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WEAK HOMOGENEITY AND PIERCE'S THEOREM FOR MV-ALGEBRAS

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Abstract. In this paper we prove a theorem on weak homogeneity of MV-algebras which generalizes a known result on weak homogeneity of Boolean algebras. Further, we consider a homogeneity condition for MV-algebras which is defined by means of an increasing cardinal property.

Keywords: MV-algebra, weak homogeneity, internal direct product decomposition

MSC 2000: 06D35

1. Introduction

A Boolean algebra B_0 is called weakly homogeneous if for each $0 < b_0 \in B_0$ the relation $\operatorname{card}[0, b_0] = \operatorname{card} B_0$ is valid.

In Sikorski [9], Section 25 the following result is presented:

Theorem (A). *Let B be a complete Boolean algebra, $B \neq \{0\}$. Then B can be represented as a direct product of weakly homogeneous Boolean algebras.*

Sikorski attributes this result to Pierce; in fact, Pierce [7] proved a theorem on Boolean algebras which are homogeneous with respect to a monotone cardinal property f; theorem (A) is a particular case of Pierce's result. Cf. also Pierce [8].

For an MV-algebra $\mathscr{A}$ we denote by A the underlying set of $\mathscr{A}$. By applying the basic operations of $\mathscr{A}$ we can define a partial order $\leqslant$ on the set A. Similarly to the case of Boolean algebras, $\mathscr{A}$ is *weakly homogeneous* if for each $0 < a \in A$, $\operatorname{card}[0, a] = \operatorname{card} A$.

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A subset $X \neq \emptyset$ of an MV-algebra $\mathscr{A}$ is *orthogonal* if $x_1 \wedge x_2 = 0$ for any two distinct elements of X. $\mathscr{A}$ is *orthogonally complete* if each its orthogonal subset has the supremum in $\mathscr{A}$. The notion of *principal polar* of $\mathscr{A}$ is defined analogously to the case of lattice ordered groups (for a detailed definition, cf. Section 2 below). $\mathscr{A}$ is *projectable* if each its principal polar is a direct factor.

We denote by $\mathscr{M}$ the class of all MV-algebras. Further, let $\mathscr{C}$ be the class of all MV-algebras which are semisimple, orthogonally complete and projectable. Each complete MV-algebra belongs to the class $\mathscr{C}$, but not conversely.

The class $\mathscr{C}$ was dealt with in [6] by investigating MV-algebras which are homogeneous with respect to a decreasing generalized cardinal property.

In this paper we prove

Theorem (B). *let $\mathscr{A}$ be an MV-algebra belonging to the class $\mathscr{C}$. Then $\mathscr{A}$ can be represented as a direct product $\prod_{i\in I} \mathscr{A}_i$ such that, for each $i \in I$, some of the following conditions is valid:*

(i) *$\mathscr{A}_i$ is weakly homogeneous;*

(ii) *$\mathscr{A}_i$ is a finite chain.*

In Section 5 we investigate the relation between Theorem (A) and Theorem (B). It turns out that (B) is a generalization of (A).

In Section 6 we prove a result concerning increasing cardinal properties on MV-algebras.

2. Preliminaries

An *MV-algebra* is an algebraic structure $\mathscr{A} = (A; \oplus, \neg, 1)$ of type $(2,1,0)$ such that the conditions (MV1)–(MV6) from [1] are satisfied. We put $\neg 1 = 0$.

(The symbol 0 denotes also the neutral element of a lattice ordered group, the least element of a Boolean algebra and the real zero; the meaning of this symbol will be always clear from the context.)

Let G be an abelian lattice ordered group with a strong unit u. We put $A = [0,u]$ and for $x, y \in A$ we set $x \oplus y = (x+y) \wedge u$, $\neg x = u - x$, $1 = u$. Then $(A; \oplus, \neg, 1)$ is an MV-algebra which is denoted by $\Gamma(G,u)$. For each MV-algebra $\mathscr{A}$ there exists an abelian lattice ordered group G with a strong unit u such that $\mathscr{A} = \Gamma(G,u)$. (Cf. [1].) In what follows we always assume that $\mathscr{A}$ is an MV-algebra, $A \neq \{0\}$ and $\mathscr{A} = \Gamma(G,u)$. Let $a \in A$, $A_1 = [0,a]$. For $x, y \in A_1$ we put $x \oplus_1 y = (x+y) \wedge a$, $\neg_1 a = a - x$. Then the structure $\mathscr{A}_1 = (A_1; \oplus_1, \neg_1, a)$ is an MV-algebra; we denote it by the symbol $[0,a]_{\mathscr{A}}$.

