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# ON THE LAPLACIAN ENERGY OF A GRAPH

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*Abstract.* In this paper we consider the energy of a simple graph with respect to its Laplacian eigenvalues, and prove some basic properties of this energy. In particular, we find the minimal value of this energy in the class of all connected graphs on $n$ vertices ($n = 1, 2, \ldots$). Besides, we consider the class of all connected graphs whose Laplacian energy is uniformly bounded by a constant $\alpha \geqslant 4$, and completely describe this class in the case $\alpha = 40$.

*Keywords*: simple graphs, Laplacian spectrum, energy of a graph

*MSC 2000*: 05C50

## 1. Introduction

First we will repeat in short some elementary facts about the Laplacian spectrum of a simple finite graph which we will use in the sequel.

Let $G$ be a simple graph on $n$ vertices with the vertex set $V(G) = \{v_1, \ldots, v_n\}$. Next, let $A(G) = [a_{ij}]$ be its $(0,1)$ adjacency matrix, and $D(G) = \operatorname{diag}(d_1, \ldots, d_n)$ the diagonal matrix with the vertex degrees $d_1, \ldots, d_n$ of its vertices $v_1, \ldots, v_n$. Then $L(G) = D(G) - A(G)$ is called the Laplacian matrix of the graph $G$. It is symmetric, singular and positive semi-definite. All its eigenvalues are real and nonnegative and form the Laplacian spectrum $\sigma_L(G) = \{\lambda_1, \ldots, \lambda_n\}$ of the graph $G$. We will always assume that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$. It is well known that $\lambda_n = 0$ and the multiplicity of 0 equals the number of (connected) components of $G$.

**Theorem A.** *We have* $\lambda_1(G) \leqslant n$.

**Theorem B.** *If $G$ has at least one edge and $d = \max\{d_1, \ldots, d_n\}$, then $d + 1 \leqslant \lambda_1(G) \leqslant 2d$.*

**Theorem C.** *If $H$ is a (not necessarily induced) subgraph of a finite graph $G$ then*

$$\lambda_i(H) \leqslant \lambda_i(G) \quad (i = 1, \ldots, |H|).$$

Next, let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two finite graphs with disjoint sets of vertices $V(G_1)$ and $V(G_2)$. Then the direct sum $G = G_1 \dotplus G_2$ of these graphs is defined by $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

**Theorem D.** *If $G = G_1 \dotplus G_2$ is the direct sum of graphs $G_1$, $G_2$, then*

$$\sigma_L(G_1 \dotplus G_2) = \sigma_L(G_1) \cup \sigma_L(G_2),$$

*including the multiplicities.*

A similar statement holds for any number of components $G_1, \ldots, G_m$ $(m \geqslant 2)$.

**Theorem E.** *If $\overline{G}$ is the complementary graph of a finite graph $G$ then*

$$\lambda_k(\overline{G}) = n - \lambda_{n-k}(G) \quad (k = 1, \ldots, n-1).$$

If $H$ is an arbitrary subgraph (not necessarily induced) of a graph $G$, we will denote it simply by $H \subset G$. As usual, $P_n$ $(n \in \mathbb{N} = \{1, 2, \ldots\})$ is the path on $n$ vertices, and $K_n$ is the complete graph on $n$ vertices.

Throughout the paper, we will often use the lists of all connected graphs with 2, 3, 4 or 5 vertices, which can be found in [1], the list of all non-isomorphic connected graphs with 6 vertices (112 graphs, see e.g. [3]), and the list of all connected non-isomorphic graphs with 7 vertices (824 graphs, see e.g. [2]).

## 2. Laplacian energy of a graph

If $G$ is a simple graph on $n$ vertices and $\{\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n\}$ is its Laplacian spectrum, then by the *Laplacian energy of $G$* we mean the invariant

$$E_L(G) = \sum_{i=1}^{n} \lambda_i^2(G) = \sum_{i=1}^{n-1} \lambda_i^2(G).$$

It is obvious that $E_L(G) \geqslant 0$ for every graph $G$, and $E_L(G) > 0$ for every connected graph on $n \geqslant 2$ vertices. Since $\lambda_1(G) \geqslant 2$ for any connected graph, we also have that $E_L(G) \geqslant 4$ if $n \geqslant 2$.

By Theorem C we immediately get the following statement.

