Marta Vrábelová

On the extension of subadditive measures in lattice ordered groups

Czechoslovak Mathematical Journal, Vol. 57 (2007), No. 1, 95–103

Persistent URL: `http://dml.cz/dmlcz/128157`

Czechoslovak Mathematical Journal, 57 (132) (2007), 95–103

ON THE EXTENSION OF SUBADDITIVE MEASURES IN LATTICE ORDERED GROUPS

Marta Vrábelová, Nitra

(Received December 10, 2004)

Abstract. A lattice ordered group valued subadditive measure is extended from an algebra of subsets of a set to a σ-algebra.

Keywords: subadditive measure, lattice ordered groups

MSC 2000: 28B15

Introduction

The problems of extensions of real-valued exhausting subadditive measures has been solved in [1], [3], [4]. In the present paper a lattice ordered group G is taken as the range of a subadditive measure μ_0 defined on an algebra $\mathscr{A}$ of subsets of a set X. In order to prove an extension theorem the condition (v) below is used instead of the exhaustion property of μ_0. The construction from [6] is used for the extension of μ_0.

Recall that a lattice ordered group G (l-group) is called conditionally complete (σ-complete), if every upper bounded (countable) subset of G has the supremum in G.

An l-group G is weakly σ-distributive, if for every bounded double sequence $(a_{ij})_{i,j} \subset G$ such that $a_{ij} \searrow 0$ ($j \to \infty$, $i = 1, 2, \ldots$) (the sequence $(a_{ij})_{i,j}$ is called a regulator in G) we have

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \bigvee_i a_{i\varphi(i)} = 0.$$

This work was supported by grant VEGA 1/2002/05.

1. Theorem. *Let G be a conditionally σ-complete l-group. Let $(a_{nij})_{n,i,j}$ be a bounded sequence of elements of G such that $a_{nij} \searrow 0$ ($j \to \infty$, $n, i = 1, 2, \ldots$). Then for every $a \in G$, $a > 0$ there exists a bounded sequence $(a_{ij})_{i,j} \subset G$, $a_{ij} \searrow 0$ ($j \to \infty$, $i = 1, 2, \ldots$) such that*

$$a \wedge \left(\sum_{n=1}^{\infty} \bigvee_{i=1}^{\infty} a_{ni\varphi(i+n)} \right) \leqslant \bigvee_{i=1}^{\infty} a_{i\varphi(i)}$$

for every $\varphi\colon \mathbb{N} \to \mathbb{N}$.

P r o o f. For the proof see [5], [7] and [8]. □

Assumptions

A. A set X and an algebra $\mathscr{A}$ of subsets of X are given.

B. An l-group G, which is conditionally complete and weakly σ-distributive, is given.

C. A mapping (a subadditive measure) $\mu_0\colon \mathscr{A} \to G$ satisfying the following conditions is given:

(i) $\mu_0(\emptyset) = 0$.

(ii) If $A \subset B$, $A, B \in \mathscr{A}$, then $\mu_0(A) \leqslant \mu_0(B)$.

(iii) $\mu_0(A \cup B) \leqslant \mu_0(A) + \mu_0(B)$ for all $A, B \in \mathscr{A}$.

(iv) If $A_n \in \mathscr{A}$, $A_n \searrow \emptyset$ (that is $A_n \supset A_{n+1}$ ($n = 1, 2, \ldots$), $\bigcap\limits_{n=1}^{\infty} A_n = \emptyset$), then $\mu_0(A_n) \searrow 0$ (that is $\mu_0(A_n) \geqslant \mu_0(A_{n+1})$ ($n = 1, 2, \ldots$) and $\bigwedge\limits_{n=1}^{\infty} \mu_0(A_n) = 0$).

