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NODAL SOLUTIONS FOR A SECOND-ORDER m-POINT BOUNDARY VALUE PROBLEM

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Abstract. We study the existence of nodal solutions of the m-point boundary value problem

$$u'' + f(u) = 0, \quad 0 < t < 1,$$
$$u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i)$$

where $\eta_i \in \mathbb{Q}$ $(i = 1, 2, \ldots, m-2)$ with $0 < \eta_1 < \eta_2 < \ldots < \eta_{m-2} < 1$, and $\alpha_i \in \mathbb{R}$ $(i = 1, 2, \ldots, m-2)$ with $\alpha_i > 0$ and $0 < \sum_{i=1}^{m-2} \alpha_i < 1$. We give conditions on the ratio $f(s)/s$ at infinity and zero that guarantee the existence of nodal solutions. The proofs of the main results are based on bifurcation techniques.

Keywords: multiplicity results, eigenvalues, bifurcation methods, nodal zeros, multi-point boundary value problems

MSC 2000: 34B10, 34G20

1. Introduction

Recently, the existence and multiplicity of positive solutions of the m-point boundary value problem

$$u'' + h(t)f(u) = 0,$$
$$u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i)$$

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have also been studied by several authors, see Ma [4] and Webb [11] for some references. However research for existence of nodal solutions of multi-point boundary value problems has proceeded very slowly. To the best of our knowledge, no results on the existence of nodal solutions have been established for multi-point boundary value problems. The likely reason is that the *spectrum structure* of the linear problem

$$u'' + \lambda u = 0, \quad u \in D(L), \tag{1.1}$$

$$u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \tag{1.2}$$

is not clear.

It is the purpose of this paper to study the *spectrum structure* of (1.1), (1.2), and investigate the existence and multiplicity of nodal solutions of

$$u'' + f(u) = 0, \quad 0 < t < 1, \tag{1.3}$$

$$u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i). \tag{1.4}$$

We make the following assumptions:

(C0) $\eta_i = p_i/q_i \in \mathbb{Q} \cap (0,1)$ $(i = 1, \dots, m-2)$ with $p_i, q_i \in \mathbb{N}$ and $(p_i, q_i) = 1$;

(C1) $\alpha_i \in (0, \infty)$, $(i = 1, 2, \dots, m-2)$ with $0 < \sum_{i=1}^{m-2} \alpha_i < 1$;

(C2) $f \in C^1(\mathbb{R}, \mathbb{R})$ with $sf(s) > 0$ for $s \neq 0$, and $f_0, f_\infty \in (0, \infty)$ exist, where

$$f_0 = \lim_{s \to 0} \frac{f(s)}{s}, \quad f_\infty = \lim_{s \to \infty} \frac{f(s)}{s}.$$

Here $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{N}$ are the sets of rational, real, and natural numbers, respectively.

We give conditions on the ratio $f(s)/s$ at infinity and zero that guarantee the existence of nodal solutions. The main tool we use is the bifurcations theory of Rabinowitz [7].

For the results on the existence and multiplicity of positive solutions and nodal solutions of second-order and higher-order two-point boundary value problems, see Ambrosetti and Hess [1], Erbe and Wang [3], Ma and Thompson [5], Naito and Tanaka [6], Rabinowitz [7], Ruf and Srikanth [9], Rynne [10] and the references therein. For the results on the existence of sign-changing solutions of elliptic problems and m-point boundary value problems for ordinary differential equations, see Castro, Drábek and Neuberger [2] and Xu [12], respectively.

For a set $D \subset \mathbb{R}$, we denote by $\#D$ the number of elements in D.

The rest of the paper is organized as follows: In Section 2, we define an auxiliary function $\Gamma(s)$ and prove some elementary properties of $\Gamma(s)$ which will be needed in the study of the spectrum of multi-point boundary value problems. Section 3 studies the linear eigenvalue problem (1.1), (1.2), and we will describe the distribution of $\{\lambda_n\}$. In Section 4, (1.1), (1.2) is reduced to an equivalent integral equation, and there we prove a result on the algebraic multiplicity of the eigenvalue of the corresponding integral operator. Finally in Section 5, we state and prove the main results.

2. Elementary properties of $\Gamma(s)$

Set

$$\Gamma(s) = \cos(s) - \sum_{i=1}^{m-2} \alpha_i \cos(\eta_i s). \tag{2.1}$$

Lemma 2.1. *Let* (C0) *hold. Then* $\Gamma(s)$ *is a periodic function.*

Proof. Let

$$\hat{q} = q_1 \dots q_{m-2}.$$

We show that $\Gamma(s)$ is a $2\hat{q}\pi$-periodic function. Using the facts that $\cos(s+2\pi) = \cos(s)$ and $\cos \eta_i(s + 2\pi q_i/p_i) = \cos(\eta_i s)$ and $\eta_i \hat{q} \in \mathbb{N}$, we conclude that

$$\begin{aligned} \Gamma(s + 2\hat{q}\pi) &= \cos(s + 2\hat{q}\pi) - \sum_{i=1}^{m-2} \alpha_i \cos(\eta_i(s + 2\hat{q}\pi)) \\ &= \cos(s) - \sum_{i=1}^{m-2} \alpha_i \cos(\eta_i s + 2\eta_i \hat{q}\pi)) \\ &= \cos(s) - \sum_{i=1}^{m-2} \alpha_i \cos(\eta_i s) = \Gamma(s). \end{aligned}$$

This completes the proof of the Lemma. □

Let

$$q^* = \min\{\hat{q} \in \mathbb{N} : \Gamma(s + 2\hat{q}\pi) = \Gamma(s),\ \forall\, s \in \mathbb{R}\}. \tag{2.2}$$

Then

$$q^* \leqslant q_1 \dots q_{m-2}. \tag{2.3}$$

Lemma 2.2. *Let* (C0) *and* (C1) *hold. Then*

$$\Gamma(s) = 0 \tag{2.4}$$

has a solution in $(0, \frac{\pi}{2})$.