The MV-algebra $\mathscr{A}$ is called *semisimple* (or *archimedean*) if there exists no $a \in A$ such that

$$0 < a < a \oplus a < a \oplus a \oplus a < \ldots < u.$$

The underlying lattice of $\mathscr{A}$ will be denoted by $\ell(\mathscr{A})$. We say that $\mathscr{A}$ is complete if $\ell(\mathscr{A})$ is complete. $\mathscr{A}$ is called a chain if the lattice $\ell(\mathscr{A})$ is linearly ordered.

The *direct product* $\prod_{i \in I} \mathscr{A}_i$ of MV-algebras $\mathscr{A}_i$ is defined in the usual way. Assume that $(a_i)_{i \in I}$ is an orthogonal indexed system of elements of A such that $\bigvee_{i \in I} a_i = u$. For each $x \in I$ we put $\mathscr{A}_i = [0, a_i]_{\mathscr{A}}$. Further, for $x \in A$ and $i \in I$ we set $x_i = x \wedge a_i$. Then the mapping $\varphi(x) = (x_i)_{i \in I}$ is an isomorphism of $\mathscr{A}$ onto the direct product $\prod_{i \in I} \mathscr{A}_i$. (Cf. [4].) We say that

$$\varphi\colon \mathscr{A} \to \prod_{i \in I} \mathscr{A}_i$$

is an *internal direct product decomposition* of $\mathscr{A}$ and that $\mathscr{A}_i$ are *internal direct factors* of $\mathscr{A}$.

Analogously we define the notion of an internal direct product decomposition of a Boolean algebra.

For $\neq X \subseteq A$ we put

$$X^{\delta} = \{a \in A\colon\ a \wedge x = 0 \quad \text{for each } x \in X\}.$$

The MV-algebra $\mathscr{A}$ is *projectable* if for each $x \in X$, $\{x\}^{\delta\delta}$ is the underlying set of an internal direct factor of $\mathscr{A}$.

3. Auxiliary results

An element a of an MV-algebra $\mathscr{A}$ is called Boolean if $a \oplus a = a$. Let $B_0(\mathscr{A})$ be the set of all Boolean elements of $\mathscr{A}$.

Lemma 3.1 (Cf. [1]). (i) *For each $\mathscr{A} \in \mathscr{M}$, $B_0(\mathscr{A})$ is a Boolean algebra.* (ii) *For each Boolean algebra B there exists an MV-algebra $\mathscr{A}$ such that $A = B_0(\mathscr{A}) = B$.*

We denote by $\mathscr{B}_0$ the class of all MV-algebras $\mathscr{A}$ with $A = B_0(\mathscr{A})$.

Lemma 3.2. *Let $\mathscr{A}$ be an orthogonally complete MV-algebra belonging to $\mathscr{B}_0$. Then $\mathscr{A} \in \mathscr{C}$.*

Proof. For each $a \in A$ we have $a \oplus a = a$, hence $\mathscr{A}$ is semisimple. There exists a complement a' of a in $\ell(\mathscr{A})$. For each $x \in A$ we put $\varphi(x) = (x \wedge a, x \wedge a')$. Then φ is an internal direct product decomposition of $\mathscr{A}$ with the internal direct factors $[0, a]$ and $[0, a']$. We obviously have $\{a\}^{\delta\delta} = [0, a]$. Thus $\mathscr{A}$ is projectable. $\square$

Lemma 3.3. *Let $\mathscr{A} \in \mathscr{C}$. Then $B_0(\mathscr{A})$ is a complete Boolean algebra.*

Proof. Let $\{b_i\}_{i\in I}$ be an orthogonal subset of $B_0(\mathscr{A})$. Then it is, at the same time, an orthogonal subset of $\mathscr{A}$. Since $\mathscr{A} \in \mathscr{C}$, there exists an element $b = \bigvee_{i\in I} b_i$ in $\mathscr{A}$. By the same method as in the proof of Lemma 3.2 in [6] we obtain $b \in B_0(\mathscr{A})$. Hence the element b is also the join of the set $\{b_i\}_{i\in I}$ in $B_0(\mathscr{A})$. Thus $B_0(\mathscr{A})$ is orthogonally complete. It is well-known that each orthogonally complete Boolean algebra is complete; therefore $B_0(\mathscr{A})$ is complete. □

We denote by A_0 the set of all atoms of the lattice $\ell(\mathscr{A})$.

