*

We conclude this section with some definitions and lemmas which will be used in Section 3 for characterization of the second class of ideals in $C(X)$.

Definition 2.7. Let R be a ring, J be an ideal in R and I be an ideal in J. We say I is a prime ideal with respect to J if whenever $a,b\in J$ and $ab\in I$, then $a\in I$ or $b\in I$. The similar definitions hold for z-ideals, z°-ideals and so on.

Lemma 2.8. *Let X be a topological space, $p\in\beta X$ and I be an ideal in $C(X)$ containing $O^p(X)$. I is a prime ideal if and only if I is a prime ideal with respect to $M^p(X)$.*

Proof. Let I be an ideal containing $O^p(X)$ and prime with respect to $M^p(X)$. Suppose $fg\in I$. If $f,g\in M^p(X)$, then we are finished. Therefore, we may suppose that f or g (say g) does not belong to $M^p(X)$ and hence there exists $h \in O^p(X)$ such that $Z(h) \cap Z(g) = \emptyset$. Defining $k = 1/(g^2 + h^2)$, we have

$$\forall x \in Z(h)\ \ 1 - g^2(x)k(x) = 0 \quad \text{therefore} \quad Z(h) \subseteq Z(1 - g^2 k)$$
$$\Rightarrow 1 - g^2 k \in O^p(X) \subseteq I \Rightarrow f - fg^2 k \in I \Rightarrow f \in I.$$

Therefore, I is a prime ideal. The converse is trivially true. □

Note that this assertion is not true in general, for instance take $I = M^p(X) \cap M^q(X)$ where $p \neq q$.

Lemma 2.9. *Suppose $O^p(X) \subseteq I \neq M^p(X)$, then I is a z°-ideal if and only if I is a z°-ideal with respect to $M^p(X)$.*

Proof. $\Rightarrow$: This is clear.

$\Leftarrow$: Suppose $O^p(X) \subseteq I \neq M^p(X)$ and I be a z°-ideal with respect to $M^p(X)$. Suppose $\operatorname{int}_X Z(f) \subseteq \operatorname{int}_X Z(g)$ and $f \in I$. It is sufficient to show that $g \in M^p(X)$. To the contrary suppose that $g \notin M^p(X)$, then $p \notin \operatorname{cl}_{\beta X} Z(g)$ and consequently $h \in O^p(X)$ exists such that $Z(h) \cap Z(g) = \emptyset$. Clearly $h^2 + f^2 \in I$ and

$$\operatorname{int}_X Z(f^2 + h^2) = \operatorname{int}_X Z(f) \cap \operatorname{int}_X Z(h) \subseteq \operatorname{int}_X Z(g) \cap \operatorname{int}_X Z(h) = \emptyset.$$

Therefore, $\operatorname{int}_X Z(f^2 + h^2) = \emptyset$ and since I is a z°-ideal with respect to $M^p(X)$, it follows that $I = M^p(X)$ and this is a contradiction. □

3. Second class of ideals

Now we define a new class of ideals that contains the class of ideals of the form $\varphi_\alpha^{-1}(J)$ and plays an interesting role in our study.

Definition 3.1. Let $\mathcal{F}$ be an ultrafilter on Λ and for every $\alpha \in \Lambda$, I_α be an ideal in $C(X_\alpha)$. We define

$$I = \{f \in C(X)\colon\ \exists\, A \in \mathcal{F} \ni \forall\, \alpha \in A,\ \ \varphi_\alpha(f) \in I_\alpha\} \tag{2}$$

and denote it by $\mathbf{I}(\mathcal{F}, \{I_\alpha\}_{\alpha\in\Lambda})$. Exactly $I = \bigcup_{A\in\mathcal{F}} \bigcap_{\alpha\in A} \varphi_\alpha^{-1} I_\alpha$. This kind of ideals is very useful and the following theorems witness this claim.

Lemma 3.2. *Suppose $I = \mathbf{I}(\mathcal{F}, \{I_\alpha\}_{\alpha\in\Lambda})$, then I is an ideal in $C(X)$ and moreover $I \subseteq M_\sigma(X)$, if and only if there exists $A \in \mathcal{F}$ such that $I_\alpha \subseteq M_\sigma(X_\alpha)$ for every $\alpha \in A$.*

Proof. Suppose $f, g \in I$, then there exist $A_1, A_2 \in \mathcal{F}$ such that $\varphi_\alpha(f) \in I_\alpha$ for each $\alpha \in A_1$ and $\varphi_\alpha(g) \in I_\alpha$ for each $\alpha \in A_2$. Therefore, $\varphi_\alpha(f) + \varphi_\alpha(g) \in I_\alpha$ for every $\alpha \in A_1 \cap A_2$ and since $A_1 \cap A_2 \in \mathcal{F}$, it follows that $f + g \in I$. Similarly if $f \in I$ and $g \in C(X)$, then $fg \in I$.

