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COMMUTATORS OF SINGULAR INTEGRALS ON SPACES OF HOMOGENEOUS TYPE

Gladis Pradolini, Oscar Salinas, Santa Fe

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Abstract. In this work we prove some sharp weighted inequalities on spaces of homogeneous type for the higher order commutators of singular integrals introduced by R. Coifman, R. Rochberg and G. Weiss in Factorization theorems for Hardy spaces in several variables, Ann. Math. 103 (1976), 611–635. As a corollary, we obtain that these operators are bounded on $L^p(w)$ when w belongs to the Muckenhoupt's class A_p, $p>1$. In addition, as an important tool in order to get our main result, we prove a weighted Fefferman-Stein type inequality on spaces of homogeneous type, which we have not found previously in the literature.

Keywords: commutators, spaces of homogeneous type, weights

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1. Introduction and main results

The higher order commutators of singular integrals were introduced by R. Coifman, R. Rochberg and G. Weiss in [7]. In a formal sense they can be defined in $\mathbb{R}^n$ for appropriate functions b and f as follows

$$T_b^m f(x) = \int_{\mathbb{R}^n} (b(x)-b(y))^m K(x,y) f(y)\,\mathrm{d}y \tag{1.1}$$

for $m=0,1,2,\ldots$, where K is a Calderón-Zygmund kernel. When $m=1$, the operator T_b^1 is usually denoted by $[b,T]f = bT(f)-T(bf)$, with T a Calderón-Zygmund operator (by the way, T is the case $m=0$ of (1.1)). The operators T_b^m have proved to be of interest in many contexts and, in particular, in the theory of P.D.E. (see [8], [9], [10]). It is well known that, in this area, many problems can be naturally seen as problems on spaces of homogeneous type ([12], [11]). So, results concerning the operators T_b^m in this general setting appears as a natural request (see, for instance, [4] and [22] where commutators appear in connection with problems about L^p-estimates for parabolic and elliptic equations with VMO coefficients respectively).

The purpose of this work is to prove weighted inequalities between L^p spaces for the commutator T_b^m on spaces of homogeneous type. In order to state them, we first recall some basic notions about these spaces and the weights we are going to use.

Let X be a set. A function $d\colon X \times X \to [0,\infty)$ is called a quasi-distance on X if the following conditions are satisfied:

i) for every x and y in X, $d(x,y) \geqslant 0$, and $d(x,y) = 0$ if and only if $x = y$,

ii) for every x and y in X, $d(x,y) = d(y,x)$,

iii) there exists a constant K such that $d(x,y) \leqslant K(d(x,z) + d(z,y))$ for every x, y and z in X.

Let μ be a positive measure on the σ-algebra of subsets of X generated by the d-balls $B(x,r) = \{y\colon\ d(x,y) < r\}$, with $x \in X$ and $r > 0$. We assume that μ satisfies a doubling condition, that is, there exists a constant A such that

$$0 < \mu(B(x,2Kr)) \leqslant A\mu(B(x,r)) < \infty \tag{1.2}$$

holds for every ball $B \subset X$. A structure (X,d,μ), with d and μ as above, is called a space of homogeneous type and it was introduced for the first time in [6] (for more details, see [14] and [15], for instance).

We say that (X,d,μ) is a space of homogeneous type regular in measure if μ is regular, that is for every measurable set E, given $\varepsilon > 0$, there exists an open set G such that $E \subset G$ and $\mu(G - E) < \varepsilon$. In what follows we always assume that the space (X,d,μ) is regular in measure.

We denote by $\mathcal{D}$ the set containing the functions $f \in L^\infty$ with bounded support.

A nonnegative function w defined on X, will be called a weight if it is a locally integrable function. If E is a measurable set we denote $w(E) = \int_E w\,\mathrm{d}\mu$.

A weight w is in the Muckenhoupt's class A_∞ respect to μ (see [16]) if there are positive constants C and ε such that the inequality

$$\frac{w(E)}{w(B)} \leqslant C\left(\frac{\mu(E)}{\mu(B)}\right)^{\varepsilon} \tag{1.3}$$

holds for every ball B and every measurable set $E \subset B$. The infimum of such C will be denoted by $[w]_{A_\infty}$.

Finally, if w is a weight, by $L^p(w)$ we mean the measurable functions f such that $\int_X |f|^p w\,\mathrm{d}\mu$ is finite.

With these definitions, we can introduce our main results as follows.

1.4. Theorem. *Let $1 < p < \infty$, w be a weight in A_∞ and b belonging to* BMO*. Then, there exists a positive constant C, depending only on the constants of the space (X, d, μ) and the A_∞ constant of w, such that*

$$\int_X |T_b^m f(x)|^p w(x)\,\mathrm{d}\mu(x) \leqslant C\|b\|_{\mathrm{BMO}}^{mp} \int_X (M^{m+1} f(x))^p w(x)\,\mathrm{d}\mu(x)$$

holds for every function $f \in \mathcal{D}$. Here M^{m+1} denotes the Hardy-Littlewood maximal operator iterated $m+1$ times.

1.5. Theorem. *Let $1 < p < \infty$, b belonging to* BMO *and w a weight. Then there exists a positive constant C, depending only on the constants of the space (X, d, μ), such that*

$$\int_X |T_b^m f(x)|^p w(x)\,\mathrm{d}\mu(x) \leqslant C\|b\|_{\mathrm{BMO}}^{mp} \int_X |f(x)|^p M^{[(m+1)p]+1} w(x)\,\mathrm{d}\mu(x) \tag{1.6}$$

holds for every function $f \in \mathcal{D}$, where $[(m+1)p]$ denotes the biggest number in $\mathbb{N} \cup \{0\}$ less than or equal to $(m+1)p$.

1.7. Remark. Theorem 1.4 has as an easy corollary (by applying a well known result about boundedness of the Hardy-Littlewood maximal operator (see [16])) that the operators T_b^k are bounded from $L^p(w)$ in $L^p(w)$ when w belongs to the A_p class (i.e. $w(B)(w^{-1/(p-1)}(B))^{p-1} \cong (\mu(B))^p$ for every ball B). In this case, it is important to note that the operator T_b^m can be continuously extended to L^p, $1 < p < \infty$ by using a well known density argument (see for instance [4]), and then, the theorem holds for every $f \in L^p$.