P r o o f. Since

$$\Gamma(0) = \cos 0 - \sum_{i=1}^{m-2} \alpha_i \cos \eta_i 0 > 0$$

and

$$\Gamma\Big(\frac{\pi}{2}\Big) = 0 - \sum_{i=1}^{m-2} \alpha_i \cos \frac{\pi \eta_i}{2} < 0$$

we see that

$$\Gamma(\tau) = 0, \quad \text{for some } \tau \in \Big(0, \frac{\pi}{2}\Big).$$

This completes the proof of the lemma. □

Set

$$A := \Big\{ s\colon\ s > 0,\ \cos s = \sum_{i=1}^{m-2} \alpha_i \cos \eta_i s \Big\}. \tag{2.5}$$

Lemma 2.3. *Let* (C0) *and* (C1) *hold. Then the set* A *is infinite.*

P r o o f. This is an immediate consequence of Lemma 2.1 and 2.2. □

Lemma 2.4. *Let* (C0) *and* (C1) *hold. Then there is no* $\{s_n\} \in A$ *with* $s_i \neq s_j$ $(i \neq j)$*, such that*

$$\lim_{n\to\infty} s_n = a, \quad \textit{for some } a \in \mathbb{R}.$$

P r o o f. Suppose on the contrary that there exists $\{s_n\} \subseteq A$ with $s_i \neq s_j$ $(i \neq j)$, such that

$$\lim_{n\to\infty} s_n = a, \quad \text{for some } a \in \mathbb{R}.$$

We may assume that

$$s_1 < s_2 < \ldots < s_n < \ldots < a.$$

By Rolle's Theorem, there exist $s_i^{(1)} \subset (s_i, s_{i+1})$ such that

$$\Gamma'(s_i^{(1)}) = 0$$

and consequently

$$\Gamma'(a) = \lim_{n\to\infty} \Gamma'(s_i^{(1)}) = 0.$$

Similarly we have that for each $n \in \mathbb{N}$

$$\Gamma^{(n)}(a) = 0.$$

Combining this with the Taylor Formula for Γ at $s = a$ and using the fact that

$$|\Gamma^{(n)}(s)| \leqslant 2, \quad s \in \mathbb{R}$$

we conclude that

$$\Gamma(s) \equiv 0, \quad s \in \mathbb{R}$$

which contradicts (2.1). This completes the proof of the lemma. □

Now we can arrange the elements of the set A as follows:

$$s_1 < s_2 < \ldots < s_n < \ldots. \tag{2.6}$$

Lemma 2.5. *Let* (C0) *and* (C1) *hold, and let*

$$s_1 < s_2 < \ldots < s_n < \ldots$$

be the sequence of the elements of A. Let

$$l = \#\{t\colon\ \Gamma(t) = 0,\ t \in (0, 2q^*\pi]\}. \tag{2.7}$$

Then for each $n = kl + j$ with $k \in \mathbb{N} \cup \{0\}$ and $j \in \{1, \ldots, l\}$

$$s_{kl+j} = 2kq^*\pi + s_j. \tag{2.8}$$

P r o o f. Lemma 2.4 yields that l is finite. (2.8) can be directly deduced from Lemma 2.1. □

Lemma 2.6. *Let* (C0) *and* (C1) *hold. Then*

$$s_1 < \frac{\pi}{2}.$$

P r o o f. This is an immediate consequence of Lemma 2.2. □

Lemma 2.7. *Let* (C0) *and* (C1) *hold. Then*

$$s_2 > \frac{\pi}{2}.$$

P r o o f. Suppose on the contrary that $0 < s_2 \leqslant \frac{1}{2}\pi$. Then $\Gamma(s_1) = \Gamma(s_2) = 0$ implies that

$$\Gamma'(\tau) = 0, \quad \text{for some } \tau \in (s_1, s_2). \tag{2.9}$$

However

$$\Gamma'(s) = -\sin s + \sum_{i=1}^{m-2} \alpha_i \eta_i \sin(\eta_i s) < 0, \quad s \in \Big(0, \frac{\pi}{2}\Big).$$

This contradicts (2.9). □

3. Linear eigenvalue problems

Lemma 3.1. *Let* (C0) *and* (C1) *hold. Let* q^* *and* l *be as in* (2.2) *and* (2.7), *respectively. Assume that the sequence of positive solutions of* $\Gamma(s) = 0$ *is*

$$s_1 < s_2 < \ldots < s_n < \ldots \tag{3.1}$$

Then

(1) *The sequence of positive eigenvalues of* (1.1), (1.2) *is exactly given by*

$$\lambda_n = s_n^2, \quad n = 1, 2, \ldots; \tag{3.2}$$

(2) *For each* $n \in \mathbb{R}$, *the eigenfunction corresponding to* λ_n *is*

$$\varphi_n(t) = \cos\big(\sqrt{\lambda_n}\, t\big); \tag{3.3}$$

(3) *For each* $n = kl + j$ *with* $k \in \mathbb{N}$ *and* $j \in \{1, \ldots, l\}$,

$$\sqrt{\lambda_{lk+j}} = 2kq^*\pi + \sqrt{\lambda_j}. \tag{3.4}$$

P r o o f. It is easy to check that $\lambda \in (0, \infty)$ is an eigenvalue of (1.1), (1.2) if and only if

$$\Gamma\big(\sqrt{\lambda}\big) = 0.$$

Hence the desired results follow from Lemmas 2.1–2.7. The proof is completed. □

Let

$$Z_n = \{t \in (0,1)\colon\ \cos\big(\sqrt{\lambda_n}\, t\big) = 0\} \tag{3.5}$$

and let

$$\mu_n := \# Z_n \tag{3.6}$$

which is the number of elements in Z_n.