Now suppose $I \subseteq M_\sigma(X)$. Taking B to be $\{\alpha \in \Lambda\colon\ I_\alpha \nsubseteq M_\sigma(X_\alpha)\}$, it is enough to show $B \notin \mathcal{F}$. To see this, for every $\alpha \in B$, we choose $f_\alpha \in I_\alpha \setminus M_\sigma(X_\alpha)$ such that $f_\alpha(\sigma) = 1$ and define $f = \overline{\bigcup_{\alpha\in B} f_\alpha}$, then clearly $f \notin M_\sigma(X)$ and so $f \notin I$. Therefore $B \notin \mathcal{F}$. The converse is trivial. □

Hence, when we consider the ideals over $O_\sigma(X)$ we can suppose $I_\alpha \subseteq M_\sigma(X_\alpha)$ for every $\alpha \in \Lambda$ and when we consider the ideals over $O^p(X)$, where $p \neq \sigma$, we can suppose $I_\alpha \nsubseteq M_\sigma(X_\alpha)$ for every $\alpha \in \Lambda$.

Next, we show that the class of ideals which is defined by (2) contains the class of ideals of the form $\varphi_\alpha^{-1}(J)$.

Proposition 3.3. *Let $I = \mathbf{I}(\mathcal{F}, \{I_\alpha\}_{\alpha\in\Lambda})$ be a nonmaximal ideal, then I is of the form $\varphi_{\alpha_\circ}^{-1}(J)$ if and only if the ultrafilter $\mathcal{F}$ is fixed and contains $\{\alpha_\circ\}$.*

Proof. $\Rightarrow$: Suppose $I = \varphi_{\alpha_\circ}^{-1}(J)$ is not a maximal ideal. On the contrary if $\{\alpha_\circ\} \notin \mathcal{F}$, then assuming $f_{\alpha_\circ} \in M_\sigma(X_{\alpha_\circ})$ it follows that $\overline{f_{\alpha_\circ}} \in I$. Since $I = \varphi_{\alpha_\circ}^{-1}(J)$, $f_{\alpha_\circ} \in J$ and consequently $J = M_\sigma(X_{\alpha_\circ})$. This is a contradiction, since $I = \varphi_{\alpha_\circ}^{-1}(J)$ is not a maximal ideal.

$\Leftarrow$: Suppose $\mathcal{F}$ is fixed. Hence there exists $\alpha_\circ \in \Lambda$ such that $\{\alpha_\circ\} \in \mathcal{F}$ and clearly in this case $I = \mathbf{I}(\mathcal{F}, \{I_\alpha\}_{\alpha\in\Lambda}) = \varphi_{\alpha_\circ}^{-1}(I_{\alpha_\circ})$. □

Now, we are ready to show the properties of the ideals of the form $\mathbf{I}(\mathcal{F}, \{I_\alpha\}_{\alpha\in\Lambda})$.

Lemma 3.4. *Let $I = \mathbf{I}(\mathcal{F}, \{I_\alpha\}_{\alpha\in\Lambda})$ and $J = \mathbf{I}(\mathcal{F}, \{J_\alpha\}_{\alpha\in\Lambda})$, then $I \subseteq J$ if and only if $A \in \mathcal{F}$ exists such that $I_\alpha \subseteq J_\alpha$ for every $\alpha \in A$.*

Proof. $\Leftarrow$: Suppose $f \in I$, then $A_\circ \in \mathcal{F}$ exists such that $\varphi_\alpha(f) \in I_\alpha$ for every $\alpha \in A_\circ$. Hence, by hypothesis $\varphi_\alpha(f) \in I_\alpha \subseteq J_\alpha$ for every $\alpha \in A_\circ \cap A$. This implies $f \in J$. Therefore $I \subseteq J$.