1.8. Remark. The Euclidean case of theorems 1.4 and 1.5 (i.e. $X = \mathbb{R}^n$ with the usual distance and the Lebesgue measure) were proved for the first time by C. Pérez in [18] and in addition, this author showed there that the number of iterations of the maximal operator needed in both theorems is optimal. On the other hand note that the case $m = 0$ recovers well known results about Calderón-Zygmund operators ([5], [25]).

1.9. Remark. As another consequence of our results we can obtain a generalization of the un-weighted results proved by M. Bramanti and M. C. Cerutti ([3]) for the case T_b^1 (see [4], too).

1.10. Remark. In [18] the Euclidean versions of the above theorems are used by C. Pérez to obtain weighted boundedness results about the following non linear commutator

$$Nf = T(f \log |f|) - T(f \log |Tf|),$$

where T is a Calderón-Zygmund operator and f belongs to an appropriate set of functions. This commutator was introduced by R. Rochberg and G. Weiss in [23] and it is related to some problems in nonlinear P.D.E. Following a similar reasoning to that applied in [18], Theorems 1.4 and 1.5 allow to get extensions of the results obtained there (and concerning Nf) to a general setting of spaces of homogeneous type. For instance, given $p \in (1, \infty)$ and $w \in A_p$ we have

$$\int_X |Nf(x)|^p w(x)\,\mathrm{d}\mu(x) \leqslant C \int_X |f(x)|^p w(x)\,\mathrm{d}\mu(x).$$

The techniques we are going to use in the proofs of Theorems 1.4 and 1.5 are based on those used in [18]. They require a weighted version of the well known Fefferman-Stein inequality on spaces of homogeneous type which is proved in Section 3 and it is interesting in itself. The structure of the paper is as follows: Section 2 contains basic facts and some notation, Section 3 is devoted, as we said, to the Fefferman-Stein type inequality; finally, Sections 4 and 5 contain the proofs of Theorems 1.4 and 1.5, respectively.

2. Preliminaries

Let (X, d, μ) be a space of homogeneous type. It is always possible to find a continuous quasi-distance d' equivalent to d (see [14]) in the sense that there exist positive constants C_1 and C_2 such that

$$C_1 d'(x, y) \leqslant d(x, y) \leqslant C_2 d'(x, y).$$

With this result in mind, we will assume that the quasi-metric d is continuous.

The spaces L^p and L^p_{loc} on (X, d, μ) are defined as usual. The weighted versions of $L^p(w)$ for a non negative function w in L^1_{loc} are obtained by taking the measure $w\,\mathrm{d}\mu$.

Given $f \in L^1_{\mathrm{loc}}$ and Ω a measurable set, we denote $m_\Omega(f) = \mu(\Omega)^{-1} \int_\Omega f\,\mathrm{d}\mu$.

We say that T is a Calderón-Zygmund operator on (X, d, μ) if the following conditions are satisfied (see [1] and [3], for instance):

i) $T\colon\ L^p(X) \to L^p(X)$ is linear and continuous for every $p \in (1, \infty)$;

ii) there exists a measurable function $k\colon\ X \times X \to \mathbb{R}$ such that for every $f \in \mathcal{D}$,

$$Tf(x) = \int_X k(x, y) f(y)\,\mathrm{d}\mu(y),$$

for a.e. $x \notin \operatorname{supp} f$;

iii) the kernels k and k^* (defined by $k^*(x,y) = k(y,x)$) satisfy the following pointwise Hörmander condition: There exist positive constants C, β and $M > 1$ such that

$$|k(x_0,y) - k(x,y)| \leqslant C \frac{d(x_0,x)^\beta}{\mu(B(x_0,2d(x_0,y)))d(x_0,y)^\beta}$$

holds for every $x_0 \in X$, $r > 0$, $x \in B(x_0,r)$, $y \in X - B(x_0,Mr)$;

iv) the kernel k also satisfies the inequality $|k(x,y)| \leqslant C/\mu(B(x,2d(x,y)))$ for every x, $y \in X$.

It is well known that if T is a Calderón-Zygmund operator on (X,d,μ), then T is of weak type $(1,1)$ (see [15]), that is

$$\mu(\{x \in X\colon\ |Tf(x)| > \lambda\}) \leqslant \frac{C}{\lambda} \int_X f(x)\,\mathrm{d}\mu(x) \tag{2.1}$$

for every $f \in L^1(X)$.

For f in L^1_{loc} we consider the ε-maximal function and the sharp function of f defined, respectively, by

$$M_\varepsilon f(x) = \sup_{r>0} \left(\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|^\varepsilon \,\mathrm{d}\mu(y) \right)^{1/\varepsilon} \tag{2.2}$$

and

$$M^\sharp f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f_{B(x,r)}|\,\mathrm{d}\mu(y). \tag{2.3}$$

The case $\varepsilon = 1$ of M_ε is the classical Hardy-Littlewood maximal operator, and M_1 will be denoted by M. Related to $M^\sharp$, we will say that f belongs to BMO if $f \in L^1_{\mathrm{loc}}$ and $M^\sharp f \in L^\infty$. We shall denote by $\|f\|_{\mathrm{BMO}}$ the semi-norm given by $\|M^\sharp f\|_\infty$.

In addition to M_ε and $M^\sharp$ we consider the operator $(M^\sharp(|f|^\delta))^{1/\delta}$, which will be denoted by $M^\sharp_\delta(f)$, and a maximal operator related to Orlicz norms. Before introducing this operator we recall that a function $\varphi\colon\ [0,\infty) \to [0,\infty)$ is called a Young function if it is continuous, convex, increasing and satisfies $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. We say that φ satisfies a doubling condition if $\varphi(2t) \leqslant C\varphi(t)$ for every $t > 0$, (i.e. φ satisfies the Δ_2 condition).