(v) If $(a_{ij})_{i,j}$ is a regulator in G, $\varphi \in \mathbb{N}^{\mathbb{N}}$ and if there are nondecreasing (resp. nonincreasing) sequences $(K_n)_n \subset \mathscr{A}$, $(L_n)_n \subset \mathscr{A}$ such that $\mu_0(K_n \setminus L_n) \leqslant \bigvee\limits_i a_{i\varphi(i)}$ (resp. $\mu_0(L_n \setminus K_n) \leqslant \bigvee\limits_i a_{i\varphi(i)}$) for all n, then there exists $n_0 \in \mathbb{N}$ such that $\bigvee\limits_{m=1}^{\infty} \mu_0(K_m \setminus K_n) \leqslant \bigvee\limits_i a_{i\varphi(i)}$ (resp. $\bigvee\limits_{m=1}^{\infty} \mu_0(K_n \setminus K_m) \leqslant \bigvee\limits_i a_{i\varphi(i)}$) for every $n > n_0$.

Further properties of μ_0 are obtained in the following lemma.

2. Lemma.

(vi) *If $A, B \in \mathscr{A}$, then $\mu_0(B) \leqslant \mu_0(B \setminus A) + \mu_0(A)$.*

(vii) *If $A_n \nearrow A$, $A_n, A \in \mathscr{A}$ ($n = 1, 2, \ldots$), then $\mu_0(A_n) \nearrow \mu_0(A)$.*

(viii) *If $B_n \searrow B$, $B_n, B \in \mathscr{A}$ ($n = 1, 2, \ldots$), then $\mu_0(B_n) \searrow \mu_0(B)$.*

P r o o f. The conditions (ii) and (iii) imply (vi). In (vii), $\mu_0(A) \leqslant \mu_0(A_n) + \mu_0(A \setminus A_n)$ for all n by (vi); and hence $\mu_0(A) \leqslant \bigvee_n \mu_0(A_n) + \bigwedge_n \mu_0(A \setminus A_n) = \bigvee_n \mu_0(A_n)$ by (iv), (ii) implies $\mu_0(A) = \bigvee_n \mu_0(A_n)$ and (viii) can be obtained similarly. □

3. Lemma. *If $A_n, B_n \in \mathscr{A}$ $(n = 1, 2, \ldots)$, $A_n \nearrow A$, $B_n \nearrow B$, $A \subset B$ $(A_n \searrow A$, $B_n \searrow B$, $A \subset B)$, then*

$$\bigvee_n \mu_0(A_n) \leqslant \bigvee_n \mu_0(B_n) \quad \Big(\text{or } \bigwedge_n \mu_0(A_n) \leqslant \bigwedge_n \mu_0(B_n)\Big).$$

P r o o f. By (vii) (resp. (viii)) and (ii) we have

$$\mu_0(A_n) = \mu_0(A_n \cap B) = \bigvee_m \mu_0(A_n \cap B_m) \leqslant \bigvee_m \mu_0(B_m)$$
$$\Big(\text{or } \mu_0(B_n) = \mu_0(B_n \cup A) = \bigwedge_m \mu_0(B_n \cup A_m) \geqslant \bigwedge_m \mu_0(A_m)\Big)$$

for all n, hence

$$\bigvee_n \mu_0(A_n) \leqslant \bigvee_m \mu_0(B_m) \quad \Big(\text{or } \bigwedge_n \mu_0(B_n) \geqslant \bigwedge_m \mu_0(A_m)\Big).$$

□

Extension

4. Definition. We put $\mathscr{A}^+ = \{B \subset X \colon \exists B_n \in \mathscr{A}$ $(n = 1, 2, \ldots)$, $B_n \nearrow B\}$, $\mathscr{A}^- = \{C \subset X \colon \exists C_n \in \mathscr{A}$ $(n = 1, 2, \ldots)$, $C_n \searrow C\}$ and define mappings $\mu^+ \colon \mathscr{A}^+ \to G$ and $\mu^- \colon \mathscr{A}^- \to G$ by the formulas