Lemma 3.2. *Let* (C0) *and* (C1) *hold. Then for each* $k \in \mathbb{N}$,

$$\mu_{kl+1} < \mu_{kl+2}. \tag{3.7}$$

Proof. By (3.4), we only need to show that

$$\mu_1 < \mu_2. \tag{3.8}$$

Using Lemma 2.6 and 2.7, we conclude that $\mu_1 = 0$ and $\mu_2 \geqslant 1$.

Example 3.1. Let's consider the linear three-point problem

$$u'' + \lambda u = 0, \quad 0 < t < 1, \tag{3.9}$$

$$u'(0) = 0, \quad u(1) = \frac{1}{2} u\Big(\frac{1}{4}\Big). \tag{3.10}$$

It is easy to check that

$$\Gamma(s) = \cos s - \frac{1}{2}\cos\Big(\frac{s}{4}\Big)$$

is a 8π-periodic function, and consequently,

$$q^* = 4.$$

Moreover Γ has exactly eight zeros in $(0, 8\pi]$. They are

$$\begin{aligned} s_1 &\doteq 1.06752, & s_2 &\doteq 4.88453,\\ s_3 &\doteq 8.07192, & s_4 &\doteq 10.5429,\\ s_5 &\doteq 14.5898, & s_6 &\doteq 17.0608,\\ s_7 &\doteq 20.2482, & s_8 &\doteq 24.0652, \end{aligned}$$

and accordingly $l = 8$, and

$$\begin{aligned} \mu_1 &= 0, & \mu_2 &= 2, \\ \mu_3 &= 3, & \mu_4 &= 3, \\ \mu_5 &= 5, & \mu_6 &= 5, \\ \mu_7 &= 6, & \mu_8 &= 8. \end{aligned}$$

Clearly

$$\mu_1 < \mu_2 < \mu_3, \quad \mu_6 < \mu_7 < \mu_8. \tag{3.11}$$

Γ has exactly eight zeros in $(8\pi, 16\pi]$. They are

$$\begin{aligned} s_9 &\doteq 26.2003, & s_{10} &\doteq 30.0173, \\ s_{11} &\doteq 33.2047, & s_{12} &\doteq 35.6756, \\ s_{13} &\doteq 39.7225, & s_{14} &\doteq 42.1935, \\ s_{15} &\doteq 45.3809, & s_{16} &\doteq 49.1979. \end{aligned}$$

Example 3.2. Let's consider the linear three-point problem

$$u'' + \lambda u = 0, \quad 0 < t < 1, \tag{3.12}$$

$$u'(0) = 0, \quad u(1) = u(\eta) \tag{3.13}$$

where $\eta \in (0,1)$ is given. A simple computation yields that λ is a real eigenvalue of (1.1), (1.2) if and only if

$$\lambda \in \Big\{ \Big(\frac{2k\pi}{1+\eta}\Big)^2 \colon k = 0, 1, \dots \Big\} \cup \Big\{ \Big(\frac{2k\pi}{1-\eta}\Big)^2 \colon k = 0, 1, \dots \Big\} \tag{3.14}$$

and the eigenfunction corresponding to λ_n is

$$\varphi_n(t) = \cos\big(\sqrt{\lambda_n}\, t\big).$$

If we take $\eta = \frac{1}{2}$, then

$\lambda_1 = 0^2$, $\varphi_1(t) = 1$ has no zero in $(0,1)$;
$\lambda_2 = \big(\frac{4}{3}\pi\big)^2$, $\varphi_2(t) = \cos \frac{4}{3}\pi t$ has 1 zero $\frac{3}{8}$ in $(0,1)$;
$\lambda_3 = \big(\frac{8}{3}\pi\big)^2$, $\varphi_3(t) = \cos \frac{8}{3}\pi t$ has 3 zeros $\frac{3}{16}, \frac{9}{16}, \frac{15}{16}$ in $(0,1)$;

$\lambda_4 = (4\pi)^2$, $\varphi_4(t) = \cos 4\pi t$ has 4 zeros $\frac{1}{8}, \frac{8}{3}, \frac{5}{8}, \frac{7}{8}$ in $(0,1)$;
$\lambda_5 = \left(\frac{16}{3}\pi\right)^2$, $\varphi_5(t) = \cos \frac{163}{\pi} t$ has 5 zeros $\frac{3}{32}, \frac{9}{32}, \frac{15}{32}, \frac{21}{32}, \frac{27}{32}$ in $(0,1)$;
$\lambda_6 = \left(\frac{20}{3}\pi\right)^2$, $\varphi_6(t) = \cos \frac{20}{3}\pi t$ has 7 zeros $\frac{3}{40}, \frac{9}{40}, \frac{15}{40}, \frac{21}{40}, \frac{27}{40}, \frac{33}{40}, \frac{39}{40}$ in $(0,1)$;
$\lambda_7 = (8\pi)^2$, $\varphi_7(t) = \cos(8\pi t)$ has 8 zeros $\frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \frac{7}{16}, \frac{9}{16}, \frac{11}{16}, \frac{13}{16}, \frac{15}{16}$ in $(0,1)$;
$\lambda_8 = \left(\frac{28}{3}\pi\right)^2$, $\varphi_8(t) = \cos \frac{28}{3}\pi t$ has 9 zeros $\frac{3}{56}, \frac{9}{56}, \frac{15}{56}, \frac{21}{56}, \frac{27}{56}, \frac{33}{56}, \frac{39}{56}, \frac{45}{56}, \frac{51}{56}$ in $(0,1)$;
$\lambda_9 = \left(\frac{32}{3}\pi\right)^2$, $\varphi_9(t) = \cos \frac{32}{3}\pi t$ has 11 zeros $\frac{3}{64}, \frac{9}{64}, \frac{15}{64}, \frac{21}{64}, \frac{27}{64}, \frac{33}{64}, \frac{39}{64}, \frac{45}{64}, \frac{51}{64}, \frac{57}{64}, \frac{63}{64}$ in $(0,1)$;
. . . .