We define the φ-average of a function f over a ball B by means of the Luxemburg norm

$$\|f\|_{\varphi,B} = \inf\left\{ \lambda > 0\colon\ \frac{1}{\mu(B)} \int_B \varphi(|f(y)|/\lambda) \leqslant 1 \right\}.$$

Also we have the following generalized Hölder inequality

$$\frac{1}{\mu(B)} \int_B |f(y)g(y)|\,\mathrm{d}\mu(y) \leqslant \|f\|_{\varphi,B} \|g\|_{\tilde{\varphi},B} \tag{2.4}$$

where $\tilde{\varphi}$ is the complementary Young function associated to φ (for more details on Orlicz spaces, see for instance [23]). There is a further generalization that will be useful for our purposes (see [17]): Let φ_1, φ_2 and ψ be Young functions such that

$$\varphi_1^{-1}\varphi_2^{-1} \leqslant \psi^{-1}$$

then

$$\|fg\|_{\psi,B} \leqslant C\|f\|_{\varphi_1,B}\|g\|_{\varphi_2,B}.$$

The maximal operator M_φ associated to Young function φ is defined by

$$M_\varphi f(x) = \sup_{B \ni x} \|f\|_{\varphi,B}. \tag{2.5}$$

The main example of Young functions we shall consider is $\varphi(t) = t(1+\log^+ t)^m$, $m = 1, 2, 3, \ldots$ with the corresponding maximal function denoted by $M_{L(\log L)^m}$. The complementary Young function is given by $\tilde{\varphi}(t) \cong \exp(t^{1/m})$ with the corresponding maximal function denoted by $M_{(\exp L)^{1/m}}$.

Another important result we are going to apply is the fundamental estimate due to John and Nirenberg (see [13] and [2]) for a function b in BMO

$$\frac{1}{\mu(B)}\int_B \exp\Big(\frac{|b(y)-b_B|}{C\|b\|_{\mathrm{BMO}}}\Big)\,\mathrm{d}\mu(y) \leqslant C$$

which is equivalent to

$$\|b - b_{2B}\|_{\exp L,B} \leqslant C\|b\|_{\mathrm{BMO}}. \tag{2.6}$$

In addition, concerning BMO, it was proved in [2] that there exist positive constants C_1 and C_2 such that the following inequality

$$C_1\|b\|_{\mathrm{BMO}} \leqslant \Big(\frac{1}{\mu(B)}\int_B |b(x)-b_B|^p\,\mathrm{d}\mu(x)\Big)^{1/p} \leqslant C_2\|b\|_{\mathrm{BMO}} \tag{2.7}$$

holds for $b \in \mathrm{BMO}$ and $1 < p < \infty$. If $p < 1$, the second inequality in (2.7) still holds, because of Hölder's inequality.

Finally, we remark that C will denote a positive constant which may be different even in a single chain of inequalities.

3. A Fefferman-Stein type inequality

The proof of our weighted version of Fefferman-Stein's inequality on spaces of homogeneous type is based on the ideas of Prof. H. Aimar for the un-weighted case. We would like to thank him for sharing them with us. We need the following classical covering lemmas on spaces of homogeneous type (both of them hold without the condition that the measure is regular). The proof of the first of them is in [6].

3.1. Lemma. *Let E be a bounded set in X, being (X, d, μ) a space of homogeneous type. Let $\{B(x, r(x)) \colon x \in E\}$ be a covering of E by balls centered at each point of E. Then there exists a sequence of points $\{x_i\}_{\mathbb{N}} \subset E$ such that $B(x_i, r(x_i)) \cap B(x_j, r(x_j)) = \emptyset$ if $i \neq j$ and $E \subset \bigcup_{i=1}^{\infty} B(x_i, 4Kr(x_i))$.*

In the next lemma (see [1]), the hypotheses of boundedness of E is replaced by $\mu(E) < \infty$.

3.2. Lemma. *Let (X, d, μ) be a space of homogeneous type. Let $\mathcal{B} = \{\mathcal{B}_\alpha \colon \alpha \in \Gamma\}$ be a family of balls in X such that $E = \bigcup_{\alpha \in \Gamma} B_\alpha$ is measurable and $\mu(E) < \infty$. Then there exists a disjoint sequence $\{B(x_i, r_i)\} \subset \mathcal{B}$, possibly finite, such that $E \subset \bigcup_{i=1} B(x_i, Cr_i)$ for some constant C (that only depends on K, the constant of the quasi-metric). Moreover, every $B \in \mathcal{B}$ is contained in some $B(x_i, Cr_i)$.*

Another result we need is the following extension to spaces of homogeneous type of the well known Calderón-Zygmund decomposition. The proof can be found in [1].

3.3. Lemma (Calderón Zygmund decomposition)**.** *Let (X, d, μ) be a space of homogeneous type. Let $f \geqslant 0$ be an integrable function on X. Then, for every $\lambda \geqslant m_X(f)$ ($m_X(f) = 0$ if $\mu(X) = +\infty$), there exists a sequence $\{B_i\}$ of pairwise disjoint balls such that, if $\tilde{B}_i$ is a dilation of B_i by the constant C of the covering Lemma 3.2, we get*

(a) *$m_{\tilde{B}_i}(f) \leqslant \lambda \leqslant m_{B_i}(f)$;*
(b) *for every $x \in X - \bigcup_i \tilde{B}_i$ and for every ball B centered at x, holds $m_B(f) \leqslant \lambda$.*

Now, we will prove a relation between weighted L^p-norms of the operators M and $M^\sharp$ (see (2.2) and (2.3)), i.e. a Fefferman-Stein type inequality.

3.4. Proposition. *Let (X, d, μ) be a space of homogeneous type regular in measure. Let $f \in \mathcal{D}$ be a positive function and $w \in A_\infty$. Then, for every p, $1 < p < \infty$, there exists a positive constant $C = C([w]_{A_\infty})$ such that if $\|Mf\|_{L^p(w)} < +\infty$, then*

$$\|Mf\|^p_{L^p(w)} \leqslant \begin{cases} C\|M^\sharp f\|^p_{L^p(w)} & \text{if } \mu(X) = \infty, \\ C(w(X)(m_X(f))^p + \|M^\sharp f\|^p_{L^p(w)}) & \text{if } \mu(X) < \infty. \end{cases} \tag{3.5}$$

P r o o f. We consider $f \geqslant 0$, $f \in \mathcal{D}$. Let $t \in \mathbb{R}$ be such that

$$\frac{t}{2} \geqslant m_X(f) = \frac{1}{\mu(X)} \int_X f \,\mathrm{d}\mu.$$

We define

$$\Omega^t = \{x \in X\colon \exists\, r > 0 \text{ such that } m_{B(x,r)}(f) > t\}$$

and

$$\Omega^{t/2} = \{x \in X\colon \exists\, r > 0 \text{ such that } m_{B(x,r)}(f) > t/2\}.$$