**Theorem 1.** *If $H$ is an arbitrary subgraph of a graph $G$, then $E_L(H) \leqslant E_L(G)$.*

By Theorem D we also have the next statement.

**Theorem 2.** *If $G$ is a disconnected graph whose components are $G_1, \ldots, G_m$, then*

$$E_L(G) = \sum_{i=1}^{m} E_L(G_i).$$

**Theorem 3.** *For any graph $G$ on $n$ vertices ($n \in \mathbb{N}$) whose vertex-degrees are $d_1, \ldots, d_n$, we have*

$$E_L(G) = \sum_{i=1}^{n} (d_i^2 + d_i). \tag{1}$$

Proof. By the Viète rules we easily find that

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} d_i, \tag{2}$$

$$\sum_{i<j} \lambda_i \lambda_j = \sum_{i<j} d_i d_j - \sum_{i<j} a_{ij}^2. \tag{3}$$

Since $a_{ij}^2 = a_{ij}$ for every $i < j$, we also find that

$$\sum_{i \neq j} \lambda_i \lambda_j = 2 \sum_{i<j} \lambda_i \lambda_j = \sum_{i \neq j} d_i d_j - \sum_{i \neq j} a_{ij} = \sum_{i \neq j} d_i d_j - \sum_{i=1}^{n} d_i.$$

Therefore

$$\begin{aligned} E_L(G) = \sum_{i=1}^{n} \lambda_i^2 &= \left( \sum_{i=1}^{n} \lambda_i \right)^2 - \sum_{i \neq j} \lambda_i \lambda_j \\ &= \left( \sum_{i=1}^{n} d_i \right)^2 - \left[ \sum_{i \neq j} d_i d_j - \sum_{i=1}^{n} d_i \right] \\ &= \sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i = \sum_{i=1}^{n} (d_i^2 + d_i). \end{aligned}$$

$\square$

**Corollary 1.** *For any graph $G$, its Laplacian energy $E_L(G)$ is an even integer (including zero).*

P r o o f. Since all numbers $d_1, \ldots, d_n \geqslant 0$ are integers, we have that all numbers $d_i(d_i+1)$ are even. Hence, their sum $\sum_{i=1}^{n} d_i(d_i+1)$ is also an even integer. □

**Corollary 2.** *If $H$ is a proper subgraph of a connected graph $G$ ($|G| \geqslant 2$), then $E_L(H) < E_L(G)$.*

P r o o f. Since $G$ has at least one edge, we find that $E_L(G) > 0$. Next, since $H$ is obtained by removing at least one edge from $G$, using (1) we easily conclude that $E_L(H) < E_L(G)$. □

As is also almost obvious by Theorem 3, the maximal Laplacian energy in the class of graphs with $n$ vertices is achieved for the complete graph $K_n$ when $E_L(K_n) = (n-1)n^2$. Only the graph $K_n$ has this maximal energy. Further, we will consider connected graphs with the minimal Laplacian energy in the class of all connected graphs with $n \geqslant 2$ vertices.

**Theorem 4.** *For any connected graph $G$ on $n \geqslant 2$ vertices, we have*

$$E_L(G) \geqslant 6n - 8. \tag{4}$$

*Equality holds in* (4) *if and only if $G$ is the path $P_n$ on $n$ vertices.*

P r o o f. We will prove this theorem by induction.

First, it is obviously true for $n = 2$. Next, let us assume that $n \in \mathbb{N}$ is fixed, this theorem is true for any connected graph with $n-1$ vertices, and let us prove it for an arbitrary connected graph with $n$ vertices.

Let $G$ be any connected graph with $n$ vertices. Then, as is well-known, there is an induced subgraph $H \subset G$ on $n-1$ vertices which is also connected. Denote $V(H) = \{v_1, \ldots, v_{n-1}\}$ and $V(G) = V(H) \cup \{v_n\}$. Then $v_n$ is adjacent to at least one vertex from $v_1, \ldots, v_{n-1}$. Assume that $v_n$ is adjacent to $v_{n-1}$. Denote by $F$ the graph with the same vertex set as $G$, induced in $G$ by $H$ and the pendant edge $(v_{n-1}, v_n)$. We obviously have $H \subset F \subset G$, and therefore $E_L(G) \geqslant E_L(F)$. Since by the inductive hypothesis $E_L(H) \geqslant 6n - 14$, and evidently $E_L(F) = E_L(H) + 6$, we obtain that

$$E_L(G) \geqslant E_L(F) = E_L(H) + 6 \geqslant 6n - 14 + 6 = 6n - 8.$$

Hence, inequality (4) is also true for $G$.