Lemma 3.4. *Assume that $\mathscr{A}$ is a semisimple MV-algebra. Let $a_0 \in A_0$. Then there exists an internal direct factor $X(a_0)$ of $\mathscr{A}$ such that $a_0 \in X(a_0)$ and the lattice $\ell(\mathscr{A}_0)$ is a finite chain.*

Proof. Let $\mathscr{A} = \Gamma(G, u)$. Then G is archimedean. Hence in view of [2] there exists an internal direct factor $Y(a_0)$ of G such that $a_0 \in Y(a_0)$ and $Y(a_0)$ is linearly ordered; moreover, $Y(a_0) = \{na_0\}_{n\in Z}$. Put $X(a_0) = Y(a_0) \cap A$. According to [3], $X(a_0)$ is an internal direct factor of $\mathscr{A}$. It is obvious that $\ell(X(a_0))$ is a finite chain. □

Lemma 3.5. *Let $\mathscr{A}$ be as in 3.4. Let $a_1, a_2 \in A_0$, $a_1 \neq a_2$. Then $X(a_1) \cap X(a_2) = \{0\}$.*

Proof. We apply the notation analogous to that used in the proof of 3.4. According to [2] we have

$$Y(a_1) \cap Y(a_2) = \{0\}, \quad \text{hence } X(a_1) \cap X(a_2) = \{0\}.$$

□

Lemma 3.6. *Assume that $\mathscr{A}$ is an orthogonally complete MV-algebra. Let $\{X_i\}_{i\in I}$ be a system of internal direct factors of $\mathscr{A}$ such that $X_{i(1)} \cap X_{i(2)} = \{0\}$ whenever $i(1)$ and $i(2)$ are distinct elements of I. Then $\mathscr{A}$ can be expressed as an internal direct product of the form*

$$\mathscr{A} = Y \times \prod_{i\in I} X_i. \tag{1}$$

Proof. For $i \in I$ we denote by x^i the greatest element of X_i. Hence $x^i \in B_0(\mathscr{A})$. Thus in view of 3.3 there exists $x^0 = \bigvee_{i\in I} x^i$ in $B_0(\mathscr{A})$. Put $y^0 = u - x^0$. We have $y^0 \wedge x^0 = 0$ and $y^0 \in B_0(\mathscr{A})$. Let Y be the interval $[0, y^0]$ of $\ell(\mathscr{A})$. Then according to the definition of the internal direct product, the internal direct decomposition (1) is valid. □

Lemma 3.7. *Assume that $\mathscr{A}$ is an MV-algebra which is semisimple and orthogonally complete. Let $\emptyset \neq A_0 = \{a_i\}_{i \in I}$. Then*

(i) *$\mathscr{A}$ can be expressed in the form* (1) *where $X_i = X(a_i)$ for each $i \in I$;*

(ii) *the lattice $\ell(Y)$ has no atom.*

P r o o f. The assertion (i) is a consequence of 3.4, 3.5 and 3.6. Let a be an atom of $\ell(\mathscr{A})$. Then there is $i \in I$ such that $a = a_i$. Hence in view of (1), $a \wedge y = 0$ for each $y \in Y$. Thus a does not belong to Y. □

Corollary 3.8. *Assume that $\mathscr{A}$ is as in* 3.7 *and that $0 < y \in Y$. Then the interval $[0, y]$ is infinite.*

In view of 3.7 we conclude that for proving Theorem (B) it suffices to consider the case when in the expression (1) we have $\mathscr{A} = Y$. Hence according to 3.8 we can assume that

$$(*) \qquad \operatorname{card}[0, a] \geqslant \aleph_0 \quad \text{for each } 0 < a \in A.$$

Further, it suffices to assume that $Y \neq \{0\}$.

Lemma 3.9. *Let $\mathscr{A}$ be an MV-algebra, $a \in A$, $\operatorname{card}[0, a] = \alpha \geqslant \aleph_0$. Then $\operatorname{card}[0, a + a] = \alpha$.*

P r o o f. Under the above notation we have $\mathscr{A} = \Gamma(G, u)$. Then

$$a \leqslant a \oplus a = (a + a) \wedge u \leqslant a + a,$$

whence by considering the intervals in G we get

$$[0, a] \subseteq [0, a \oplus a] \subseteq [0, a + a].$$

Thus it suffices to verify that $\operatorname{card}[0, 2a] = \alpha$.