Then, it is obvious that $\Omega^t \subset \Omega^{t/2}$. For $x \in \Omega^t$ let

$$R^t(x) = \{r > 0\colon m_{B(x,r)}(f) > t\}$$

and

$$R^{t/2}(x) = \{r > 0\colon m_{B(x,r)}(f) > t/2\}.$$

Then $R^t(x) \subset R^{t/2}(x)$. On the other hand $R^t(x)$ and $R^{t/2}(x)$ are non empty sets of real positive numbers bounded from above. This fact is obvious if $\mu(X) < \infty$. If $\mu(X) = +\infty$, since $f \in \mathcal{D}$, we have

$$0 < \frac{t}{2} < \frac{1}{\mu(B(x,r))} \int_{B(x,r)\cap \operatorname{supp} f} f \leqslant C \frac{\mu(B(x,r) \cap \operatorname{supp} f)}{\mu(B(x,r))} \tag{3.6}$$

which tends to zero when r tends to $+\infty$.

Thus we can choose $r^t(x) \in R^t(x)$ such that $Cr^t(x) \notin R^t(x)$ where C is the constant of the Lemma 3.2. So

$$m_{B(x,r^t(x))}(f) > t \geqslant m_{B(x,Cr^t(x))}(f).$$

Let $r^{t/2}(x) \in R^{t/2}(x)$ be such that

$$m_{B(x,r^{t/2}(x))}(f) > \frac{t}{2} \geqslant m_{B(x,Cr^{t/2}(x))}(f)$$

and

$$r^t(x) < r^{t/2}(x).$$

It is clear that for $x \in \Omega^{t/2}$ we have $B(x, r^{t/2}(x)) \subset \{Mf > t/2\}$. Then

$$\bigcup_{x\in\Omega^t} B(x, r^t(x)) \subset \bigcup_{x\in\Omega^{t/2}} B\big(x, r^{t/2}(x)\big) \subset \{Mf > t/2\} \tag{3.7}$$

and thus, since M is of weak type (1,1) (see [6]), we obtain

$$\mu\bigg(\bigcup_{x\in\Omega^t} B(x, r^t(x))\bigg) \leqslant \mu(\{Mf > t/2\}) \leqslant \frac{C}{t}\int_X |f|\,\mathrm{d}\mu < +\infty.$$

Applying Lemma 3.2 to the families

$$\mathcal{B}^t = \{B(x, r^t(x))\colon\ x \in \Omega^t\} \quad \text{and} \quad \mathcal{B}^{t/2} = \{B(x, r^{t/2}(x))\colon\ x \in \Omega^{t/2}\}$$

we get two collections of balls $\{B_i^s\colon\ i \in \mathbb{N}\}$ with $s = t, t/2$, such that

i) $B_i^s \cap B_j^s = \emptyset$, $i \neq j$;

ii) for every $x \in \Omega^s$ there exists $i \in \mathbb{N}$ such that $B(x, r^s(x)) \subset B(x_i, Cr^s(x_i)) = \tilde{B}_i^s$;

iii) $m_{B_i^s}(f) > s \geqslant m_{\tilde{B}_i^s}(f)$;

iv) $\Omega^s \subset \bigcup_i \tilde{B}_i^s$, if $x \notin \bigcup_i \tilde{B}_i^s$ then $f(x) \leqslant s$,

and, in addition,

v) for every $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $B_i^t \subset \tilde{B}_j^{t/2}$;

vi) for each $j \in \mathbb{N}$ let $I_j = \{i \in \mathbb{N}\colon\ B_i^t \subset \tilde{B}_j^{t/2},\ \text{but}\ B_i^t \not\subset \tilde{B}_l^{t/2},\ l = 1, \ldots, j-1\}$. Then $\{I_j,\ j \in \mathbb{N}\}$ is a disjoint partition of $\mathbb{N}$.

Concerning $\mathcal{B}^t$ and $\mathcal{B}^{t/2}$, we can prove that there exists C such that the following inequality

$$\sum_{i\in\mathbb{N}} w(B_i^t) \leqslant Cw(\{M^\sharp f > t/A\}) + \frac{C}{A^\delta}\sum_{j\in\mathbb{N}} w(B_j^{t/2}) \tag{3.8}$$

holds for every $A > 1$. Indeed, let $J_1 = \{j\colon\ \tilde{B}_j^{t/2} \subset \{M^\sharp f > t/A\}\}$ and $J_2 = \{j\colon\ \tilde{B}_j^{t/2} \not\subset \{M^\sharp f > t/A\}\}$. Then

$$\begin{aligned} \sum_{i\in\mathbb{N}} w(B_i^t) &= \sum_{j\in\mathbb{N}}\sum_{i\in I_j} w(B_i^t) \\ &= \sum_{j\in J_1}\sum_{i\in I_j} w(B_i^t) + \sum_{j\in J_2}\sum_{i\in I_j} w(B_i^t) = I + II. \end{aligned} \tag{3.9}$$

It is easy to check that

$$I \leqslant Cw(\{M^\sharp f > t/A\}). \tag{3.10}$$

On the other hand, if $j \in J_2$ there exists $x \in \tilde{B}_j^{t/2}$ such that $M^\sharp f(x) < t/A$, and, consequently,

$$\frac{1}{\mu(\tilde{B}_j^{t/2})} \int_{\tilde{B}_j^{t/2}} |f - m_{\tilde{B}_j^{t/2}}(f)| \,\mathrm{d}\mu \leqslant t/A.$$

Then, by recalling that $m_{B_i^t}(f) > t$ and $m_{\tilde{B}_j^{t/2}}(f) \leqslant t/2$, we have

$$\sum_{i\in I_j} (t - t/2)\mu(B_i^t) \leqslant \sum_{i\in I_j} \int_{B_i^t} (f - m_{\tilde{B}_j^{t/2}}(f)) \,\mathrm{d}\mu \leqslant (t/A)\mu(\tilde{B}_j^{t/2}),$$

which implies

$$\mu\Big(\bigcup_{i\in I_j} B_i^t\Big) = \sum_{i\in I_j} \mu(B_i^t) \leqslant (2/A)\mu(\tilde{B}_j^{t/2}). \tag{3.11}$$