Clearly

(i) $q^* = 2$, $\Gamma(s) = \cos s - \cos \frac{1}{2}s$ is a 4π-periodic function which has 3 zeros 0, $\frac{4}{3}\pi$, $\frac{8}{3}\pi$ in $[0, 4\pi)$, and consequently $l = 3$;
(ii) $\mu_{3k+1} < \mu_{3k+2} < \mu_{3k+3}$ for each $k \in \mathbb{N} \cup \{0\}$;
(iii) $\sqrt{\lambda_{3k+j}} = 4k\pi + \sqrt{\lambda_j}$ for $j \in \{1,2,3\}$ and $k \in \mathbb{N} \cup \{0\}$.

Example 3.3. Let's consider the linear two-point problem

$$u'' + \lambda u = 0, \quad 0 < t < 1, \tag{3.15}$$

$$u'(0) = 0, \quad u(1) = 0. \tag{3.16}$$

It is well-known that $\lambda_n = ((n - \frac{1}{2})\pi)^2$, $n = 1, 2, \ldots$, and the corresponding eigenfunction $\varphi_n(s) = \cos(n - \frac{1}{2})\pi t$ has exactly $n - 1$ simple zeros in (0,1). In this case,

(i) $\Gamma(s) = \cos s$ is a 2π-periodic function which has only 2 zeros in $[0, 2\pi)$, and consequently $l = 2$;
(ii) $\mu_{2k} < \mu_{2k+1} < \mu_{2k+2}$ for each $k \in \mathbb{N} \cup \{0\}$;
(iii) $\sqrt{\lambda_{2k+j}} = 2k\pi + \sqrt{\lambda_j}$ for $j \in \{1,2\}$ and $k \in \mathbb{N} \cup \{0\}$.

4. The algebraic multiplicity of the eigenvalue

Let $Y = C[0,1]$ with the norm

$$\|u\|_\infty = \max_{t \in [0,1]} |u(t)|.$$

Let $E = C^1[0,1]$ with the norm

$$\|u\| = \max_{t \in [0,1]} |u(t)| + \max_{t \in [0,1]} |u'(t)|.$$

Let $G(t,s)$ be the Green function for the second-order boundary value problem

$$-u''(t)=0, \quad t\in(0,1), \tag{4.1}$$

$$u'(0)=u(1)=0, \tag{4.2}$$

which is explicitly given by

$$G(t,s)=\begin{cases}1-t, & 0\leqslant s\leqslant t\leqslant 1,\\ 1-s, & 0\leqslant t\leqslant s\leqslant 1.\end{cases} \tag{4.3}$$

Define $K\colon E\to E$ by

$$(Ku)(t)=\int_0^1 G(t,s)u(s)\,\mathrm{d}s+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^1 G(\eta_i,s)u(s)\,\mathrm{d}s. \tag{4.4}$$

Set

$$H(t,s)=G(t,s)+\frac{\sum_{i=1}^{m-2}\alpha_i G(\eta_i,s)}{1-\sum_{i=1}^{m-2}\alpha_i\eta_i}, \tag{4.5}$$

then (4.4) can be rewritten as

$$(Ku)(t)=\int_0^1 H(t,s)u(s)\,\mathrm{d}s. \tag{4.6}$$

Lemma 4.1. *Let* (C0) *and* (C1) *hold. Then* (1.1), (1.2) *is equivalent to the operator equation*

$$u=\lambda Ku. \tag{4.7}$$

Moreover $K\colon E\to E$ is completely continuous.

It follows from Lemma 4.1 that λ is a *characteristic value* of K if and only if λ is a eigenvalue of (1.1), (1.2). This together with Lemma 3.1 implies that K has a strictly increasing sequence of characteristic values $\lambda_n=s_n^2$, $n=1,2,\ldots$, each with *geometric multiplicity* one (the geometric multiplicity of the characteristic values λ_n is defined to be the dimension of the subspace $\ker(I_E-\lambda_n K)$). However to apply the global bifurcation results of [7] it is necessary that the characteristic values of K have odd *algebraic multiplicity*. (The *algebraic multiplicity* of the characteristic values λ_n is defined to be the dimension of the subspace $\bigcup_{r=1}^{\infty}(\ker(I_E-\lambda_n K))^r$. See [7, p. 490].)

Lemma 4.2. *Let* (C0) *and* (C1) *hold. Assume that the sequence of positive solutions of* $\Gamma(s) = 0$ *is*

$$s_1 < s_2 < \ldots < s_n < \ldots. \tag{4.8}$$

Then the sequence of positive characteristic values of the operator K *is*

$$s_1^2 < s_2^2 < \ldots < s_n^2 < \ldots. \tag{4.9}$$

Moreover, the characteristic values s_n^2 *have algebraic multiplicity one, and the corresponding eigenfunction is*

$$\varphi_n(t) = \cos(s_n t). \tag{4.10}$$

Proof. We only need to show that

$$\ker(I - s_n^2 K) = \ker(I - s_n^2 K)^2.$$

Obviously, it is sufficient to show that

$$\ker(I - s_n^2 K)^2 \subseteq \ker(I - s_n^2 K).$$

For any $y \in \ker(I - s_n^2 K)^2$, $(I - s_n^2 K)y$ is the characteristic function of the linear operator K corresponding to the eigenvalue s_n^2 if $(I - \lambda_n K)y \neq \theta$. Then there exists a nonzero constant γ such that