$\Rightarrow$: Suppose that for all $A \in \mathcal{F}$ there exists an $\alpha \in A$ such that $I_\alpha \nsubseteq J_\alpha$. Put $B = \{\alpha \in \Lambda\colon\ I_\alpha \nsubseteq J_\alpha\}$. We can easily see that $B \in \mathcal{F}$ and for every $\alpha \in B$ there exists an $f_\alpha \in I_\alpha \setminus J_\alpha$. Without loss of generality we can suppose $f_\alpha(\sigma) = 0$ (or $f_\alpha(\sigma) = 1$) for every $\alpha \in B$. Now we define $f = \overline{\bigcup_{\alpha\in B} f_\alpha}$. Clearly $f \in I \setminus J$ and consequently $I \nsubseteq J$. □

It follows from above that $I = J$ if and only if there exists $A \in \mathcal{F}$ such that $I_\alpha = J_\alpha$ for every $\alpha \in A$.

Proposition 3.5. *Let $I = \mathbf{I}(\mathcal{F}, \{I_\alpha\}_{\alpha\in\Lambda})$ and $J = \mathbf{I}(\mathcal{F}, \{J_\alpha\}_{\alpha\in\Lambda})$, then*

$$I \cap J = \mathbf{I}(\mathcal{F}, \{I_\alpha \cap J_\alpha\}_{\alpha\in\Lambda}),$$
$$I + J = \mathbf{I}(\mathcal{F}, \{I_\alpha + J_\alpha\}_{\alpha\in\Lambda}).$$

Proof. This is easy. □

Proposition 3.6. *Let $\mathcal{F}_1 \neq \mathcal{F}_2$, $I = \mathbf{I}(\mathcal{F}_1, \{I_\alpha\}_{\alpha\in\Lambda})$ and $J = \mathbf{I}(\mathcal{F}_2, \{J_\alpha\}_{\alpha\in\Lambda})$.*
(a) *If $I, J \subseteq M_\sigma(X)$, then $I + J = M_\sigma(X)$.*
(b) *If $I \nsubseteq M_\sigma(X)$ or $J \nsubseteq M_\sigma(X)$, then $I + J = C(X)$.*

Proof. The proof is routine. □

In the main result of this section which follows, we can see as well that there exists a close connection between a big class of ideals of $C(X)$ and the ideals of $C(X_\alpha)$'s.

Theorem 3.7. *Suppose $I = \mathbf{I}(\mathcal{F}, \{I_\alpha\}_{\alpha\in\Lambda})$, then:*
(a) *If $O_\sigma(X) \subseteq I$, then there exists $A \in \mathcal{F}$ such that $O_\sigma(X_\alpha) \subseteq I_\alpha$ for every $\alpha \in A$.*
(b) *I is a prime ideal if and only if there exists $A \in \mathcal{F}$ such that I_α is a prime ideal for every $\alpha \in A$.*
(c) *I is a maximal ideal if and only if there exists $A \in \mathcal{F}$ such that I_α is a maximal ideal for every $\alpha \in A$.*
(d) *Let $I \neq M_\sigma(X)$. I is a minimal prime ideal if and only if there exists $A \in \mathcal{F}$ such that I_α is a minimal prime ideal for every $\alpha \in A$.*
(e) *I is convex (absolutely convex) if and only if there exists $A \in \mathcal{F}$ such that I_α is convex (absolutely convex) for every $\alpha \in A$.*
(f) *I is a z-ideal if and only if there exists $A \in \mathcal{F}$ such that I_α is a z-ideal for every $\alpha \in A$.*
(g) *If $O_\sigma(X) \subseteq I \neq M_\sigma(X)$, then I is a z°-ideal if and only if there exists $A \in \mathcal{F}$ such that I_α is a z°-ideal for every $\alpha \in A$.*
(h) *If $I \nsubseteq M_\sigma(X)$, then I is a z°-ideal if and only if there exists $A \in \mathcal{F}$ such that I_α is a z°-ideal for every $\alpha \in A$.*

Proof. The proof of (a) is routine.

(b) $\Rightarrow$: Suppose $I = \mathbf{I}(\mathcal{F}, \{I_\alpha\}_{\alpha\in\Lambda})$ is a prime ideal. Let $B = \{\alpha \in \Lambda\colon I_\alpha$ is not a prime ideal$\}$. It is enough to show that $B \notin \mathcal{F}$. By hypothesis, for every $\alpha \in B$ there exist $f_\alpha, g_\alpha \notin I_\alpha$ such that $f_\alpha g_\alpha \in I_\alpha$. Of course we can choose f_α, g_α such that there are fixed real numbers r and s such that $f_\alpha(\sigma) = r$ and $g_\alpha(\sigma) = s$ for every $\alpha \in B$. Define $f = \overline{\bigcup_{\alpha\in B} f_\alpha}$ and $g = \overline{\bigcup_{\alpha\in B} g_\alpha}$. On the contrary if $B \in \mathcal{F}$, then $f, g \notin I$ while $fg \in I$ and this is a contradiction.