Since $\bigcup_{i\in I_j} B_i^t \subset \tilde{B}_j^{t/2}$, and from the fact that w satisfies the A_∞ condition (1.3), there exist positive constants C and δ such that

$$w\Big(\bigcup_{i\in I_j} B_i^t\Big) \Big/ w(\tilde{B}_j^{t/2}) \leqslant C\bigg(\mu\Big(\bigcup_{i\in I_j} B_i^t\Big) \Big/ \mu(\tilde{B}_j^{t/2})\bigg)^\delta.$$

Therefore, from (3.11) we obtain

$$w\Big(\bigcup_{i\in I_j} B_i^t\Big) \Big/ w(\tilde{B}_j^{t/2}) \leqslant C/A^\delta. \tag{3.12}$$

But $\{B_i^t\}_{i\in\mathbb{N}}$ are disjoint, then, this inequality allows us to get

$$\sum_{i\in I_j} w(B_i^t) \leqslant \frac{C}{A^\delta} w(\tilde{B}_j^{t/2}) \leqslant \frac{C}{A^\delta} w(B_j^{t/2}).$$

Thus, from the above estimate, (3.9) and (3.10) we have

$$\mu\Big(\bigcup_{i\in I_j} B_i^t\Big) \leqslant (C/A)\mu(\tilde{B}_j^{t/2}).$$

Setting $\alpha(t) = \sum_{i\in\mathbb{N}} w(B_i^t)$, the inequality (3.8) can be written as

$$\alpha(t) \leqslant Cw(\{M^\sharp f > t/A\}) + \frac{C}{A^\delta}\alpha(t/2). \tag{3.13}$$

Now, let $\beta(t) = w(\{Mf > t\})$. Since $B_i^t \subset \{Mf > t\}$ if $t \geqslant 2m_X(f)$, we have $\alpha(t) \leqslant \beta(t)$, $t \geqslant 2m_X f$. On the other hand we get $\{Mf > t\} \subset \bigcup_{i\in\mathbb{N}} \tilde{B}_i^t$ so it follows that $\beta(t) \leqslant C\alpha(t)$ for every t.

Let us observe that, if $N > 0$,

$$\int_{2m_X(f)}^{N} pt^{p-1}\alpha(t)\,\mathrm{d}t \leqslant \int_{2m_X(f)}^{N} pt^{p-1}\beta(t)\,\mathrm{d}t \leqslant C\|Mf\|_{L^p(w)} < +\infty. \tag{3.14}$$

Thus, from (3.13) we can obtain

$$\begin{aligned}
&\int_{2m_X(f)}^{N} pt^{p-1}\alpha(t)\,\mathrm{d}t \\
&\qquad \leqslant C\int_{2m_X(f)}^{N} pt^{p-1}w(\{M^\sharp f > t/A\})\,\mathrm{d}t + \frac{C}{A^\delta}\int_{2m_X(f)}^{N} pt^{p-1}\alpha(t/2)\,\mathrm{d}t \\
&\qquad \leqslant C\int_{2m_X(f)}^{N} pt^{p-1}w(\{M^\sharp f > t/A\})\,\mathrm{d}t + \frac{C}{A^\delta}\int_{m_X(f)}^{N} pt^{p-1}\alpha(t)\,\mathrm{d}t.
\end{aligned}$$

Writing the last integral as

$$\int_{m_X(f)}^{2m_X(f)} pt^{p-1}\alpha(t)\,\mathrm{d}t + \int_{2m_X(f)}^{N} pt^{p-1}\alpha(t)\,\mathrm{d}t,$$

choosing A such that $C/A^\delta = 1/2$ and taking into account (3.14) we get

$$\begin{aligned}
&\frac{1}{2}\int_{2m_X(f)}^{N} pt^{p-1}\alpha(t)\,\mathrm{d}t \\
&\qquad \leqslant C\int_{2m_X(f)}^{N} pt^{p-1}w(\{M^\sharp f > t/A\})\,\mathrm{d}t + \frac{1}{2}\int_{m_X(f)}^{2m_X(f)} pt^{p-1}\alpha(t)\,\mathrm{d}t \\
&\qquad \leqslant C\int_{2m_X(f)}^{\infty} pt^{p-1}w(\{M^\sharp f > t/A\})\,\mathrm{d}t + Cw(X)(m_X(f))^p.
\end{aligned}$$

Finally, by using that $\alpha(t) \leqslant \beta(t)$ when $t > 2m_X(f)$, and $\beta(t) \leqslant w(X)$ for all $t > 0$, we have the estimate

$$\begin{aligned}
\int_X |Mf|^p w(x)\,\mathrm{d}\mu &\leqslant \int_0^{2m_X(f)} pt^{p-1}\beta(t)\,\mathrm{d}t + \int_{2m_X(f)}^{\infty} pt^{p-1}\alpha(t)\,\mathrm{d}t \\
&\leqslant Cw(X)(m_X(f))^p + \int_X |M^\sharp f|^p w(x)\,\mathrm{d}\mu,
\end{aligned}$$

which proves our result (note that $m_X(f) = 0$ when $\mu(X) = \infty$). □

4. Proof of theorem 1.4

In order to give a rigorous definition of T_b^m, note that for the case $m = 1$, the operator

$$T_b^1 f = bTf - T(bf) \tag{4.1}$$

is well defined if b belongs to L^∞ and $f \in \mathcal{D}$. If $b \in \mathrm{BMO}$, by the John-Nirenberg lemma, $b \in L^p_{\mathrm{loc}}$, $1 < p < \infty$. Then, for $f \in \mathcal{D}$ and $b \in \mathrm{BMO}$, $T_b^1 f$ is well defined.

For the general case, it is easy to see that, in a formal sense, the operator (1.1) satisfies the following identity

$$T_b^m f = \sum_{i=0}^{m-1} C_i(b(x) - \lambda)^{m-i} T_b^i f(x) + T((b-\lambda)^m f)(x), \tag{4.2}$$

for any λ in $\mathbb{R}$ and constants C_i (from the Newton's formula). Then, for b in L^∞ and f in $\mathcal{D}$, $T_b^m f$ can be defined inductively from the case $m = 1$. The extension for b in BMO and a wider set of f will be obtained by an argument of density from the inequalities proved in our main results.