$$(I - s_n^2 K)y = \gamma \cos s_n t, \quad t \in [0,1]. \tag{4.11}$$

By direct computation, we have

$$y''(t) + s_n^2 y = -s_n^2 \gamma \cos s_n t, \quad t \in [0,1], \tag{4.12}$$

$$y'(0) = 0, \quad y(1) = \sum_{i=1}^{m-2} \alpha_i y(\eta_i). \tag{4.13}$$

Since (C1) and the fact $y(1) = \sum\limits_{i=1}^{m-2} \alpha_i y(\eta_i)$ imply

$$\sum_{i=1}^{m-2} \alpha_i \min_{1 \leqslant i \leqslant m-2} y(\eta_i) \leqslant \sum_{i=1}^{m-2} \alpha_i y(\eta_i) \leqslant \sum_{i=1}^{m-2} \alpha_i \max_{1 \leqslant i \leqslant m-2} y(\eta_i),$$

we have from the fact that $y \in C[0,1]$ that there exists $\eta \in [\eta_1, \eta_{m-2}]$ such that

$$y(\eta) = \frac{\sum_{i=1}^{m-2} \alpha_i y(\eta_i)}{\sum_{i=1}^{m-2} \alpha_i}.$$

Set

$$\alpha = \sum_{i=1}^{m-2} \alpha_i; \tag{4.14}$$

then by (4.13), we get

$$y(1) = \alpha y(\eta). \tag{4.15}$$

Now (4.12), (4.13) yield

$$y''(t) + s_n^2 y = -s_n^2 \gamma \cos s_n t, \quad t \in [0,1], \tag{4.16}$$

$$y'(0) = 0, \quad y(1) = \alpha y(\eta). \tag{4.17}$$

It is easy to verify that the general solution of (4.16) is of the form

$$\begin{aligned} y(t) &= C_1 \cos s_n t + C_2 \sin s_n t \\ &\quad + \Big(\frac{-\gamma}{4} \cos 2 s_n t\Big) \cos s_n t + \Big(-\frac{s_n \gamma}{2} t - \frac{\gamma}{4} \sin 2 s_n t\Big) \sin s_n t. \end{aligned} \tag{4.18}$$

That is,

$$y(t) = C_1 \cos s_n t + C_2 \sin s_n t - \frac{\gamma}{4} \cos s_n t - \frac{s_n \gamma}{2} t \sin s_n t. \tag{4.19}$$

Applying the condition $y'(0) = 0$ and

$$\begin{aligned} y'(t) &= -s_n C_1 \sin s_n t + s_n C_2 \cos s_n t \\ &\quad + \frac{s_n \gamma}{4} \sin s_n t - \frac{s_n \gamma}{2} \sin s_n t - \frac{s_n^2 \gamma}{2} t \cos s_n t \end{aligned} \tag{4.20}$$

we obtain that $C_2 = 0$. This together with (4.19) implies that

$$y(1) = C_1 \cos s_n - \frac{\gamma}{4} \cos s_n - \frac{s_n \gamma}{2} \sin s_n \tag{4.21}$$

and

$$\alpha y(\eta) = \alpha C_1 \cos s_n \eta - \frac{\alpha \gamma}{4} \cos s_n \eta - \frac{s_n \gamma}{2} \alpha \eta \sin s_n \eta. \tag{4.22}$$

Since $y(1) = \alpha y(\eta)$ and

$$\cos s_n = \alpha \cos \eta s_n, \tag{4.23}$$

we have

$$\sin s_n = \alpha\eta \sin \eta s_n. \tag{4.24}$$

Combining this with (4.23), we conclude that

$$\cos^2 s_n = \frac{1-\alpha^2\eta^2}{1-\eta^2} > 1,$$

a contradiction. Therefore $(I - s_n^2 K)y = 0$, and consequently

$$\ker(I - s_n^2 K)^2 \subseteq \ker(I - s_n^2 K).$$

This completes the proof of the lemma. □

5. The main results

Assume that

(C3) $\lambda_l < \lambda_{l+1}$;

(C4) there exists $r \in \{2, \dots, l-1\}$ such that $\lambda_{r-1} < \lambda_r < \lambda_{r+1}$.

Remark 5.1. Combining (C3) with (3.4) and using Lemma 3.2, we conclude that

$$\lambda_{kl} < \lambda_{kl+1} < \lambda_{kl+2}, \quad k \in \mathbb{N}. \tag{5.1}$$

Remark 5.2. From (3.11), we know that (C4) holds for either $i_0 = 2$ or $i_0 = 7$.

Theorem 5.1. *Let* (C0), (C1), (C2) *and* (C3) *hold. Assume that either*

$$f_0 < \lambda_{kl+1} < f_\infty$$

or

$$f_\infty < \lambda_{kl+1} < f_0$$

for some $k \in \mathbb{N}$. *Then the problem* (1.3), (1.4) *has two solutions* u^+_{kl+1} *and* u^-_{kl+1}, u^+_{kl+1} *has exactly* μ_{kl+1} *zeros in* $(0,1)$ *and is positive near* $t = 0$, *and* u^-_{kl+1} *has exactly* μ_{kl+1} *zeros in* $(0,1)$ *and is negative near* $t = 0$.

Theorem 5.2. *Let* (C0), (C1), (C2) *and* (C3) *hold. Assume that either* (i) *or* (ii) *holds for some* $k \in \mathbb{N}$ *and* $j \in \{0\} \cup \mathbb{N}$*:*

(i) $f_0 < \lambda_{kl+1} < \ldots < \lambda_{(k+j)l+1} < f_\infty$;

(ii) $f_\infty < \lambda_{kl+1} < \ldots < \lambda_{(k+j)l+1} < f_0$.

Then the problem (1.3), (1.4) *has* $2(j+1)$ *solutions* $u^+_{(k+i)l+1}$, $u^-_{(k+i)l+1}$, $i = 0, \ldots, j$; $u^+_{(k+i)l+1}$ *has exactly* $\mu_{(k+i)l+1}$ *zeros in* $(0,1)$ *and is positive near* $t = 0$, *and* $u^-_{(k+i)l+1}$ *has exactly* $\mu_{(k+i)l+1}$ *zeros in* $(0,1)$ *and is negative near* $t = 0$.