Next, equality obviously holds in (4) for the path $P_n$. We will finally prove that if $G$ is a connected graph on $n \geqslant 2$ vertices such that $E_L(G) = 6n - 8$, then $G$ must be the path $P_n$.

Let $H$ have the same meaning as in the previous part of the proof. Prove that $d_n = d(v_n) = 1$. On the contrary, assume that $d_n \geqslant 2$ and let $v_n$ be adjacent not only to $v_{n-1}$ but also to some $v_{n-2} \in V(H)$. We know that $E_L(H) \geqslant 6n - 14$. Since it is easy to see that

$$E_L(G) \geqslant E_L(H) + 2d_H(v_{n-2}) + 2d_H(v_{n-1}) + 10$$

and $d_H(v_{n-2}), d_H(v_{n-1}) \geqslant 1$, we get

$$E_L(G) = 6n - 8 \geqslant 6n - 14 + 14 = 6n,$$

which is a contradiction. Hence, $d_n = d(v_n) = 1$, so $G$ is the same as $F$. However, since by Theorem 3

$$E_L(G) = E_L(F) = E_L(H) + 2d_{n-1}(H) + 4 = 6n - 8,$$

we get that

$$2d_{n-1}(H) = 6n - 12 - E_L(H) \leqslant 6n - 12 - (6n - 14) = 2$$

and consequently $d_{n-1}(H) = 1$. Therefore

$$E_L(H) = 6n - 8 - 4 - 2 = 6n - 14 = 6(n - 1) - 8.$$

By the inductive hypothesis this means that $H = P_{n-1}$. Hence, $H$ is the path $P_{n-1}$ and $v_{n-1}$ is the end-vertex of this path. This obviously yields that $G$ must be the path $P_n$. So we have proved that $E_L(G) = 6n - 8$ if and only if $G$ is the path $P_n$. This completes the proof. $\square$

## 3. Connected graphs with uniformly bounded Laplacian energy

In the sequel we will consider the problem of finding all connected graphs with the property $E_L(G) \leqslant \alpha$ for a fixed $\alpha \geqslant 4$. A similar problem for the usual energy of connected graphs (in fact, a half of the energy) has been considered in [6].

**Theorem 5.** *For any $\alpha > 4$, the class $\mathcal{P}(\alpha)$ of all non-isomorphic connected graphs with the property $E_L(G) \leqslant \alpha$ is finite.*

Proof. Let $G$ be any connected graph on $n$ vertices such that $E_L(G) \leqslant \alpha$. If its vertex-degrees are $d_1, \ldots, d_n$, then

$$E_L(G) = \sum_{i=1}^{n}(d_i^2 + d_i) \leqslant \alpha,$$

and hence

$$n \leqslant \sum_{i=1}^{n} d_i < \alpha.$$

So we obtain that $n < \alpha$. Hence, the class $\mathcal{P}(\alpha)$ must be finite. Also notice that the applied inequality is very rough. □

Next, if $G$ is any connected graph from the class $\mathcal{P}(\alpha)$ ($\alpha \geqslant 4$) and $H$ is any of its proper connected subgraphs, we have that $E_L(H) < E_L(G) \leqslant \alpha$, so that $H \in \mathcal{P}(\alpha)$ too.

So, for a particular value $\alpha \geqslant 4$, it is interesting to describe the so called maximal graphs from the class $\mathcal{P}(\alpha)$ with respect to the relation "$\subset$". We will do this for the value $\alpha = 40$.

First notice that for any connected graph $G$ with 9 vertices we have $E_L(G) \geqslant 6 \times 9 - 8 = 46 > 40$, so that all graphs from the class $\mathcal{P}(40)$ have at most 8 vertices. Next, for any connected graph with 8 vertices we have $E_L(G) \geqslant 6 \times 8 - 8 = 40$, and $E_L(G) = 40$ if and only if $G = P_8$. Hence, the class $\mathcal{P}(40)$ contains just one graph of the order 8 (the graph $P_8$), and all other graphs have a lower order (up to 7).