$\Leftarrow$: Let $A \in \mathcal{F}$ be such that I_α is prime for every $\alpha \in A$ and $fg \in I$. Define

$$A_1 = \{\alpha \in A\colon\ \varphi_\alpha(f) \in I_\alpha\}, \quad A_2 = \{\alpha \in A\colon\ \varphi_\alpha(g) \in I_\alpha\}.$$

It is easy to see that $A_1 \cup A_2 = A \in \mathcal{F}$ and since $\mathcal{F}$ is a prime filter, $A_1 \in \mathcal{F}$ or $A_2 \in \mathcal{F}$ and consequently $f \in I$ or $g \in I$. Therefore, I is a prime ideal.

(c) $\Rightarrow$: Evident.

$\Leftarrow$: Suppose $A \in \mathcal{F}$ and I_α is a maximal ideal in $C(X_\alpha)$ for every $\alpha \in A$. If $f \notin I$ and $0 \leqslant f < 1$, then there exists $B \subseteq A$ such that $B \in \mathcal{F}$ and $\varphi_\alpha(f) \notin I_\alpha$, for every $\alpha \in B$. Hence for every $\alpha \in B$, there exists $g_\alpha \in I_\alpha$ such that $\varphi_\alpha(f) + g_\alpha = 1$ on X_α for every $\alpha \in B$. Clearly $g_\alpha(\sigma) = 1 - f(\sigma)$ for every $\alpha \in B$ and so we can define $g = \overline{\bigcup_{\alpha\in B} g_\alpha}$. Clearly $g \in I$ and $f + g > 0$ and so is a unit. Therefore, I is a maximal ideal.

(d) $\Rightarrow$: Suppose $I = \mathbf{I}(\mathcal{F}, \{I_\alpha\}_{\alpha\in\Lambda})$ is a minimal prime ideal. Let $B = \{\alpha \in \Lambda\colon\ I_\alpha$ is not a minimal prime ideal$\}$. We must show that $B \notin \mathcal{F}$. To see this, by hypothesis for every $\alpha \in B$, there exists a prime ideal P_α in $C(X_\alpha)$ such that $P_\alpha \subsetneq I_\alpha$. By extending these P_α's to the family $\{P_\alpha\}_{\alpha\in\Lambda}$ we will obtain the prime ideal $P = \mathbf{I}(\mathcal{F}, \{P_\alpha\}_{\alpha\in\Lambda})$ and if on the contrary $B \in \mathcal{F}$, then P is strictly contained in I and this is a contradiction.

$\Leftarrow$: Without loss of generality, we can suppose $I = \mathbf{I}(\mathcal{F}, \{I_\alpha\}_{\alpha\in\Lambda})$ where I_α is a minimal prime ideal in $C(X_\alpha)$ and not equal to $M_\sigma(X_\alpha)$ for every $\alpha \in \Lambda$. Let $f \in I$, then there exists $A \in \mathcal{F}$ such that $\varphi_\alpha(f) \in I_\alpha$ for every $\alpha \in A$. So, there exists $g_\alpha \in M_\sigma(X_\alpha) \setminus I_\alpha$ such that $g_\alpha \varphi_\alpha(f) = 0$ for every $\alpha \in A$. Define $g = \overline{\bigcup_{\alpha\in A} g_\alpha}$, then clearly $fg = 0$ and $g \notin I$. Therefore, I is a minimal prime ideal.

The proofs of (e) and (f) are routine.