For each $x \in [0, 2a]$ we put

$$\varphi(x) = (x \wedge a, x \vee a).$$

Since the underlying lattice $\ell(G)$ of G is distributive, φ is a monomorphism of $[0, 2a]$ into the direct product $L = [0, a] \times [a, 2a]$. We have $\operatorname{card}[a, 2a] = \operatorname{card}[0, a]$, hence $\operatorname{card} L = \alpha$. Thus $\operatorname{card}[0, 2a] \leqslant \alpha$. Since $\operatorname{card}[0, 2a] \geqslant \operatorname{card}[0, a]$, we get $\operatorname{card}[0, 2a] = \alpha$. □

We denote

$$n \cdot a = a \oplus \ldots \oplus a \quad (n\text{-times}).$$

From 3.9 we obtain by induction

Lemma 3.10. *Under the assumptions of Lemma* 3.9 *we have* $\operatorname{card}[0, n \cdot a] = \alpha$ *for each positive integer* n.

Lemma 3.11. *Assume that* $\mathscr{A}$ *is an* MV*-algebra belonging to* $\mathscr{C}$. *Suppose condition* $(*)$ *holds. Let* $0 < a \in A$. *Then there exist elements* a_n, b_n $(n = 1, 2, \ldots)$ *of* A *such that*

(i) *both the indexed systems* $(a_n)_{n\in N}$ *and* $(b_n)_{n\in N}$ *are orthogonal;*
(ii) *for each* $n \in N$, $a_n \leqslant b_n$ *and there exists* $k_n \in N$ *with* $b_n \leqslant k_n a_n$;
(iii) *for each* $n \in N$, b_n *is a boolean element in* $\mathscr{A}$;
(iv) $a = \bigvee\limits_{n\in N} a_n$.

Proof. This is a consequence of the construction given in Sections 3 and 4 of [6] (in [6], the symbol u_n was used instead of b_n). □

Lemma 3.12. *Under the notation as in* 3.11 *we have* $\operatorname{card}[0, a_n] = \operatorname{card}[0, b_n]$ *for each* $n \in N$.

Proof. This follows from 3.10 and 3.11, (ii). □

4. Proof of (B)

First assume that $\mathscr{A}$ is an MV-algebra belonging to $\mathscr{C}$ such that $A \neq \{0\}$ and that condition $(*)$ from Section 3 is satisfied.

We recall that in view of 3.2, the Boolean algebra $B_0(\mathscr{A})$ is complete.

By a simple construction using Axiom of Choice and a transfinite induction argument we obtain

Lemma 4.1. *Let* $y_0 \in B_0(\mathscr{A})$, $\emptyset \neq Y = \{y_i\}_{i\in I} \subseteq B_0(\mathscr{A})$, $y_0 = \sup Y$ *in* $B_0(\mathscr{A})$. *Then for each* $i \in I$ *there exists* $y_i' \in B_0(\mathscr{A})$ *such that* $y_i' \leqslant y_i$, $y_0 = \bigvee\limits_{i\in I} y_i'$ *in* $B_0(\mathscr{A})$ *and the indexed system* $\{y_i'\}_{i\in I}$ *is orthogonal.*

Lemma 4.2. *Assume that* $(y_i)_{i\in I}$ *is an orthogonal indexed system of elements of* $B_0(\mathscr{A})$. *Let* $y_0 \in B_0(\mathscr{A})$ *and let the relation* $y_0 = \bigvee\limits_{i\in I} y_i$ *be valid in* $B_0(\mathscr{A})$. *Then this relation holds also in the lattice* $\ell(\mathscr{A})$.

Proof. According to the definition of the internal direct product we have an internal direct product decomposition

$$\mathscr{A}_1 = \prod_{i\in I} \mathscr{A}_1, \tag{1}$$

where $\mathscr{A}_i = [0, y_i]_{\mathscr{A}}$ for each $i \in I$, and $\mathscr{A}_1 = [0, y_0]_{\mathscr{A}}$. By way of contradiction, suppose that the relation $y_0 = \bigvee_{i \in I} y_i$ fails in $\ell(\mathscr{A})$. Then there exists $a \in A_1$ such that $a \geqslant y_i$ for each $i \in I$ and $a < y_0$. For $i \in I$ let a_i be the component of a in $\mathscr{A}_i$. Hence $a_i = a \wedge y_i = a$ for each $i \in I$. There exists $x \in A_1$ with $y_0 = a + x = a \oplus x$, $x > 0$. Then there is $i \in I$ such that $x_i > 0$, where x_i is the component of x in $\mathscr{A}_i$.