$$\mu^+(B) = \bigvee_n \mu_0(B_n), \quad \mu^-(C) = \bigwedge_n \mu_0(C_n).$$

Further, we put $\mathscr{S} = \big\{D \subset X \colon \exists$ bounded $(a_{ij})_{i,j} \subset G$, $a_{ij} \searrow 0$ $(j \to \infty$, $i = 1, 2, \ldots)$ such that for every $\varphi \colon \mathbb{N} \to \mathbb{N}$ there are $E^\varphi \in \mathscr{A}^-$, $F^\varphi \in \mathscr{A}^+$, with $E^\varphi \subset D \subset F^\varphi$ and $\mu^+(F^\varphi \setminus E^\varphi) \leqslant \bigvee_i a_{i\varphi(i)}\big\}$ and we define $\mu \colon \mathscr{S} \to G$ by the formula

$$\mu(D) = \bigwedge \{\mu^+(F) \colon F \supset D, \ F \in \mathscr{A}^+\}.$$

The definitions of μ^+ and μ^- are correct by virtue of Lemma 3.

5. Lemma. *Let* $B_n \in \mathscr{A}^+$, $C_n \in \mathscr{A}^-$ $(n = 1, 2, \ldots)$, $B_n \nearrow B$, $C_n \searrow C$. *Then* $B \in \mathscr{A}^+$, $C \in \mathscr{A}^-$ *and*

$$\mu^+(B) = \bigvee_n \mu^+(B_n), \quad \mu^-(C) = \bigwedge_n \mu^-(C_n).$$

Proof. There exist $B_{n,m} \in \mathscr{A}$, $B_{n,m} \nearrow B_n$ $(m \to \infty)$. Put $D_n = \bigcup_{m=1}^{n} B_{m,n}$. Then $D_n \subset B_n$, $D_n \in \mathscr{A}$, $\mu_0(D_n) = \mu^+(D_n) \leqslant \mu^+(B_n)$ $(n = 1, 2, \ldots)$, $D_n \nearrow B$, which implies $B \in \mathscr{A}^+$ and

$$\mu^+(B) = \bigvee_n \mu_0(D_n) \leqslant \bigvee_n \mu^+(B_n) \leqslant \mu^+(B).$$

Similarly the second part can be obtained. □

6. Lemma. *If* $A, B \in \mathscr{A}^+$, $C, D \in \mathscr{A}^-$, *then* $A \cup B \in \mathscr{A}^+$, $B \setminus C \in \mathscr{A}^+$, $C \setminus B \in \mathscr{A}^-$ *and*

$$\mu^+(A \cup B) \leqslant \mu^+(A) + \mu^+(B), \quad \mu^-(C \cup D) \leqslant \mu^-(C) + \mu^-(D),$$
$$\mu^+(B) \leqslant \mu^+(B \setminus C) + \mu^-(C), \quad \mu^-(C) \leqslant \mu^-(C \setminus B) + \mu^+(B).$$

If $A \subset B$, *then* $\mu^+(A) \leqslant \mu^+(B)$, *if* $C \subset D$, *then* $\mu^-(C) \leqslant \mu^-(D)$, *if* $A \subset C$, *then* $\mu^+(A) \leqslant \mu^-(C)$ *and if* $C \subset A$, *then* $\mu^-(C) \leqslant \mu^+(A)$.

Proof. The proof is evident. □

7. Lemma. *If* $A, B \in \mathscr{S}$, *then* $A \cup B \in \mathscr{S}$, $A \setminus B \in \mathscr{S}$.

Proof. Let $A_1, B_1 \in \mathscr{A}^-$, $A_2, B_2 \in \mathscr{A}^+$ with $A_1 \subset A \subset A_2$, $B_1 \subset B \subset B_2$ be such that

$$\mu^+(A_2 \setminus A_1) \leqslant \bigvee_i a_{i\varphi(i)}, \quad \mu^+(B_2 \setminus B_1) \leqslant \bigvee_i b_{i\varphi(i)}.$$

Then $A_1 \cup B_1 \subset A \cup B \subset A_2 \cup B_2$, $A_1 \setminus B_2 \subset A \setminus B \subset A_2 \setminus B_1$ and

$$(A_2 \cup B_2) \setminus (A_1 \cup B_1) \subset (A_2 \setminus A_1) \cup (B_2 \setminus B_1),$$
$$(A_2 \setminus B_1) \setminus (A_1 \setminus B_2) \subset (A_2 \setminus A_1) \cup (B_2 \setminus B_1).$$