The next two lemmas are devoted to show connections between the operators T_b^m, M_ε and $M^\sharp$ and are the key points for the reasoning. The Euclidean case of the first one is contained in [19].

4.3. Lemma. *Let $b \in \mathrm{BMO}$, with $\|b\|_{\mathrm{BMO}} = 1$ and $0 < \delta < \varepsilon < 1$. Then, there exists a positive constant $C = C_\delta$ such that*

$$M_\delta^\sharp(T_b^m f)(x) \leqslant C\left(\sum_{j=0}^{m-1} M_\varepsilon(T_b^j f)(x) + M^{m+1} f(x)\right) \tag{4.4}$$

holds for every $f \in \mathcal{D}$, for a.e. $x \in X$ and for each $m = 0, 1, 2, \ldots$

4.5. Remark. When $m = 0$ we understand (4.4) as $M_\delta^\sharp(Tf)(x) \leqslant CMf(x)$.

P r o o f. The proof follows the same lines as in the Euclidean case with obvious changes (see [19]). □

4.6. Lemma. *Let $f \in \mathcal{D}$, $b \in \mathrm{BMO}$ with $\|b\|_{\mathrm{BMO}} = 1$ and $0 < \delta < \varepsilon < 1$. Then the following estimate for T_b^m holds*

$$\int_X |T_b^m f|^\delta \,\mathrm{d}\mu \leqslant C\Big(\sum_{j=0}^{m-1} \int_X |M_\varepsilon(T_b^j f)(x)|^\delta \,\mathrm{d}\mu(x) + \mu(X)\|f\|^\delta_{L(\log L)^m,X}\Big).$$

Proof. The case $\mu(X) = \infty$ is obvious. Let us consider $\mu(X) < \infty$. Using the expression (4.2) for T_b^m we have

$$\begin{aligned}
&\int_X |T_b^m f|^\delta \,\mathrm{d}\mu \\
&\quad \leqslant C\Big(\sum_{j=0}^{m-1} \int_X |b(x)-\lambda|^{(m-j)\delta} |T_b^j f(x)|^\delta \,\mathrm{d}\mu(x) + \int_X |T((b-\lambda)^m f)(x)|^\delta \,\mathrm{d}\mu(x)\Big) \\
&\quad \leqslant C(A+B).
\end{aligned}$$

Let us first estimate B. If we take $\lambda = b_X$, since T is of weak type (1.1), then Kolmogorov's inequality, (2.4) and (2.6) allow us to get

$$\begin{aligned}
\int_X |T((b-\lambda)^m f)(x)|^\delta \,\mathrm{d}\mu(x) &\leqslant \mu(X)\Big(\frac{1}{\mu(X)}\int_X |b(x)-b_X|^m f(x)\,\mathrm{d}\mu(x)\Big)^\delta \\
&\leqslant \mu(X)\|(b-b_X)^m\|^\delta_{\exp L^{1/m},X}\|f\|^\delta_{L(\log L)^m,X} \\
&\leqslant C\mu(X)\|b\|^\delta_{\mathrm{BMO}}\|f\|^\delta_{L(\log L)^m,X} \\
&= C\mu(X)\|f\|^\delta_{L(\log L)^m,X}.
\end{aligned}$$

In order to estimate A we select r such that $1 < r < \varepsilon/\delta$ we use Hölder's inequality and the equivalence between norms in BMO to obtain

$$\begin{aligned}
&\sum_{j=0}^{m-1} \int_X |b(x)-b_X|^{(m-j)\delta} |T_b^j f(x)|^\delta \,\mathrm{d}\mu(x) \\
&\qquad \leqslant \sum_{j=0}^{m-1} \mu(X)\Big(\frac{1}{\mu(X)}\int_X |b(x)-b_X|^{(m-j)\delta r'}\,\mathrm{d}\mu(x)\Big)^{1/r'} \\
&\qquad\quad \times \Big(\frac{1}{\mu(X)}\int_X |T_b^j f(x)|^{\delta r}\,\mathrm{d}\mu(x)\Big)^{1/r} \\
&\qquad \leqslant C\mu(X)\sum_{j=0}^{m-1}\Big(\frac{1}{\mu(X)}\int_X |T_b^j f(x)|^\varepsilon \,\mathrm{d}\mu(x)\Big)^{\delta/\varepsilon} \\
&\qquad \leqslant C\mu(X)\sum_{j=0}^{m-1}(M_\varepsilon(T_b^j f)(x))^\delta
\end{aligned}$$

for almost every $x \in X$.

Then, from the estimates for A and B, we get

$$\int_X |T_b^m f|^\delta \,\mathrm{d}\mu \leqslant C\mu(X)\Big(\sum_{j=0}^{m-1} |M_\varepsilon(T_b^j f)(x)|^\delta + \|f\|^\delta_{L(\log L)^m, X}\Big).$$

Finally, integrating on X we obtain the desired result. □

Now we are in position to proceed with the proof of Theorem 1.4.

Proof of Theorem 1.4. Assuming $\|b\|_{\mathrm{BMO}} = 1$, we proceed by induction.

Let us prove the case $m = 0$ for X satisfying $\mu(X) = \infty$. Since $w \in A_\infty$, there exists $r > 1$ such that $w \in A_r$. We can select $0 < \varepsilon < 1$ such that $0 < \varepsilon < p/r$, and choose $\delta < \varepsilon$ small enough such that $p/\delta > r$ and $w \in A_{p/\delta}$. Then, we have that the Hardy Littlewood maximal operator is bounded from $L^{p/\delta}(w)$ into $L^{p/\delta}(w)$ (see [16]). Consequently, since $\|Tf\|_{L^p(w)} < \infty$, we get

$$\|M_\delta(Tf)\|_{L^p(w)} = \|M(|Tf|^\delta)\|^{1/\delta}_{L^{p/\delta}(w)} \leqslant C\|\,|Tf|^\delta\|^{1/\delta}_{L^{p/\delta}(w)} \leqslant C\|Tf\|_{L^p(w)} < \infty.$$

From Lebesgue's Differentiation theorem, Proposition 3.4 and Lemma 4.3 we have

$$\|Tf\|_{L^p(w)} \leqslant \|M_\delta(Tf)\|_{L^p(w)} \leqslant \|M^\sharp_\delta(Tf)\|_{L^p(w)} \leqslant C\|Mf\|_{L^p(w)}.$$