In view of [4], there exists an internal direct factor G_i of the lattice-ordered group G_1 (we suppose that $\mathscr{A}_0 = \Gamma(G_1, u_1)$) such that $A_i = G_i \cap A_1$. For each $z \in A_i$, the component of z in G_i is the same as the component of z in $\mathscr{A}_i$. Thus we obtain

$$y_i = (y_0)_i = a_i + x_i = a + x_i \geqslant y_i + x_i > y_i.$$

We have arrived at a contradiction. □

Now, 4.1 and 4.2 yield

Lemma 4.3. *Let $y_0 \in B_0(\mathscr{A})$ and $\{y_i\}_{i \in I} \subseteq B_0(\mathscr{A})$. If the relation $y_0 = \sup\{y_i\}_{i \in I}$ holds with respect to $B_0(\mathscr{A})$, then this relation holds also with respect to $\ell(\mathscr{A})$.*

Let $0 < a \in A$. Let a_n and b_n $(n \in N)$ be as in 3.11. Since $B_0(\mathscr{A})$ is complete, there exists $b \in B_0(\mathscr{A})$ such that the relation $b = \bigvee_{n \in N} b_n$ is valid in $B_0(\mathscr{A})$. According to 4.3, this relation is valid also in $\ell(\mathscr{A})$.

Lemma 4.4. *Under the above notation, $a \leqslant b$.*

P r o o f. This is a consequence of 3.11 (ii) and (iv). □

Let α be an infinite cardinal, $\alpha \leqslant \operatorname{card} A$. Put

$$A(\alpha) = \{a \in A \colon \operatorname{card}[0, a] \leqslant \alpha\},$$
$$B_0(\alpha) = \{b \in B_0(\mathscr{A}) \colon \operatorname{card}[0, b] \leqslant \alpha\}.$$

Since $B_0(\alpha) \subseteq B_0(\mathscr{A})$ and since $B_0(\mathscr{A})$ is complete there exists $b(\alpha) \in B_0(\mathscr{A})$ such that

$$b(\alpha) = \sup B_0(\alpha) \tag{2}$$

is valid in $B_0(\mathscr{A})$; in view of 4.3, the relation (2) holds also in $\ell(\mathscr{A})$.

We have $B_0(\alpha) \subseteq A(\alpha)$. Let $0 < a \in A(\alpha)$. Further, let a_n and b_n $(n \in N)$ be as in 3.11. According to 3.11 and 3.12, $b_n \in B_0(\alpha)$ for each $n \in N$. Let b be as above. Then $b \leqslant b(\alpha)$. Hence in view of 4.4 we obtain

$$b(\alpha) = \sup A(\alpha). \tag{3}$$

For $\alpha = \aleph_0$ we put $a^1(\alpha) = b(\alpha)$. Let $\alpha > \aleph_0$; we denote by $\overline{\alpha}$ the set of all infinite cardinals α_1 with $\alpha_1 < \alpha$. For each $\alpha_1 \in \overline{\alpha}$ we have $b(\alpha_1) \in B_0(\mathscr{A})$, hence in view of the completeness of $B_0(\mathscr{A})$ there exists $b^1(\alpha) \in B_0(\mathscr{A})$ such that $b^1(\alpha) = \sup\limits_{\alpha_1 \in \overline{\alpha}} b(\alpha_1)$ in $B_0(\mathscr{A})$. According to 4.3, this relation holds also in $\ell(\mathscr{A})$. We put

$$a^1(\alpha) = b(\alpha) - b^1(\alpha),$$

where the operation $-$ is taken with respect to the lattice ordered group G. Then, in fact, $a^1(\alpha)$ belongs to $B_0(\mathscr{A})$ and

$$a^1(\alpha) \wedge b^1(\alpha) = 0, \quad a^1(\alpha) \vee b^1(\alpha) = b(\alpha). \tag{4}$$

This yields

Lemma 4.5. *Let α_1 be an infinite cardinal, $\alpha_1 > \alpha$. Then $a^1(\alpha) \wedge a^1(\alpha_1) = 0$.*

For each cardinal α we denote by $K_0(\alpha)$ the set of all infinite cardinals β with $\beta \leqslant \alpha$. Put $\alpha_0 = \operatorname{card} A$, $K_0 = K_0(\alpha_0)$.

Lemma 4.6. *For each $\alpha \in K_0$ we have*

$$b(\alpha) = \bigvee_{\alpha_1 \in K_0(\alpha)} a^1(\alpha_1). \tag{5}$$

Proof. We proceed by transfinite induction. Let $\alpha = \aleph_0$. Then $a^1(\alpha) = b(\alpha)$, whence (5) is valid.