Using the tables of connected graphs with respectively 2, 3, 4, 5, 6 and 7 vertices, we can get the following general result. □

**Theorem 6.** *The class $\mathcal{P}(40)$ contains exactly 39 graphs. More exactly, it contains exactly 1 graph of order 2, 2 graphs of order 3, 5 graphs of order 4, 8 graphs of order 5, 14 graphs of order 6, 8 graphs of order 7, and 1 graph of order 8.*

*It also possesses exactly 19 maximal graphs whose orders run over the set $\{4, 5, 6, 7, 8\}$, which are depicted in Fig. 1.*

**Corollary 3.** *A graph $G$ belongs to $\mathcal{P}(40)$ if and only if it is a subgraph of one of the graphs depicted in Fig. 1.*

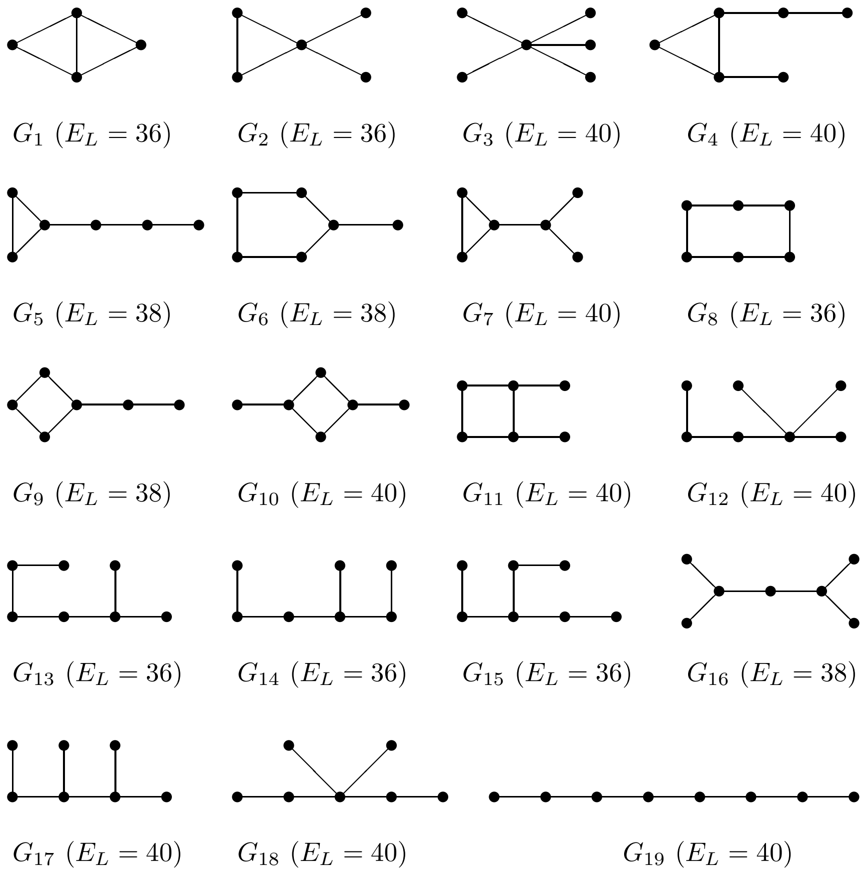


Fig. 1

*References*


[1] *D. Cvetković, M. Doob, and H. Sachs*: Spectra of Graphs—Theory and Application. VEB Deutscher Verlag der Wissenschaften, Berlin-New York, 1980. Zbl 458.05042

[2] *D. Cvetković, M. Doob, I. Gutman, and A. Torgašev*: Recent Results in the Theory of Graph Spectra. Ann. Discrete Math. 36. North-Holland, Amsterdam, 1988. Zbl 634.05054

[3] *D. Cvetković, M. Petrić*: A table of connected graphs with six vertices. Discrete Math. *50* (1984), 37–49. Zbl 533.05052

[4] *R. Grone, R. Merris, V. Sunder*: The Laplacian spectrum of a graph. SIAM J. Matrix Anal. Appl. *11* (1990), 218–238. Zbl 0733.05060

[5] *R. Merris*: Laplacian matrices of graphs. A survey. Linear Algebra and its Appl. *197, 198* (1994), 143–176. Zbl 0802.05053

[6] *A. Torgašev*: Graphs whose energy does not exceed 3. Czechoslovak Math. J. *36* (1986), 167–171. Zbl 0615.05042


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