Let $\zeta, \xi \in C(\mathbb{R})$ be such that

$$f(u) = f_0 u + \zeta(u), \quad f(u) = f_\infty u + \xi(u), \tag{5.2}$$

$$\lim_{|u| \to 0} \frac{\zeta(u)}{u} = 0, \quad \lim_{|u| \to \infty} \frac{\xi(u)}{u} = 0. \tag{5.3}$$

Let

$$\tilde{\xi}(u) = \max_{0 \leqslant |s| \leqslant u} |\xi(s)|; \tag{5.4}$$

then $\tilde{\xi}$ is nondecreasing and

$$\lim_{u \to \infty} \frac{\tilde{\xi}(u)}{u} = 0. \tag{5.5}$$

Let us consider

$$\begin{gathered} u'' + \lambda f_0 u + \lambda \zeta(u) = 0, \\ u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \end{gathered} \tag{5.6}$$

as a bifurcation problem from the trivial solution $u \equiv 0$.

In view of (4.6), Equation (5.6) can be converted to the equivalent equation

$$u(t) = \int_0^1 H(t,s)[\lambda f_0 u(s) + \lambda \zeta(u(s))] \, \mathrm{d}s. \tag{5.7}$$

Further we note that

$$\|K[\zeta(u(\cdot)]\| = o(\|u\|)$$

for u near 0 in E, since

$$\begin{aligned} \|K[\zeta(u(\cdot))]\| &= \max_{t \in [0,1]} \left| \int_0^1 H(t,s)\zeta(u(s)) \, \mathrm{d}s \right| + \max_{t \in [0,1]} \left| \int_0^1 H_t(t,s)\zeta(u(s)) \, \mathrm{d}s \right| \\ &\leqslant C \|\zeta(u(\cdot))\|_\infty. \end{aligned}$$

Let $\mathbb{E} = \mathbb{R} \times E$ with the product topology. Let S_k^+ denote the set of functions in E which have exactly $k-1$ interior nodal (i.e. nondegenerate) zeros in $(0,1)$ and are positive near $t=0$, and set $S_k^- = -S_k^+$, and $S_k = S_k^+ \cup S_k^-$. They are disjoint and open in E. Finally, let $\Phi_k^{\pm} = \mathbb{R} \times S_k^{\pm}$ and $\Phi_k = \mathbb{R} \times S_k$.

If (C3) holds, then we have from Remark 5.1 that for each $k \in \mathbb{N}$,

$$\mu_{lk} < \mu_{lk+1} < \mu_{lk+2}.$$

Thus the results of Rabinowitz [7] for (5.7) can be stated as follows: For each integer $k \geqslant 1$ and each $\nu \in \{+,-\}$, there exists a continuum of solutions $C_{kl+1}^{\nu} \subset \mathbb{R} \times E$ satisfying

$$C_{kl+1}^{\nu} \setminus \{(\lambda_{kl+1}/f_0, 0)\} \subseteq \Phi_{kl+r}^{\nu}$$

and joining $(\lambda_{kl+1}/f_0, 0)$ to infinity in Φ_{kl+1}^{ν}.

Remark 5.3. It is worth remarking that if (C3) holds, then for $p \in \{2,\ldots,l\}$ and $k \in \mathbb{N}$, there exists a connected set C_{kl+p}^{ν} of nontrivial solutions of (5.7) such that $C_{kl+p}^{\nu} \cup (\lambda_{kl+p}/f_0, 0)$ is closed and connected. However we give no information on the interesting question of which of the following cases will occur:

(i) C_{kl+p}^{ν} meets infinity in $\mathbb{R} \times E$;

(ii) $C_{kl+p}^{\nu} \cap C_{kl+p'}^{\nu'} \neq \emptyset$ for some $r' \in \{2,\ldots,l\}$ with $p' \neq p$ and $\nu' \in \{+,-\}$.

In fact, for the multi-point eigenvalue problem (1.1), (1.2), $\lambda_{kl+p} < \lambda_{kl+p'}$ does not imply

$$\mu_{kl+p} < \mu_{kl+p}.$$

Let us recall Example 3.1. In this example, $\lambda_3 < \lambda_4$. But $\mu_3 = \mu_4 = 3$. So we don't know if C_3^+ joins infinity or not.

Proof of Theorem 5.1. It is clear that any solution of (5.6) of the form $(1,u)$ yields a solutions u of (1.3), (1.4). We will show that C_{kl+1}^{ν} crosses the hyperplane $\{1\} \times E$ in $\mathbb{R} \times E$. To do this, it is enough to show that C_{kl+1}^{ν} joins $(\lambda_{kl+1}/f_0, 0)$ to $(\lambda_{kl+1}/f_\infty, \infty)$. Let $(r_n, y_n) \in C_{kl+1}^{\nu}$ satisfy

$$r_n + \|y_n\| \to \infty.$$

We note that $r_n > 0$ for all $n \in \mathbb{N}$ since $(0,0)$ is the only solution of (5.6) for $\lambda = 0$ and $C_{kl+1}^{\nu} \cap (\{0\} \times E) = \emptyset$.

Case 1. $f_0 < \lambda_{kl+1} < f_\infty$. In this case, we show that

$$\Big(\frac{\lambda_{kl+1}}{f_\infty}, \frac{\lambda_{kl+1}}{f_0}\Big) \subseteq \{\lambda \in \mathbb{R}\colon\ \exists(\lambda, u) \in C_{kl+1}^{\nu}\}$$

We divide the proof into two steps.

Step 1. We show that if there exists a constant number $M > 0$ such that

$$r_n \subset (0, M],$$

then C_{kl+1}^{ν} joins $(\lambda_{kl+1}/f_0, 0)$ to $(\lambda_{kl+1}/f_\infty, \infty)$.