Let $\alpha > \aleph_0$ and suppose that (5) holds for all infinite cardinals $\alpha' < \alpha$. In view of (4) we get

$$b(\alpha) = a^1(\alpha) \vee b^1(\alpha) = a^1(\alpha) \vee \bigvee_{\alpha_2 \in \overline{\alpha}} b(\alpha_2).$$

In view of the induction hypothesis we have

$$b(\alpha_2) = \bigvee_{\alpha_1 \in K_0(\alpha_2)} a^1(\alpha_1)$$

for each $\alpha_2 \in \overline{\alpha}$; thus we obtain

$$\bigvee_{\alpha_2 \in \overline{\alpha}} b(\alpha_2) = \bigvee_{\aleph_0 \leqslant \alpha_1 < \alpha} a^1(\alpha_1).$$

Hence the relation (5) is valid. □

Since $b(\alpha_0) = u$ we infer

Corollary 4.7. $u = \bigvee_{\alpha\in K_0} a^1(\alpha)$.

From the relation $u \neq 0$ we get that the set $K_1 = \{\alpha \in K_0 \colon a^1(\alpha) \neq 0\}$ is nonempty; hence we have

$$u = \bigvee_{\alpha\in K_1} a^1(\alpha). \tag{6}$$

Let $\alpha \in K_1$. Since $a^1(\alpha) \leqslant b(\alpha)$ and $b(\alpha) = \sup B(\alpha)$ we obtain

$$a_1(\alpha) = \sup_{t\in B(\alpha)} (a^1(\alpha) \wedge t).$$

Since $a_1(\alpha) \in B_0(\mathscr{A})$ and $B(\alpha) \subseteq B_0(\mathscr{A})$ we conclude that the elements $a^1(\alpha) \wedge t$ belong to $B_0(\mathscr{A})$. Then in view of 4.1 there exists an orthogonal indexed set $(x_{\alpha,s})_{s\in S(\alpha)}$ of nonzero elements of $B_0(\mathscr{A})$ such that

$$a_1(\alpha) = \bigvee_{s\in S(\alpha)} x_{\alpha,s} \tag{7}$$

and for each $x \in S(\alpha)$ there exists $t \in B(\alpha)$ with $x_{\alpha,s} \leqslant a^1(\alpha) \wedge t$. In view of (6) and (7) we get

$$u = \bigvee_{\alpha\in K_1} \bigvee_{s\in S(\alpha)} x_{\alpha,s}. \tag{8}$$

Moreover, the indexed system $(x_{\alpha,s})_{\alpha\in K_1, s\in S(\alpha)}$ is orthogonal. From this fact and from the definition of the internal direct product we obtain

Lemma 4.8. *The MV-algebra $\mathscr{A}$ can be expressed as an internal direct product*

$$\mathscr{A} = \prod_{\alpha\in K_1, s\in S(\alpha)} A_{\alpha,s},$$

where $\mathscr{A}_{\alpha,s} = [0, x_{\alpha,s}]_{\mathscr{A}}$.

Lemma 4.9. *Let $\alpha \in K_1$ and $s \in S(\alpha)$. Then the MV-algebra $\mathscr{A}_{\alpha,s}$ is weakly homogeneous.*

P r o o f. Let $0 < a \in A_{\alpha,s}$. Then $a \in B(\alpha)$, whence $\operatorname{card}[0,a] \leqslant \alpha$. By way of contradiction, assume that $\operatorname{card}[0,a] = \alpha_1 < \alpha$. This yields that $a \in B(\alpha_1)$, thus $a \leqslant b^1(\alpha)$. Hence $a \wedge a^1(\alpha) = 0$. But from $a \in A_{\alpha,s}$ we get $a \leqslant a^1(\alpha)$, thus $a \wedge a^1(\alpha) = a$ and we have arrived at a contradiction. Therefore $\operatorname{card}[0,a] = \alpha$ for each $0 < a \in A_{\alpha,s}$. □

From 4.8 and 4.9 we conclude

Proposition 4.10. *Let $\mathscr{A}$ be an MV-algebra belonging to the class $\mathscr{C}$, $A \neq \{0\}$. Assume that the condition $(*)$ from Section 3 is valid. Then $\mathscr{A}$ can be expressed as an internal direct product of weakly homogeneous MV-algebras.*

Now let us omit the assumption on the validity of the condition $(*)$.