We have

$$\mu^+((A_2 \cup B_2) \setminus (A_1 \cup B_1)) \leqslant \bigvee_i a_{i\varphi(i)} + \bigvee_i b_{i\varphi(i)},$$
$$\mu^+((A_2 \setminus B_1) \setminus (A_1 \setminus B_2)) \leqslant \bigvee_i a_{i\varphi(i)} + \bigvee_i b_{i\varphi(i)}.$$

Put $c_{ij} = 2(a_{i,j} + b_{ij})$ for $i, j = 1, 2, \ldots$ Then $(c_{ij})_{i,j}$ is a regulator in G and

$$\bigvee_i a_{i\varphi(i)} + \bigvee_i b_{i\varphi(i)} \leqslant \bigvee_i c_{i\varphi(i)}$$

for every $\varphi \in \mathbb{N}^{\mathbb{N}}$. Hence $A \cup B$, $A \setminus B \in \mathscr{S}$. □

8. Lemma. *If $A \in \mathscr{S}$, then $\mu(A) = \bigvee\{\mu^-(C)\colon\ C \in \mathscr{A}^-,\ C \subset A\}$.*

Proof. Given $\varphi \in \mathbb{N}^{\mathbb{N}}$ take $B \in \mathscr{A}^+$, $C \in \mathscr{A}^-$ such that $C \subset A \subset B$, $\mu^+(B \setminus C) \leqslant \bigvee_i a_{i\varphi(i)}$. Then

$$\mu(A) \leqslant \mu^+(B) \leqslant \mu^+(B \setminus C) + \mu^-(C) \leqslant \bigvee_i a_{i\varphi(i)} + \bigvee\{\mu^-(C)\colon\ C \subset A,\ C \in \mathscr{A}^-\}$$

for all $\varphi \in \mathbb{N}^{\mathbb{N}}$. Hence

$$\mu(A) \leqslant \bigvee_i a_{i\varphi(i)} + \bigvee\{\mu^-(C)\colon\ C \subset A,\ C \in \mathscr{A}^-\}$$

for all $\varphi \in \mathbb{N}^{\mathbb{N}}$. By the weak σ-distributivity of G we have $\bigwedge_\varphi \bigvee_i a_{i\varphi(i)} = 0$ and

$$\mu(A) \leqslant \bigvee_i\{\mu^-(C)\colon\ C \subset A,\ C \in \mathscr{A}^-\}.$$

Further, $\mu^-(C) \leqslant \mu^+(B)$ for every $C \in \mathscr{A}^-$, $B \in \mathscr{A}^+$, $C \subset A \subset B$ (by Lemma 6) and hence

$$\bigvee\{\mu^-(C)\colon\ C \subset A,\ C \in \mathscr{A}^-\} \leqslant \bigwedge\{\mu^+(B)\colon\ B \supset A,\ B \in \mathscr{A}^+\} = \mu(A).$$

□

9. Theorem. *If $A_n \in \mathscr{S}$, $A_n \nearrow A$, then $A \in \mathscr{S}$ and $\mu(A) = \bigvee_n \mu(A_n)$.*

Proof. There are bounded sequences $(a_{nij})_{n,i,j} \subset G$, $a_{nij} \searrow 0$ ($j \to \infty$, $i, n = 1, 2, \ldots$) such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there are $C_n^\varphi \in \mathscr{A}^-$, $B_n^\varphi \in \mathscr{A}^+$, $C_n^\varphi \subset A_n \subset B_n^\varphi$ such that

$$\mu^+(B_n^\varphi \setminus C_n^\varphi) \leqslant \bigvee_i a_{ni\varphi(i+n)}$$

for $n = 1, 2, \ldots$. Put $D_n^\varphi = \bigcup_{k=1}^n B_k^\varphi$, $E_n^\varphi = \bigcup_{k=1}^n C_k^\varphi$. Then

$$D_n^\varphi \in \mathscr{A}^+, \quad E_n^\varphi \in \mathscr{A}^-, \quad E_n^\varphi \subset \bigcup_{k=1}^n A_k = A_n \subset D_n^\varphi$$