(g) $\Rightarrow$: Let I be a nonmaximal z°-ideal containing $O_\sigma(X)$. By (a) and (c) there exists $A_\circ \in \mathcal{F}$ such that $O_\sigma(X_\alpha) \subseteq I_\alpha \neq M_\sigma(X_\alpha)$ for every $\alpha \in A_\circ$. Take $B \subseteq A_\circ$ such that I_α is not a z°-ideal for all $\alpha \in B$. We must show that $B \notin \mathcal{F}$. To see this, by Lemma 2.9, for every $\alpha \in B$ ther exists $f_\alpha \in I_\alpha$ and $g_\alpha \in M_\sigma(X_\alpha) \setminus I_\alpha$ such that $\operatorname{int}_{X_\alpha} Z(f_\alpha) \subseteq \operatorname{int}_{X_\alpha} Z(g_\alpha)$. Again, we take

$$f = \overline{\bigcup_{\alpha\in B} f_\alpha}, \quad g = \overline{\bigcup_{\alpha\in B} g_\alpha}.$$

Clearly $\operatorname{int}_X Z(f) \subseteq \operatorname{int}_X Z(g)$. Now, on the contrary if $B \in \mathcal{F}$, then $f \in I$, $g \notin I$ and this is a contradiction.

$\Leftarrow$: Let $A \in \mathcal{F}$ and I_α be a z°-ideal for every $\alpha \in A$. Suppose $\operatorname{int}_X Z(f) \subseteq \operatorname{int}_X Z(g)$ and $f \in I$. Hence, for every $\alpha \in A$

$$X_\alpha \cap \operatorname{int}_X Z(f) \subseteq X_\alpha \cap \operatorname{int}_X Z(g). \tag{3}$$

We claim for every $\alpha \in A$,

$$\operatorname{int}_{X_\alpha} Z(\varphi_\alpha(f)) \subseteq \operatorname{int}_{X_\alpha} Z(\varphi_\alpha(g)). \tag{4}$$

To prove our claim, by (3) it is enough to show that if $\sigma \notin \operatorname{int}_{X_\alpha} Z(\varphi_\alpha(g))$, then $\sigma \notin \operatorname{int}_{X_\alpha} Z(\varphi_\alpha(f))$. To see this, suppose $\sigma \notin \operatorname{int}_{X_\alpha} Z(\varphi_\alpha(g))$ and U_α is an arbitrary neighborhood of σ in X_α. By hypothesis there exists $x_\alpha \in U_\alpha$ such that $x_\alpha \notin Z(\varphi_\alpha(g))$. Then $x_\alpha \notin Z(g)$ and hence $x_\alpha \notin \operatorname{int}_X Z(f)$. Therefore, $U_\alpha \nsubseteq \operatorname{int}_{X_\alpha} Z(\varphi_\alpha(f))$ and consequently $\sigma \notin \operatorname{int}_{X_\alpha} Z(\varphi_\alpha(f))$. Hence, (4) holds and it follows that $\varphi_\alpha(g) \in I_\alpha$ for every $\alpha \in A$ and so, $g \in I$. Therefore, I is a z°-ideal.

(h) $\Rightarrow$: Let $I \nsubseteq M_\sigma(X)$ be a z°-ideal. Suppose $B \subseteq \Lambda$ is such that I_α is not a z°-ideal for every $\alpha \in B$. We must show that $B \notin \mathcal{F}$. By hypothesis, for every $\alpha \in B$, there exist $f_\alpha \in I_\alpha$ and $g_\alpha \notin I_\alpha$ such that $\operatorname{int}_{X_\alpha} Z(f_\alpha) \subseteq \operatorname{int}_{X_\alpha} Z(g_\alpha)$. Since $I \nsubseteq M_\sigma(X)$, we can choose f_α and g_α such that $f_\alpha(\sigma) = g_\alpha(\sigma) = 1$, for every $\alpha \in B$. Define

$$f = \overline{\bigcup_{\alpha \in B} f_\alpha}, \quad g = \overline{\bigcup_{\alpha \in B} g_\alpha}.$$

Clearly $\operatorname{int}_X Z(f) \subseteq \operatorname{int}_X Z(g)$ and if $B \in \mathcal{F}$, then $f \in I$ and $g \notin I$, which is a contradiction. □

Remark 3.8. Similar to part (d) of the above theorem, we can conclude that if $I = \mathbf{I}(\mathcal{F}, \{I_\alpha\}_{\alpha\in\Lambda})$ and $J = \mathbf{I}(\mathcal{F}, \{J_\alpha\}_{\alpha\in\Lambda})$, then $J \in \operatorname{Min}(I)$ if and only if there exists $A \in \mathcal{F}$ such that $J_\alpha \in \operatorname{Min}(I_\alpha)$ for every $\alpha \in A$.

By the previous discussion, a natural question is this: "Are all of the prime ideals (at least minimal prime ideals) in $C(X)$ of the form $\mathbf{I}(\mathcal{F}, \{I_\alpha\}_{\alpha\in\Lambda})$?" We leave it as an open problem. We partially answer this question in the next lemmas.