P r o o f of Theorem B. The assertion of this theorem is a consequence of 3.6 and 4.10. □

5. Concluding remarks on the weak homogeneity

Assume that $\mathscr{A}$ is an MV-algebra such that the lattice $\ell(\mathscr{A})$ is a Boolean algebra; we put $\ell(\mathscr{A}) = B$.

Each internal direct product decomposition of $\mathscr{A}$ defines, at the same time, an internal direct product decomposition of B, and conversely.

If B is complete, then in view of 3.2, $\mathscr{A}$ belongs to the class $\mathscr{C}$.

If $X \neq \{0\}$ is a finite linearly ordered internal direct factor of B, then $\operatorname{card} X = 2$. Thus if B is complete then in the relation (1) of 3.6 either $I = \emptyset$ or each X_i is a two-element MV-algebra. Since each two-element MV-algebra is weakly homogeneous, from (B) we infer that each complete MV-algebra is an internal direct product of weakly homogeneous MV-algebras. Thus we have verified that Theorem (B) is a generalization of Theorem (A).

For an MV-algebra $\mathscr{A}$ we denote by $(s), (oc)$ and (p) the condition that $\mathscr{A}$ is semisimple, orthogonally complete or projectable, respectively. These conditions were used in the definition of the class $\mathscr{C}$, hence they are assumed to be valid in Theorem (B).

Example 1. This example shows that the condition (s) cannot be omitted in Theorem (B).

Let $\mathbb{Z}$ and $\mathbb{R}$ be the additive group of all integers or of all reals, respectively, with the natural linear order. Consider the lexicographic product $G = \mathbb{R} \circ \mathbb{Z}$ and put $u = (1, 0)$. Then u is a strong unit of G, thus we can construct the MV-algebra $\mathscr{A} = \Gamma(G, u)$. This MV-algebra is linearly ordered, hence it has the properties (oc) and (p). It is easy to verify that it has not the property (s). Further, being linearly ordered, it is directly indecomposable. Put $a = [0, 1]$. We have $\operatorname{card}[0, a] = 2$ and the set A is infinite. Hence $\mathscr{A}$ cannot be represented in the form from Theorem (B).

Example 2. Let M be an infinite set. For a mapping $f\colon M \to \{0, 1\}$ we put

$$M_1(f) = \{i \in M\colon f(i) = 1\}, \quad M_2(f) = M \setminus M_1(f).$$

We denote by B the set of all f having the property that either $M_1(f)$ is finite or $M_2(f)$ is finite. The set B is partially ordered coordinate-wise. Then B is a Boolean algebra. Hence there is an MV-algebra $\mathscr{A}$ with $\ell(\mathscr{A}) = B$. This MV-algebra has the properties (s) and (p) (cf. Section 3). On the other hand, $\mathscr{A}$ has not the property (oc).

Let $0 < f \in B$. Then the interval $[0, f]$ of B is weakly homogeneous if and only if f is an atom of B. It is easy to verify that $\mathscr{A}$ cannot be represented as a direct product of two-element MV-algebras. Thus the condition (oc) cannot be omitted in Theorem (B).

The question whether the condition (p) can be omitted in Theorem (B) remains open.

6. A homogeneity condition defined by an increasing cardinal property

Pierce [7] defined a cardinal property f on Boolean algebras as a rule that assigns to each Boolean algebra B a cardinal $f(B)$ such that, whenever B_1 and B_2 are isomorphic Boolean algebras, then $f(B_1) = f(B_2)$. Cf. also Pierce [8].

Analogously we can define the notion of a cardinal property for other types of ordered algebraic structures. Cardinal properties and generalized cardinal properties on lattice ordered groups were studied in [3], [5].

Let f be a cardinal property on the class $\mathscr{B}$ of all Boolean algebras. A Boolean algebra B is *homogeneous with respect* to f (shortly: f-homogeneous) if $f(B) = f([0, b])$ for each $0 < b \in B$. We say that f is *increasing* (or *monotone*, cf. Sikorski [9]) if for each $B \in \mathscr{B}$ and each $0 < b \in B$ the relation $f(B) \geqslant f([0, b])$ is valid.