and

$$\mu^+(D_n^\varphi \setminus E_n^\varphi) = \mu^+\Big(\bigcup_{k=1}^n B_k^\varphi \setminus \bigcup_{k=1}^n C_k^\varphi\Big) \leqslant \mu^+\Big(\bigcup_{k=1}^n (B_k^\varphi \setminus C_k^\varphi)\Big)$$
$$\leqslant \sum_{k=1}^n \mu^+(B_k^\varphi \setminus C_k^\varphi) \leqslant \sum_{k=1}^n \bigvee_i a_{ki\varphi(i+k)} \leqslant \sum_{k=1}^\infty \bigvee_i a_{ki\varphi(i+k)}.$$

Therefore

$$\mu^+(D_n^\varphi \setminus E_n^\varphi) = a \wedge \Big(\sum_{k=1}^\infty \bigvee_i a_{ki\varphi(i+k)}\Big),$$

where $a = \mu_0(X)$, $a \in G$. By Theorem 1 there is a regulator $(a_{ij})_{i,j}$ in G such that

$$a \wedge \Big(\sum_{k=1}^\infty \bigvee_i a_{ki\varphi(i+k)}\Big) \leqslant \bigvee_i a_{i\varphi(i)}.$$

Further put $B^\varphi = \bigcup_{n=1}^\infty B_n^\varphi$. Then $D_n^\varphi \nearrow B^\varphi$ and hence $B^\varphi \in \mathscr{A}^+$ by Lemma 5. That is, there exist $K_n \in \mathscr{A}$ such that $K_n \subset D_n^\varphi$, $K_n \nearrow B^\varphi$. Then $B^\varphi \setminus K_n \searrow 0$, $B^\varphi \setminus K_n \in \mathscr{A}^+$. Now

$$\begin{aligned}\mu^+(B^\varphi \setminus E_n^\varphi) &\leqslant \mu^+((B^\varphi \setminus D_n^\varphi) \cup (D_n^\varphi \setminus E_n^\varphi))\\ &\leqslant \mu^+(B^\varphi \setminus D_n^\varphi) + \mu^+(D_n^\varphi \setminus E_n^\varphi)\\ &\leqslant \mu^+(B^\varphi \setminus K_n) + \mu^+(D_n^\varphi \setminus E_n^\varphi)\\ &\leqslant \mu^+\Big(\bigcup_{m=1}^\infty K_m \setminus K_n\Big) + \bigvee_i a_{i\varphi(i)}.\end{aligned}$$

The sequence $(E_n^\varphi)_n \in \mathscr{A}^-$ is nondecreasing and hence there exists a nondecreasing sequence $(L_n)_n \subset \mathscr{A}$ such that $E_n^\varphi \subset L_n$ for every n. Now

$$\mu_0(K_n \setminus L_n) \leqslant \mu^+(D_n^\varphi \setminus E_n^\varphi) < \bigvee_i a_{i\varphi(i)}$$

for all n. By the assumption (v) of C there is n_0 such that

$$\bigvee_{m=1}^\infty \mu_0(K_m \setminus K_n) < \bigvee_i a_{i\varphi(i)}$$

whenever $n > n_0$. Put $b_{ij} = 2a_{ij}$, $i, j = 1, 2, \ldots$. Then $(b_{ij})_{i,j}$ is a regulator and for every $\varphi \in \mathbb{N}^\mathbb{N}$ there are $B^\varphi \in \mathscr{A}^+$, $E_n^\varphi \in \mathscr{A}^-$, $E_n^\varphi \subset A \subset B^\varphi$ such that

$$\mu^+(B^\varphi \setminus E_n^\varphi) \leqslant \bigvee_i b_{i\varphi(i)}.$$

Then $A \in \mathscr{S}$ and

$$\mu(A) \leqslant \mu^+(B^\varphi) \leqslant \mu^+(B^\varphi \setminus E_n^\varphi) + \mu^-(E_n^\varphi) \leqslant \bigvee_i b_{i\varphi(i)} + \bigvee_n \mu(A_n)$$