Lemma 3.9. *Let P be a prime ideal containing $O_\sigma(X)$, then there exists an ultrafilter $\mathcal{F}$ on Λ such that $\mathbf{I}(\mathcal{F}, \{O_\sigma(X_\alpha)\})_{\alpha\in\Lambda} \subseteq P$.*

Proof. Without loss of generality, we can suppose that P is not a maximal ideal. Put $\mathcal{F} = \Big\{A \subseteq \Lambda \colon \bigcap_{\alpha\in A} \varphi_\alpha^{-1} O_\sigma(X_\alpha) \subseteq P\Big\}$. It suffices to show that $\mathcal{F}$ is an ultrafilter on Λ. We first show that

$$A \in \mathcal{F} \Leftrightarrow A^c \notin \mathcal{F}, \tag{5}$$

$$A \cup B \in \mathcal{F} \Rightarrow A \in \mathcal{F} \quad \text{or} \quad B \in \mathcal{F}. \tag{6}$$

Suppose $A, A^c \in \mathcal{F}$, then we can easily that see

$$\bigcap_{\alpha\in A} \varphi_\alpha^{-1} O_\sigma(X_\alpha) + \bigcap_{\alpha\in A^c} \varphi_\alpha^{-1} O_\sigma(X_\alpha) = M_a(X)$$

and this a contradiction. Hence, (5) holds. To prove (6), suppose that $A \cup B \in \mathcal{F}$, then

$$\Big(\bigcap_{\alpha\in A} \varphi_\alpha^{-1} O_\sigma(X_\alpha)\Big) \cap \Big(\bigcap_{\alpha\in B} \varphi_\alpha^{-1} O_\sigma(X_\alpha)\Big) = \bigcap_{\alpha\in A\cup B} \varphi_\alpha^{-1} O_\sigma(X_\alpha) \subseteq P$$
$$\Rightarrow \bigcap_{\alpha\in A} \varphi_\alpha^{-1} O_\sigma(X_\alpha) \subseteq P \quad \text{or} \quad \bigcap_{\alpha\in B} \varphi_\alpha^{-1} O_\sigma(X_\alpha) \subseteq P \Rightarrow A \in \mathcal{F} \quad \text{or} \quad B \in \mathcal{F}.$$

Therefore, (6) holds and it remains to be shown that $\mathcal{F}$ is a filter on Λ. Suppose $A, B \in \mathcal{F}$. By (5) $A^c, B^c \notin \mathcal{F}$ and by (6) $(A \cap B)^c = A^c \cup B^c \notin \mathcal{F}$, thus by (5) $A \cap B \in \mathcal{F}$. Finally suppose that $A \subseteq B$ where $A \in \mathcal{F}$ and $B \subseteq \Lambda$, then clearly $B \in \mathcal{F}$. □

Lemma 3.10. *Suppose that $I = \mathbf{I}(\mathcal{F}, \{I_\alpha\}_{\alpha\in\Lambda}) \subseteq J_\circ$ and $f \in J_\circ \setminus I$, then there exists $J = \mathbf{I}(\mathcal{F}, \{J_\alpha\}_{\alpha\in\Lambda})$ such that $(I, f) \subseteq J \subseteq J_\circ$.*

Proof. The proof is routine. □

4. Some applications

In this section we consider some applications and we answer the two questions already mentioned. Also at the end of this section we will give an example which shows that the answer to an open question in [11] is negative.

Proposition 4.1.

(a) *σ is a P-point with respect to X if and only if σ is a P-point with respect to X_α for every $\alpha \in \Lambda$.*

(b) *Assuming σ is a nonisolated-point with respect to X_α's, then σ is an almost P-point with respect to X if and only if σ is an almost P-point with respect to at least one of the X_α's.*

(c) *σ is an F-point with respect to X if and only if σ is an F-point with respect to one of the X_α's and is a P-point with respect to the others.*