Let us now consider the case $\mu(X) < \infty$. Proposition 3.4 yields

$$\begin{aligned}\|Tf\|_{L^p(w)} &\leqslant \|M_\delta(Tf)\|_{L^p(w)} \\ &\leqslant C\big(w(X)^{1/p}(m_X(|Tf|^\delta))^{1/\delta} + \|M^\sharp_\delta(Tf)\|_{L^p(w)}\big) \\ &= I + II.\end{aligned}$$

We estimate II as in the previous case. For I we use Kolmogorov's inequality to obtain

$$\begin{aligned} I = C\frac{w(X)^{1/p}}{\mu(X)^{1/\delta}}\Big(\int_X |Tf|^\delta \,\mathrm{d}\mu\Big)^{1/\delta} &\leqslant w(X)^{1/p}\frac{1}{\mu(X)}\int_X |f| \,\mathrm{d}\mu \\ \leqslant \Big(\int_X \Big(\frac{1}{\mu(X)}\int_X |f| \,\mathrm{d}\mu\Big)^p w \,\mathrm{d}\mu\Big)^{1/p} &\leqslant \Big(\int_X |Mf|^p w \,\mathrm{d}\mu\Big)^{1/p}, \end{aligned}$$

which completes the case $m = 0$.

Now, suppose the result holds for $0, 1, \ldots, m-1$. Let X be such that $\mu(X) = +\infty$. Reasoning as in the case $m = 0$ we get that $\|M_\delta(T_b^m f)\|_{L^p(w)} < \infty$. Then, we can apply Proposition 3.4 and Lemma 4.3 to get

$$\begin{aligned}\|T_b^m f\|_{L^p(w)} &\leqslant \|M_\delta(T_b^m f)\|_{L^p(w)} \leqslant \|M^\sharp_\delta(T_b^m f)\|_{L^p(w)} \\ &\leqslant C\Big(\sum_{j=0}^{m-1} \|M_\varepsilon(T_b^j f)\|_{L^p(w)} + \|M^{m+1} f\|_{L^p(w)}\Big).\end{aligned}$$

Since $w \in A_{p/\varepsilon}$ and from the inductive hypothesis, we have that the last expression in the above inequality is bounded by

$$\begin{aligned}C\Big(\sum_{j=0}^{m-1}\|T_b^j f\|_{L^p(w)} + \|M^{m+1}f\|_{L^p(w)}\Big)&\\ &\leqslant C\Big(\sum_{j=0}^{m-1}\|M^{j+1}f\|_{L^p(w)} + \|M^{m+1}f\|_{L^p(w)}\Big)\\ &\leqslant C\|M^{m+1}f\|_{L^p(w)},\end{aligned}$$

as we wanted to prove.

Let us now consider the case $\mu(X) < \infty$. By Proposition 3.4 for this case we have

$$\begin{aligned}\|T_b^m f\|_{L^p(w)} &\leqslant \|M_\delta(T_b^m f)\|_{L^p(w)}\\ &\leqslant C\big(w(X)(m_X(|T_b^m f|^\delta))^{p/\delta} + \|M_\delta^\sharp(T_b^m f)\|_{L^p(w)}^p\big)^{1/p}\\ &\leqslant C\big((w(X)(m_X(|T_b^m f|^\delta))^{p/\delta})^{1/p} + \|M_\delta^\sharp(T_b^m f)\|_{L^p(w)}\big)\\ &= C(A+B).\end{aligned}$$

An estimate for B can be obtained following similar lines as in the previous case. In order to get the estimate of A, we first apply Lemma 4.6 to obtain

$$A \leqslant \frac{w(X)^{1/p}}{\mu(X)^{1/\delta}}\Big(\sum_{j=0}^{m-1}\int_X |M_\varepsilon(T_b^j f)(x)|^\delta \,\mathrm{d}\mu(x) + \mu(X)\|f\|_{L(\log L)^m,X}^\delta\Big)^{1/\delta}. \tag{4.7}$$

Let ε and δ be as before. Then, since $w \in A_{p/\varepsilon}$ we have that

$$\begin{aligned}\int_X |M_\varepsilon^\delta(T_b^j f)(x)|\,\mathrm{d}\mu(x) &\leqslant \Big(\int_X |M(|T_b^j f|^\varepsilon)(x)|^{p/\varepsilon} w(x)\,\mathrm{d}\mu(x)\Big)^{\delta/p}\\ &\qquad\times\Big(\int_X w^{-\delta/(p-\delta)}\Big)^{1-\delta/p}\\ &\leqslant C\Big(\int_X |T_b^j f|^p w(x)\,\mathrm{d}\mu(x)\Big)^{\delta/p}\Big(\int_X w^{-1/(p/\delta-1)}\Big)^{1-\delta/p}.\end{aligned}$$

So, from (4.7) we obtain

$$\begin{aligned}A \leqslant C\frac{w(X)^{1/p}}{\mu(X)^{1/\delta}}\Big(\Big(\int_X w^{-1/(p/\delta-1)}\Big)^{1-\delta/p}&\\ \times\sum_{j=0}^{m-1}\|T_b^j f)\|_{L^p(w)}^\delta + \mu(X)\|f\|_{L(\log L)^m,X}^\delta\Big)^{1/\delta}&.\end{aligned} \tag{4.8}$$

Using an induction argument like in the case $\mu(X)=\infty$ we obtain that the last expression is bounded by

$$
\begin{aligned}
(4.9)\quad & C\frac{w(X)^{1/p}}{\mu(X)^{1/\delta}}\bigg(\|M^{m+1}f)\|^{\delta}_{L^p(w)}\Big(\int_X w^{-1/(p/\delta-1)}\Big)^{1-\delta/p}+\mu(X)\|f\|^{\delta}_{L(\log L)^m,X}\bigg)^{1/\delta}\\
&\leqslant C\bigg(\frac{w(X)^{\delta/p}}{\mu(X)}\|M^{m+1}f)\|^{\delta}_{L^p(w)}\Big(\int_X w^{-1/(p/\delta-1)}\Big)^{1-\delta/p}\\
&\qquad\qquad + w(X)^{\delta/p}\|f\|^{\delta}_{L(\log L)^m,X}\bigg)^{1/\delta}\\
&=(I+II)^{1/\delta}.
\end{aligned}
$$