Theorem ($\mathbf{A}_1$) (Cf. Pierce [7]). *Let $B \neq \{0\}$ be a complete Boolean algebra. Let f be an increasing cardinal property on the class $\mathscr{B}$. Then B can be represented as a direct product of f-homogeneous Boolean algebras.*

The above Boolean algebraic definitions can be straighthforwardly adapted to MV-algebras. As above, let $\mathscr{M}$ be the class of all MV-algebras. For $\mathscr{A} \in \mathscr{M}$ let $\ell(\mathscr{A})$ be the corresponding lattice. Let f be a cardinal property on $\mathscr{M}$. Thus for each $\mathscr{A} \in \mathscr{M}$, $f(\mathscr{A})$ is a cardinal such that, whenever $\mathscr{A}_1, \mathscr{A}_2 \in \mathscr{M}$ and $\ell(\mathscr{A}_1) \simeq \ell(\mathscr{A}_2)$, then $f(\mathscr{A}_1) = f(\mathscr{A}_2)$. An MV-algebra $\mathscr{A}$ is *f-homogeneous* if $f(\mathscr{A}) = f([0, a]_{\mathscr{A}})$ for each $0 < a \in A$. The cardinal property f is *increasing (decreasing)* if for each $\mathscr{A} \in \mathscr{M}$ and each $0 < a \in A$ the relation $f([0, a]_{\mathscr{A}}) \leqslant f(\mathscr{A})$ (or $f([0, a]_{\mathscr{A}}) \geqslant f(\mathscr{A})$, respectively) is valid.

Decreasing cardinal properties (and, also, decreasing generalized cardinal properties) on the class $\mathscr{M}$ were dealt with in [6].

Consider the following condition for a cardinal property f on $\mathscr{M}$:

(γ) For each $\mathscr{A} \in \mathscr{M}$ and for each $0 < a \in A$, if $f([0,a]_{\mathscr{A}})$ is infinite, then $f([0,a]_{\mathscr{A}}) = f([0, a \oplus a]_{\mathscr{A}})$.

In view of 3.9, the cardinal property f defined by $f(\mathscr{A}) = \operatorname{card} \mathscr{A}$ for each $\mathscr{A} \in \mathscr{M}$ satisfies the condition (γ).

Theorem ($\mathbf{B}_1$). *Let $\mathscr{A}$ be an MV-algebra belonging to the class $\mathscr{C}$. Let f be an increasing cardinal property on $\mathscr{M}$ satisfying the condition (γ). Then $\mathscr{A}$ can be represented as an internal direct product of MV-algebras $\mathscr{A}_i$ ($i \in I$) such that for each $i \in I$ some of the following conditions is satisfied:*

(i) *$\mathscr{A}_i$ is a finite chain;*

(ii) *$\mathscr{A}_i$ is f-homogeneous.*

When presenting the proof of (A), Sikorski ([9], p. 107) remarks that the proof of (A_1) is the same as the proof of (A).

Analogously, for proving (B_1) it suffices to apply minor modifications in the proof of (B).

Proof of (B_1). Let $\mathscr{A}$ be an MV-algebra belonging to the class $\mathscr{C}$. Further, let f be an increasing cardinal property on $\mathscr{M}$ satisfying the condition (γ).

First let us suppose that for each $0 < a \in \mathscr{A}$, the interval $[0,a]$ is infinite (i.e., we suppose that the condition ($*$) is satisfied). We apply the same steps as in 4.4–4.9 with the distinction that

(i) if $0 < x \in A$, then instead of considering the cardinal

$$\operatorname{card}[0,x]$$

we consider the cardinal

$$f([0,x]_{\mathscr{A}});$$

(ii) instead of applying Lemma 3.9 we apply the assumption on the validity of the condition (γ);

(iii) instead of speaking about weak homogeneity we speak about f-homogeneity.

In this way we obtain that the assertion analogous to Proposition 4.10 is valid; the distinction is in the fact that the weak homogeneity of direct factors is replaced by f-homogeneity. The next step consists in omitting the condition on the validity of ($*$); similarly to Section 4, it suffices to apply 3.6 and the assertion analogous to 4.10. This completes the proof. □

Also, by the same argument as in Section 5, we obtain that (B_1) is a generalization of (B).

We conclude by the following example.

For each MV-algebra $\mathscr{A}$ let $\mathscr{X}(\mathscr{A})$ be the set of all chains X which are sublattices of the lattice $\ell(\mathscr{A})$. We put

$$f_1(\mathscr{A}) = \max\{\aleph_0, \sup\{\operatorname{card} X\}_{X\in\mathscr{X}(\mathscr{A})}\}.$$

Then f_1 is an increasing cardinal property on $\mathscr{M}$; f_1 satisfies the condition (γ). Hence the assertion of (B_1) can be applied for f_1.

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