Proof. (a) and (b) are clear. (c) Suppose σ is an F-point with respect to X, then $O_\sigma(X) = \bigcap_{\alpha\in\Lambda} \varphi_\alpha^{-1}(O_\sigma(X_\alpha))$ is a prime ideal. But, $\varphi_\alpha^{-1}(O_\sigma(X_\alpha))$ is a z-ideal containing $O_\sigma(X)$ for every $\alpha \in \Lambda$. Hence $\varphi_\alpha^{-1}(O_\sigma(X_\alpha))$ is a prime ideal, for every $\alpha \in \Lambda$. Then, $O_\sigma(X_\alpha) = \varphi_\alpha \varphi_\alpha^{-1}(O_\sigma(X_\alpha))$ is a prime ideal. In addition, for every $\alpha \neq \beta$, $\varphi_\alpha^{-1}(I_\alpha)$ and $\varphi_\beta^{-1}(I_\beta)$ are comparable if and only if I_α or I_β is a maximal ideal. Therefore, all of the $O_\sigma(X_\alpha)$'s are maximal ideals, except at most one. The converse is obvious. □

Proposition 4.2. *Let σ be an F-point with respect to each of the X_α's and m be the cardinal number of α's for which σ is not a P-point with respect to X_α, then:*

(a) *Each $P \in \mathrm{Min}(O_\sigma(X))$ is of the form $\mathbf{I}(\mathcal{F}, \{I_\alpha\}_{\alpha\in\Lambda})$.*
(b) *If m is finite, then $|\mathrm{Min}(O_\sigma(X))| = m$.*
(c) *If m is infinite, then $|\mathrm{Min}(O_\sigma(X))| = 2^{2^m}$.*

Proof. (a) follows by Lemma 3.9. (b) is clear. To prove (c), without loss of generality, we can suppose σ is not a P-point with respect to X_α for every $\alpha \in \Lambda$. By (a) every minimal prime ideal containing $O_\sigma(X)$ is of the form $\mathbf{I}(\mathcal{F}, \{P_\alpha\}_{\alpha\in\Lambda})$ where $P_\alpha = O_\sigma(X_\alpha)$ for every $\alpha \in \Lambda$. Since for every $\mathcal{F}_1 \neq \mathcal{F}_2$, we can see that $\mathbf{I}(\mathcal{F}_1, \{P_\alpha\}_{\alpha\in\Lambda}) \neq \mathbf{I}(\mathcal{F}_2, \{P_\alpha\}_{\alpha\in\Lambda})$, we get

$$|\mathrm{Min}(O_\sigma(X))| = \mathrm{card}\{\mathcal{F}\colon\ \mathcal{F} \text{ is an ultrafilter on } \Lambda\} = 2^{2^{|\Lambda|}} = 2^{2^m}.$$

We already mentioned that if Λ is finite, then any prime ideal containing $O_\sigma(X)$ is of the form $\varphi_\alpha^{-1}(P_\alpha)$. Hence if $O_\sigma(X) \neq M_\sigma(X)$, then

$$\mathrm{Min}(O_\sigma(X)) = \{\varphi_\alpha^{-1}(P_\alpha)\colon\ \alpha \in \Lambda,\ \ P_\alpha \in \mathrm{Min}(O_\sigma(X_\alpha)),\ \ P_\alpha \neq M_\sigma(X_\alpha)\}.$$

Therefore, if Λ is finite, then X is an FMP-space if and only if X_α is an for every $\alpha \in \Lambda$. □

Recall that if $p \in X$ and the number of minimal prime ideals contained in M_p is equal to $n \in \mathbb{N}$, then the rank of p is n and we write $\mathrm{rk}(p) = n$. For more information about an FMP-points or points with finite rank, see [8], [9] and [11]. We need the following generalization of this concept.

Definition 4.3. Let $p \in \beta X$ and m be a cardinal number. We say that $p \in \beta X$ is an *mMP-point* with respect to X if there exists a subfamily $\{P_\beta\}_{\beta\in B}$ of $\mathrm{Min}(O^p(X))$ such that $O^p(X) = \bigcap\limits_{\beta\in B} P_\beta$, this intersection is irreducible and m is the smallest cardinal with such property.

In the following as the main theorem of this section we will show that there exists an mMP-point for any given cardinal number m. In addition, by the following theorem we will answer the first two questions automatically.