Let us consider I. We recall that $w\in A_{p/\delta}$, so, since X becomes a ball,

$$\frac{1}{\mu(X)}\int_X w\Big(\frac{1}{\mu(X)}\int_X w^{-1/(p/\delta-1)}\Big)^{p/\delta-1}\leqslant C.$$

Thus we have

$$I\leqslant C\|M^{m+1}f\|^{\delta}_{L^p(w)}. \tag{4.10}$$

To estimate II we use the fact that there exists positive constants C_1 and C_2 such that

$$C_1 M_{L(\log L)^m}f(x)\leqslant M^{m+1}f(x)\leqslant C_2 M_{L(\log L)^m}f(x) \tag{4.11}$$

(for $\mathbb{R}^n$ this result is due to C. Pérez ([19]). In the general setting of spaces of homogeneous type, the left inequality was proved in [20]. The right one can be proved by reasoning as in the Euclidean case ([19], p. 174) with minor changes.

So, using (2.5) and (4.11), we obtain

$$
\begin{aligned}
w(X)^{\delta/p}\|f\|^{\delta}_{L(\log L)^m,X}&=\Big(\int_X \|f\|^{p}_{L(\log L)^m,X}w(x)\,\mathrm{d}\mu(x)\Big)^{\delta/p}\\
&\leqslant\Big(\int_X (M_{L(\log L)^m}f(x))^p w(x)\,\mathrm{d}\mu(x)\Big)^{\delta/p}\\
&\leqslant C\|M^{m+1}f\|^{\delta}_{L^p(w)}.
\end{aligned}
$$

Finally, from (4.7), (4.8), (4.9), (4.10) and the last inequality we get the desired result.

If $\|b\|_{\mathrm{BMO}}\neq 1$ we apply the above case with $b/\|b\|_{\mathrm{BMO}}$ to conclude the result, taking in account that $T^m_{b/\|b\|_{\mathrm{BMO}}}f=T^m_b f/\|b\|^m_{\mathrm{BMO}}$. □

5. Proof of Theorem 1.5

In order to prove this theorem we are going to apply, like in [18], a duality argument. For this method we need the following result concerning the operator M_φ defined in (2.5). The proof is in [21]. We remark that the same result is proved by R. Wheeden and C. Pérez in [20], but under the additional hypothesis that the annuli in (X, d, μ) are nonempty.

We recall that a doubling Young function φ satisfies the B_p condition, $1 < p < \infty$, if there is a positive constant c such that

$$\int_c^\infty \frac{\varphi(t)}{t^p}\frac{\mathrm{d}t}{t} \cong \int_c^\infty \Big(\frac{t^{p'}}{\tilde{\varphi}(t)}\Big)^{p-1}\frac{\mathrm{d}t}{t} < \infty.$$

5.1. Theorem. *Let $1 < p < \infty$, φ be a doubling Young function and (X, d, μ) a space of homogeneous type with $\mu(X) = \infty$. Then the following statements are equivalent:*

i) $\varphi \in B_p$.

ii) *There exists a constant C such that*

$$\int_X (M_\varphi f(x))^p \,\mathrm{d}\mu(x) \leqslant C \int_X |f(x)|^p \,\mathrm{d}\mu(x)$$

for all nonnegative functions f.

iii) *There exists a constant C such that*

$$\int_X (M_\varphi f(x))^p w(x) \,\mathrm{d}\mu(x) \leqslant C \int_X |f(x)|^p M w(x) \,\mathrm{d}\mu(x)$$

for all nonnegative functions f and all weights w, where M is the Hardy-Littlewood maximal operator.

iv) *There exists a constant C such that*

$$\int_X (Mf(x))^p \frac{w(x)}{(M_{\tilde{\varphi}}(u^{1/p})(x))^p} \,\mathrm{d}\mu(x) \leqslant C \int_X |f(x)|^p \frac{Mw(x)}{u(x)} \,\mathrm{d}\mu(x) \tag{5.2}$$

for all nonnegative functions f and all weights w and u.

5.3. Remark. If $\mu(X) < \infty$, the implications i) $\Rightarrow$ ii) $\Rightarrow$ iii) $\Rightarrow$ iv) still hold but the converses results are not true (see [21]).

P r o o f of Theorem 1.5. The proof can be done by reasoning as in the case $X = \mathbb{R}^n$ (see [18]), with minor changes. The main steps are:

Step 1. For simplicity, we denote $k(p) = [(k+1)p] + 1$. Instead of proving (1.6), and from the fact that the adjoint operator to T_b^k is essentially the same, we consider the corresponding dual inequality

$$\int_X |T_b^k f(x)|^{p'} (M^{k(p)} w(x))^{1-p'} \, \mathrm{d}\mu(x) \leqslant C \int_X |f(x)|^{p'} w(x)^{1-p'} \, \mathrm{d}\mu(x). \tag{5.4}$$

Step 2. Since $(M^{k(p)} w)^{1-p'}$ belongs to the A_∞ class of Muckenhoupt, from Theorem 1.4 we have that

$$\int_X |T_b^k f(x)|^{p'} (M^{k(p)} w)^{1-p'} \, \mathrm{d}\mu(x) \leqslant C \int_X (M^{k+1} f(x))^{p'} (M^{k(p)} w)^{1-p'} \, \mathrm{d}\mu(x).$$

Step 3. Then, the proof is reduced to proving the next two weighted norm inequalities for the maximal operator M^{m+1}

$$\int_X (M^{k+1} f(x))^{p'} (M^{k(p)} w)^{1-p'} \, \mathrm{d}\mu(x) \leqslant C \int_X |f(x)|^{p'} w(x)^{1-p'} \, \mathrm{d}\mu(x).$$

This type of inequality follows as an application of Theorem 5.1. $\square$

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Authors' address: Gladis Pradolini, Oscar Salinas, Consejo Nacional de Investigaciones Científicas y Técnicas, Instituto de Matemática Aplicada Litoral-Conicet Guemes 3450, Santa Fe-La Capital, 3000-Santa Fe, Argentina, e-mails: `gprado@math.unl.edu.ar`, `salinas@ceride.gov.ar`.