for every $\varphi \in \mathbb{N}^\mathbb{N}$. Now,

$$\mu(A) \leqslant \bigvee_i b_{i\varphi(i)} + \bigvee_n \mu(A_n)$$

for every $\varphi \in \mathbb{N}^\mathbb{N}$ and by the weak σ-distributivity of G we get $\mu(A) \leqslant \bigvee_n \mu(A_n)$. Since $A_n \subset A$ $(n = 1, 2, \ldots)$, the reverse inequality holds by Lemma 6 and Lemma 8, hence

$$\mu(A) = \bigvee_n \mu(A_n).$$

□

10. Theorem. *The mapping $\mu\colon \mathscr{S} \to G$ is subadditive.*

Proof. Let $A, B \in \mathscr{S}$. Then there are regulators $(a_{ij})_{i,j}$, $(b_{ij})_{i,j}$ in G such that for every $\varphi \in \mathbb{N}^\mathbb{N}$ there are $A_1^\varphi, B_1^\varphi \in \mathscr{A}^-$, A_2^φ, $B_2^\varphi \in \mathscr{A}^+$, $A_1^\varphi \subset A \subset A_2^\varphi$, $B_1^\varphi \subset B \subset B_2^\varphi$ with $\mu^+(A_2^\varphi \setminus A_1^\varphi) < \bigvee_i a_{i\varphi(i)}$, $\mu^+(B_2^\varphi \setminus B_1^\varphi) < \bigvee_i b_{i\varphi(i)}$. Then

$$\bigvee_i a_{i\varphi(i)} > \mu^+(A_2^\varphi \setminus A_1^\varphi) \geqslant \mu^+(A_2^\varphi) - \mu^-(A_1^\varphi) \geqslant \mu^+(A_2^\varphi) - \mu(A),$$

$$\bigvee_i b_{i\varphi(i)} > \mu^+(B_2^\varphi \setminus B_1^\varphi) \geqslant \mu^+(B_2^\varphi) - \mu^-(B_1^\varphi) \geqslant \mu^+(B_2^\varphi) - \mu(B).$$

We get

$$\mu(A) + \mu(B) + \bigvee_i a_{i\varphi(i)} + \bigvee_i b_{i\varphi(i)} \geqslant \mu^+(A_2^\varphi) + \mu^+(B_2^\varphi) \geqslant \mu^+(A_2^\varphi \cup B_2^\varphi) \geqslant \mu(A \cup B).$$

Put $c_{ij} = 2(a_{ij} + b_{ij})$ for $i, j, \ldots$ Then $(c_{ij})_{i,j}$ is a regulator in G and

$$\bigvee_i a_{i\varphi(i)} + \bigvee_i b_{i\varphi(i)} \leqslant \bigvee_i c_{i\varphi(i)}$$

for every $\varphi \in \mathbb{N}^\mathbb{N}$. Hence

$$\mu(A \cup B) \leqslant \mu(A) + \mu(B) + \bigvee_i c_{i\varphi(i)}$$

for every $\varphi \in \mathbb{N}^\mathbb{N}$. By the weak σ-distributivity we have

$$\mu(A \cup B) \leqslant \mu(A) + \mu(B).$$

□

11. Theorem. *The set $\mathscr{S}$ is a σ-algebra of subsets of the set X. The mapping $\mu\colon \mathscr{S} \to G$ is an extension of μ_0, μ satisfies the conditions* (i)–(iii) *and* (vii), (viii). *If μ' is an extension of μ_0 and μ' satisfies* (ii), (vii) *and* (viii), *then $\mu' = \mu$.*