Theorem 4.4. *Let m be a cardinal number. There exist a topological space X and $\sigma \in X$ such that σ is an mMP-point.*

Proof. Let Λ be an arbitrary set such that $|\Lambda| = m$. Put $X_\alpha = \Sigma$ and $x_\alpha = \sigma$ for every $\alpha \in \Lambda$ (where $\Sigma = \mathbf{N} \cup \{\sigma\}$, see [7] 4M). Clearly by (1) $O_\sigma(X) = \bigcap_{\alpha\in\Lambda} \varphi_\alpha^{-1}(O_\sigma(X_\alpha))$ where $O_\sigma(X_\alpha)$ is a prime ideal in $C(X_\alpha)$ for every $\alpha \in \Lambda$. For each $\beta \in \Lambda$, suppose $f_\alpha \in O_\sigma(X_\alpha)$ for every $\alpha \neq \beta$ and $f_\beta \in M_\sigma(X_\beta) \setminus O_\sigma(X_\beta)$, then we can see easily that

$$\bigcup_{\alpha\in\Lambda} f_\alpha \in \bigcap_{\alpha\in\Lambda} \varphi_\alpha^{-1}(O_\sigma(X_\alpha)) \setminus \varphi_\beta^{-1}(O_\sigma(X_\beta)).$$

It follows that the above intersection is irreducible. To complete the proof, suppose $\{P_\beta\}_{\beta\in B}$ is a family of prime ideals such that $O_\sigma(X) = \bigcap_{\beta\in B} P_\beta$. It is enough to show that this family contains the family $\{\varphi_\alpha^{-1}(O_\sigma(X_\alpha))\}_{\alpha\in\Lambda}$. On the contrary, let $\alpha_\circ \in \Lambda$ be such that $\varphi_{\alpha_\circ}^{-1}(O_\sigma(X_{\alpha_\circ}))$ does not belong to the family $\{P_\beta\}_{\beta\in B}$. By Proposition 4.2 every P_β is of the form $\mathbf{I}(\mathcal{F}_\beta, \{I_{\alpha\beta}\}_{\alpha\in\Lambda})$. Also we can clearly suppose that $I_{\alpha\beta} = M_\sigma(X_\alpha)$ where $\alpha = \alpha_\circ$ for every $\beta \in B$ and $I_{\alpha\beta}$ is a prime ideal over $O_\sigma(X_\alpha)$ for every $\alpha \in \Lambda$ and every $\beta \in B$. Taking $f_{\alpha_\circ} \in M_\sigma(X_{\alpha_\circ}) \setminus O_\sigma(X_{\alpha_\circ})$, $f_\alpha \in O_\sigma(X_\alpha)$ and $f = \bigcup_{\alpha\in\Lambda} f_\alpha$, clearly $f \in \bigcap_{\beta\in B} P_\beta \setminus O_\sigma(X)$ and this is a contradiction. □

Remark 4.5. It is good to know that if $|\Lambda| = m$, then since the number of ultrafilters on Λ equals to 2^{2^m}, according to what has been asserted in Section 4, σ is an mMP-point while the number of minimal prime ideals containing $O_\sigma(X)$ equals to 2^{2^m}. Also it follows from the above that for any given cardinal number m there exists a ring R and an ideal I in R such that $|\mathrm{Min}(I)| = 2^{2^m}$. Now, we have a natural question. For any given infinite cardinal number m, is there a ring R and an ideal I in it such that $|\mathrm{Min}(I)| = m$? This is an open problem.

We conclude the paper by an example that answers the following open question which has occurred in [11] following Lemma 3.3.

"Let X be a completely regular space such that every maximal ℓ-ideal of $C(X)$ has finite rank less than some m (i.e., $\mathrm{rk}(p) < m$ for every $p \in \beta X$). It is an open question as to whether or not the set of points of X of rank greater than one is closed."

Example 4.6. By Theorem 4.4 for any $n \in \mathbb{N}$, we can construct a space $Y = D \cup \{\sigma\}$ where D is discrete and $\mathrm{rk}(\sigma) = n > 1$. Now, let Λ be an uncountable set and $\{Y_\lambda\}_{\lambda\in\Lambda}$ be a family of mutually disjoint copies of Y. Suppose X is the disjoint union of Y_λ's and $T = X \cup \{\alpha\}$ where X is open in T and the open neighborhood system of α consists of co-countable subsets of T containing α. We can easily see that T is a normal space and $\mathrm{rk}(\alpha) = 1$ (exactly α is a P-point). Since every point of T has rank less than or equal to n, by Theorem 5.10 of [9] every maximal ideal of $C(X)$ has finite rank less than $m = n + 1$. But $\mathrm{rk}(\sigma_\lambda) = n$ for every $\lambda \in \Lambda$ and α is a limit point of $\{\sigma_\lambda \colon \lambda \in \Lambda\}$, therefore the set of points of T of rank greater than one is not closed.

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