In this case it follows that $\|y_n\| \to \infty$. We divide the equation

$$y_n'' + r_n f_\infty y_n + r_n \xi(y_n(t)) = 0 \tag{5.8}$$

by $\|y_n\|$ and set $\bar{y}_n = \frac{y_n}{\|y_n\|}$. Since $\bar{y}_n$ is bounded in $C^2[0,1]$, choosing a subsequence and relabelling if necessary, we see that $\bar{y}_n \to \bar{y}$ for some $\bar{y} \in E$ with $\|\bar{y}\| = 1$. Moreover, from (5.3) and the fact that $\tilde{\xi}$ is nondecreasing, we have

$$\lim_{n\to\infty} \frac{|\xi(y_n(t))|}{\|y_n\|} = 0 \tag{5.9}$$

since $|\xi(y_n(t))|/\|y_n\| \leqslant \tilde{\xi}(|y_n(t)|)/\|y_n\| \leqslant \tilde{\xi}(\|y_n\|_\infty)/\|y_n\| \leqslant \tilde{\xi}(\|y_n\|)/\|y_n\|$. Thus

$$\bar{y}(t) = \int_0^1 H(t,s)\bar{r} f_\infty \bar{y}(s)\,\mathrm{d}s$$

where $\bar{r} := \lim\limits_{n\to\infty} r_n$, again choosing a subsequence and relabelling if necessary. Thus

$$\begin{gathered} \bar{y}'' + \bar{r} f_\infty \bar{y} = 0, \\ \bar{y}'(0) = 0, \quad \bar{y}(1) = \sum_{i=1}^{m-2} \alpha_i \bar{y}(\eta_i). \end{gathered} \tag{5.10}$$

We claim that

$$\bar{y} \in S_{kl+1}^{\nu}. \tag{5.11}$$

Suppose on the contrary that $\bar{y} \notin S_{kl+1}^{\nu}$. Since $\bar{y} \neq 0$ is a solution of (5.10), all zeros of $\bar{y}$ in $[0,1]$ are non-degenerate. It follows that $\bar{y} \in S_h^{\iota} \neq S_{kl+1}^{\nu}$ for some $h \in \mathbb{N}$ and $\iota \in \{+,-\}$. By the openness of S_h^{ι}, we know that there exists a neighborhood $U(\bar{y}, \delta)$ such that

$$U(\bar{y}, \delta) \subset S_h^{\iota}$$

which contradicts the facts that $\bar{y}_n \to \bar{y}$ in E and $\bar{y}_n \in C_{kl+1}^{\nu}$. Therefore $\bar{y} \in S_{kl+1}^{\nu}$.

By Lemma 3.1 and 3.2, $\bar{r} f_\infty = \lambda_{kl+1}$, so that

$$\bar{r} = \frac{\lambda_{kl+1}}{f_\infty}.$$

Thus C_{kl+1}^{ν} joins $(\lambda_{kl+1}/f_0, 0)$ to $(\lambda_{kl+1}/f_\infty, \infty)$.

Step 2. We show that there exists a constant M such that $r_n \in (0, M]$, for all n.

Suppose there is no such M. Choosing a subsequence and relabelling if necessary, it follows that

$$\lim_{n\to\infty} r_n = \infty. \tag{5.12}$$

Let

$$\tau(1,n) < \tau(2,n) < \ldots < \tau(\mu_{kl+1} - 1, n)$$

denote the zeros of y_n in $(0,1)$, and set

$$\tau(0,n) = 0, \quad \tau(\mu_{kl+1}, n) = 1$$

for convenience. Then there exists a subsequence $\{\tau(1,n_m)\} \subseteq \{\tau(1,n)\}$ such that

$$\lim_{m\to\infty} \tau(1,n_m) := \tau(1,\infty).$$

Clearly

$$\lim_{m\to\infty} \tau(0,n_m) := \tau(0,\infty) = 0.$$

We claim that

$$\tau(1,\infty) - \tau(0,\infty) = 0. \tag{5.13}$$

Suppose on the contrary that

$$\tau(0,\infty) < \tau(1,\infty). \tag{5.14}$$

Define a function $p\colon (0,\infty)\times\mathbb{R} \to \mathbb{R}$ by

$$p(r,u) := \begin{cases} r\dfrac{f(u)}{u}, & u \neq 0, \\ rf_0, & u = 0. \end{cases} \tag{5.15}$$

Then by (C2), there exist two positive numbers ϱ_1 and ϱ_2, such that

$$r\varrho_1 \leqslant r\frac{f(u)}{u} \leqslant r\varrho_2, \quad \text{for all } u \geqslant 0. \tag{5.16}$$

Using (5.14), (5.16), and the fact that $\lim\limits_{m\to\infty} r_{n_m} = \infty$, we conclude that there exists a closed interval $I_1 \subset (\tau(0,\infty), \tau(1,\infty))$ such that

$$\lim_{m\to\infty} p(r_{n_m}, y_{n_m}(t)) = \infty, \quad \text{uniformly for } t \in I_1.$$

It follows that the solution y_{n_m} of the equation

$$y''_{n_m}(t) = p(r_{n_m}, y_{n_m}(t))y_{n_m}(t)$$

must change sign on I_1. However, this contradicts the fact that for all m sufficiently large we have $I_1 \subset (\tau(0, n_m), \tau(1, n_m))$ and

$$\nu y_{n_m}(t) > 0, \quad t \in (\tau(0, n_m), \tau(1, n_m)).$$

Therefore, (5.13) holds.