P r o o f. By Lemma 7 and Theorem 9 the set $\mathscr{S}$ is a σ-algebra and contains $\mathscr{A}$. It is evident that the mapping μ satisfies (i) and (ii). The subadditivity of μ, i.e. (iii), is proved in Theorem 10. The manner of the proof of (viii) is dual to the proof of Theorem 9. We prove uniqueness. Put

$$N = \{A \in \mathscr{S}\colon\ \mu(A) = \mu'(A)\}.$$

Then $N \supset \mathscr{A}^+$ and $N \supset \mathscr{A}^-$. Indeed, if $A_n \in \mathscr{A}$ for $n = 1, 2, \ldots$, $A_n \nearrow A$ (resp. $A_n \searrow A$), then

$$\mu'(A) = \bigvee_{n=1}^{\infty} \mu'(A_n) = \bigvee_{n=1}^{\infty} \mu(A_n) = \mu(A)$$
$$\Big(\text{resp. } \mu'(A) = \bigwedge_{n=1}^{\infty} \mu'(A_n) = \bigwedge_{n=1}^{\infty} \mu(A_n) = \mu(A)\Big).$$

Let $A \in \mathscr{S}$. Then there is a regulator $(a_{ij})_{i,j}$ in G such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there are $D_1^\varphi \in \mathscr{A}^-$, $D_2^\varphi \in \mathscr{A}^+$, $D_1^\varphi \subset A \subset D_2^\varphi$ with $\mu^+(D_2^\varphi \setminus D_1^\varphi) \leqslant \bigvee_i a_{i\varphi(i)}$. We have

$$\begin{aligned}\mu(A) &\leqslant \mu^+(D_2^\varphi) \leqslant \mu^+(D_2^\varphi \setminus D_1^\varphi) + \mu^-(D_1^\varphi)\\ &\leqslant \bigvee_i a_{i\varphi(i)} + \mu^-(D_1^\varphi) = \bigvee_i a_{i\varphi(i)} + \mu'(D_1^\varphi) \leqslant \bigvee_i a_{i\varphi(i)} + \mu'(A)\end{aligned}$$

for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and by the weak σ-distributivity,

$$\mu(A) \leqslant \mu'(A).$$

On the other hand,

$$\mu^+(D_2^\varphi) \leqslant \mu^+(D_2^\varphi \setminus D_1^\varphi) + \mu^-(D_1^\varphi) \leqslant \bigvee_i a_{i\varphi(i)} + \mu(A),$$

which yields

$$\mu(A) \geqslant \mu^+(D_2^\varphi) - \bigvee_i a_{i\varphi(i)} = \mu'(D_2^\varphi) - \bigvee_i a_{i\varphi(i)} \geqslant \mu'(A) - \bigvee_i a_{i\varphi(i)}$$

for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and we get $\mu(A) \geqslant \mu'(A)$. Hence

$$\mu(A) = \mu'(A).$$

□

References

[1] *V. N. Alexiuk, F. D. Beznosikov*: Extension of continuous outer measure on a Boolean algebra. Izv. VUZ *4(119)* (1972), 3–9. (In Russian.)

[2] *I. Dobrakov*: On subadditive operators on $C_0(T)$. Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. *20* (1972), 561–562. Zbl 0237.47035

[3] *L. Drewnowski*: Topological rings of sets, continuous set functions, integration I. Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. *20* (1972), 269–276. Zbl 0249.28004

[4] *L. Drewnowski*: Topological rings of sets, continuous set functions, integration II. Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. *20* (1972), 277–286. Zbl 0249.28005

[5] *D. H. Fremlin*: A direct proof of the Mathes-Wright integral extension theorem. J. London Math. Soc., II. Ser. 11 (1975), 267–284. Zbl 0313.06016

[6] *B. Riečan*: An extension of the Daniell integration scheme. Mat. Čas. *25* (1975), 211–219. Zbl 0308.28005

[7] *B. Riečan, T. Neubrunn*: Integral, Measure and Ordering. Mathematics and its Applications, 411. Kluwer, Dordrecht, 1997. Zbl 0916.28001

[8] *B. Riečan, P. Volauf*: On a technical lemma in lattice ordered groups. Acta Math. Univ. Comenianae *44–45* (1984), 31–35. Zbl 0558.06019

Author's address: Marta Vrábelová, Department of Mathematics, Constantin the Philosopher University, Faculty of Natural Sciences, Tr. A. Hlinku 1, 949 74 Nitra, Slovak Republic, e-mail: `mvrabelova@ukf.sk`.