Next, we work with $\{(\tau(1, n_m), \tau(2, n_m))\}$. It is easy to see that there is a subsequence $\{\tau(2, n_{m_j})\} \subseteq \{\tau(2, n_m)\}$, such that

$$\lim_{j\to\infty} \tau(2, n_{m_j}) := \tau(2, \infty).$$

Clearly

$$\lim_{j\to\infty} \tau(1, n_{m_j}) = \tau(1, \infty). \tag{5.17}$$

We claim that

$$\tau(2, \infty) - \tau(1, \infty) = 0. \tag{5.18}$$

Suppose on the contrary that $\tau(1, \infty) < \tau(2, \infty)$. Then from (5.15) and (5.16) and the fact that $\lim_{j\to\infty} r_{n_{m_j}} = \infty$, we know that there exists a closed interval $I_2 \subset (\tau(0, \infty), \tau(1, \infty))$ such that

$$\lim_{j\to\infty} p(r_{n_{m_j}}, y_{n_{m_j}}) = \infty, \quad \text{uniformly for } t \in I_2.$$

This implies that the solution $y_{n_{m_j}}$ of the equation

$$y''_{n_{m_j}}(t) = p(r_{n_{m_j}}, y_{n_{m_j}}(t))y_{n_{m_j}}(t)$$

must change sign on I_2. However, this contradicts the fact that for all j sufficiently large we have $I_2 \subset (\tau(1, n_{m_j}), \tau(2, n_{m_j}))$ and

$$\nu y_{n_{m_j}}(t) < 0, \quad t \in (\tau(1, n_{m_j}), \tau(2, n_{m_j})).$$

This proves that (5.18) holds.

By a similar argument to that used to obtain (5.13) and (5.18), we can show that for each $s \in \{2, \ldots, \mu_{lk+1} - 1\}$

$$\tau(s+1, \infty) - \tau(s, \infty) = 0. \tag{5.19}$$

Taking a subsequence and relabelling it as $\{(r_n, y_n)\}$ if necessary, it follows that for each $s \in \{0, \ldots, \mu_{lk+1} - 1\}$

$$\lim_{n\to\infty} \big(\tau(s+1, n) - \tau(s, n)\big) = 0. \tag{5.20}$$

But this is impossible since

$$1 = \tau(\mu_{lk+1}, n) - \tau(0, n) = \sum_{s=0}^{\mu_{lk+1}-1} \big(\tau(s+1, n) - \tau(s, n)\big)$$

for all n.

Therefore

$$|r_n| \leqslant M$$

for some constant number $M > 0$, independent of $n \in \mathbb{N}$.

Case 2. $f_\infty < \lambda_{kl+1} < f_0$.

In this case, we have

$$\frac{\lambda_{kl+1}}{f_0} < 1 < \frac{\lambda_{kl+1}}{f_\infty}.$$

If $(r_n, y_n) \in C^\nu_{kl+1}$ is such that

$$\lim_{n\to\infty} (r_n + \|y_n\|) = \infty$$

and

$$\lim_{n\to\infty} r_n = \infty,$$

then

$$\Big(\frac{\lambda_{kl+1}}{f_0}, \frac{\lambda_{kl+1}}{f_\infty}\Big) \subseteq \{\lambda \in (0,\infty)\colon\ (\lambda, u) \in C^\nu_{kl+1}\}$$

and consequently

$$(\{1\} \times E) \cap C^\nu_{kl+1} \neq \emptyset.$$

Assume that there exists $M > 0$, such that for all $n \in \mathbb{N}$,

$$r_n \in (0, M].$$

Applying a similar argument to that used in Step 1 of Case 1, after taking a subsequence and relabelling, if necessary, it follows that

$$(r_n, y_n) \to \Big(\frac{\lambda_{kl+1}}{f_\infty}, \infty\Big), \quad n \to \infty.$$

Again C^ν_{kl+1} joins $(\lambda_{kl+1}/f_0, 0)$ to $(\lambda_{kl+1}/f_\infty, \infty)$ and the result follows. □

P r o o f of Theorem 5.2. Repeating the arguments used in the proof of Theorem 1, we see that for each $\nu \in \{+,-\}$ and each $i \in \{0, 1, \ldots, j\}$

$$C^\nu_{l(k+i)+1} \cap (\{1\} \times E) \neq \emptyset.$$

The result follows. This completes the proof of Theorem 5.2. □

By using the similar method, we can establish the following results under the condition (C4).

Theorem 5.3. *Let* (C0), (C1), (C2) *and* (C4) *hold. Assume that either*

$$f_0 < \lambda_{kl+r} < f_\infty$$

or

$$f_\infty < \lambda_{kl+r} < f_0$$

for some $k \in \mathbb{N}$. *Then the problem* (1.3), (1.4) *has two solutions* u^+_{kl+r} *and* u^-_{kl+r}, u^+_{kl+1} *has exactly* μ_{kl+r} *zeros in* $(0,1)$ *and is positive near* $t = 0$, *and* u^-_{kl+r} *has exactly* μ_{kl+1} *zeros in* $(0,1)$ *and is negative near* $t = 0$.

Theorem 5.4. *Let* (C0), (C1), (C2) *and* (C4) *hold. Assume that either* (i) *or* (ii) *holds for some* $k \in \mathbb{N}$ *and* $j \in \{0\} \cup \mathbb{N}$:

(i) $f_0 < \lambda_{kl+r} < \ldots < \lambda_{(k+j)l+r} < f_\infty$;

(ii) $f_\infty < \lambda_{kl+r} < \ldots < \lambda_{(k+j)l+r} < f_0$.

Then the problem (1.3), (1.4) *has* $2(j+1)$ *solutions* $u^+_{(k+i)l+r}$, $u^-_{(k+i)l+r}$, $i = 0, \ldots, j$, $u^+_{(k+i)l+r}$ *has exactly* $\mu_{(k+i)l+r}$ *zeros in* $(0,1)$ *and is positive near* $t = 0$, *and* $u^-_{(k+i)l+r}$ *has exactly* $\mu_{(k+i)l+r}$ *zeros in* $(0,1)$ *and is negative near* $t = 0$.

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